Solution of an Initial Value Problem for the Multi-Velocity Neutron Transport Equation with a Slab Geometry

GEORGE H. PIMBLEY, JR.

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This paper treats an initial value problem which arises in several branches of physical science, namely, the problem stemming from the velocity dependent integro-differential transport equation. Perhaps most notably, it occurs in the study of neutron behavior in reactor physics. Here it appears usually as a linear equation because of neglect of interactions of neutrons with neutrons.

The general linear velocity-dependent equation is seldom used directly. Rather, systems of equations of integro-differential type are obtained for discrete velocity groups, as described in the literature (Ref. 2, 3, 7 and 10). Then the initial value problem is redefined in terms of this system.

In the past, two approaches have found employment, at least in a practical way: reduction by approximation to a classical diffusion problem and numerical integration by forward differences. To the writer's knowledge, however, no penetrating analysis of the nature of this problem and its solution was ever made until the appearance in 1955-56 for two papers (Ref. 8 and 9) by J. LEHNER & G. M. WING. Dealing only with the one-velocity case, they proved existence and uniqueness for both the infinite slab and the sphere. For the slab, they were able to discuss completely the structure of the solution. The present paper is an attempt to extend the work of LEHNER & WING to the more general n -velocity case.

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1. Let $N_i(x, \mu, t)$ be the density of neutrons at a point with space coordinate x moving in a direction with angle θ with respect to the x axis, with speed in velocity group i at time $t > 0$. Let $\mu = \cos \theta$. The x axis is taken perpendicular

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to the sides of the slab, from an origin at the center of the slab. Along the x axis, the slab extends from $-a$ to a , while the y and z axes are suppressed, the slab being of infinite extent along these coordinates. In the interior of the slab, the neutron behavior is governed by the linear "transport equation":

$$\frac{1}{V_i} \frac{\partial N_i}{\partial t} + \mu \frac{\partial N_i}{\partial x} + \sigma_i N_i = \frac{1}{2} \sum_{j=1}^n C_{ij} \int_{-1}^{+1} N_j(x, \mu', t) d\mu', \quad i = 1, 2, \dots, n,$$

where n is the number of velocity groups, and the positive quantities V_i , σ_i are respectively the constant neutron speed and the total cross-section associated with velocity group i . The positive constants C_{ij} are called transfer coefficients, the exact definition of which takes into account all the processes that lead to the transfer of neutrons from velocity group j to velocity group i (Ref. 2, page 168; see also 3, Chapter XIX).

The boundary conditions for the problem are homogeneous:

$$\begin{aligned} N_i(a, \mu, t) &= 0, & \mu < 0, & & i = 1, 2, \dots, n, \\ N_i(-a, \mu, t) &= 0, & \mu > 0, & \end{aligned}$$

for $t \geq 0$. In other words, neutrons do not enter the slab from the surrounding perfect absorber. An initial condition is prescribed:

$$N_i(x, \mu, 0) = f_i(x, \mu), \quad -a \leq x \leq a, \quad -1 \leq \mu \leq +1, \quad i = 1, 2, \dots, n.$$

Thus, starting with an initial density at $t = 0$, we follow the neutron behavior for $t > 0$.

For conciseness it is well to introduce vector matrix notation. We write the multi-group transport equations as one vector transport equation:

$$(1) \quad V^{-1} \frac{\partial N}{\partial t} + \mu \frac{\partial N}{\partial x} + \sigma N = \frac{1}{2} C \int_{-1}^{+1} N(x, \mu', t) d\mu'$$

where $N(x, \mu, t)$ is an n -component vector whose i^{th} component is the neutron density N_i for the i^{th} velocity group. V and σ are positive diagonal matrices of velocities and cross-sections respectively, while C is a generally non-symmetric matrix of positive transfer coefficients. The boundary and initial conditions assume a vector form:

$$\begin{aligned} (2) \quad & N(a, \mu, t) = 0, & \mu < 0, & & t \geq 0; \\ & N(-a, \mu, t) = 0, & \mu > 0, & \\ & N(x, \mu, 0) = f(x, \mu), & -a \leq x \leq +a, & & -1 \leq \mu \leq +1, \end{aligned}$$

where 0 is a vector with null components and $f(x, \mu)$ is a vector of initial density functions.

The vector transport equation may be regarded as an operator differential equation defined over a certain Hilbert space:

$$\frac{\partial N}{\partial t} = AN$$

where

$$(3) \quad A = -\mu V \frac{\partial}{\partial x} \cdot -\sigma V \cdot + \frac{1}{2} VC \int_{-1}^{+1} \cdot d\mu'.$$

A family of solution operators $T(t)$ is sought such that $N = T(t)f$ where f is a suitably restricted initial element.

2. The question of existence, uniqueness and the explicit representation of a family of solution operators for this problem is directly related to the spectral properties of the unbounded operator A , which we regard as being defined on a Hilbert space H .

Let H_1 be the space of Lebesgue square integrable functions defined on the fundamental rectangle: $|x| \leq a$, $|\mu| \leq 1$, with the usual scalar product $(f, g) = \int_{-1}^{+1} \int_{-a}^{+a} f\bar{g} \, dx \, d\mu$ and quadratic norm

$$||f|| = \left(\int_{-1}^{+1} \int_{-a}^{+a} |f|^2 \, dx \, d\mu \right)^{\frac{1}{2}}.$$

Then we define A on the Hilbert product space $H = H_1 \times H_1 \times \cdots \times H_1$ (taken n times for the n -velocity problem) with scalar product

$$(f, g) = \sum_{i=1}^n \int_{-1}^{+1} \int_{-a}^{+a} f_i \bar{g}_i \, dx \, d\mu$$

and norm

$$||f|| = \left(\sum_{i=1}^n \int_{-1}^{+1} \int_{-a}^{+a} |f_i|^2 \, dx \, d\mu \right)^{\frac{1}{2}}.$$

The integral part of the operator A , namely $\frac{1}{2} \int_{-1}^{+1} f(x, \mu) \, d\mu$, is a projection on H to a proper subspace defined as follows: Let L_2 be the space of square integrable functions defined on the interval $(-a, a)$ with scalar product $(f, g) = \int_{-a}^{+a} f\bar{g} \, dx$. Then define L as the product space $L = L_2 \times L_2 \times \cdots \times L_2$ (taken n times) with the scalar product $(f, g) = \sum_{i=1}^n \int_{-a}^{+a} f_i \bar{g}_i \, dx$. Since it seems there can be no confusion, we use the same scalar product notation for both H and L .

The operator (3) on H has as domain $D(A)$ the linear manifold of all n -tuples in H , with components absolutely continuous in x for each fixed μ in $|\mu| \leq 1$, which satisfy the homogeneous boundary conditions set for the problem. It will be shown that $D(A)$ is dense in H .

3. We now proceed to examine the spectral properties of A on H . Let λ be the complex spectral variable. Consider the system of integro-differential equations so ordered that $\min_i \sigma_i V_i = \sigma_n V_n$. So as not to complicate unduly the discussion, we suppose this to be a proper minimum. Then $-\sigma_n V_n > -\sigma_i V_i$, $i = 1, 2, \dots, n-1$. We treat, first of all, the half plane $\text{Re } \lambda \leq -\sigma_n V_n$.

Employing the following well known result (Ref. 6, page 31), it is possible to show that the half plane $\operatorname{Re} \lambda \leq -\sigma_n V_n$ is contained in the spectrum σA .

Theorem. Suppose there exists a sequence of elements $W_\delta \in D(A)$, $\delta \rightarrow 0$, such that

- (a) $\|W_\delta\| \geq k > 0$,
 (b) $\|(\lambda - A)W_\delta\| \rightarrow 0$ as $\delta \rightarrow 0$.

Then λ belongs to the spectrum of A , $\lambda \in \sigma A$.

For $0 < \delta < \frac{1}{2}$ and $\operatorname{Re} \lambda = \beta < -\sigma_n V_n$ consider the sequence of n -component vectors, elements of $D(A) \subset H$, which have null components: $(W_\delta)_i \equiv 0$, $i = 1, 2, \dots, n - 1$, but for which

$$(W_\delta)_n = e^{-(\sigma_n + \lambda/V_n)(x-a)/\mu} b_\delta(\mu) f(x),$$

where $f(x) = (x + a)/a$ and

$$b_\delta(\mu) = \begin{cases} \frac{1}{\delta}, & \delta^2 \leq \mu \leq \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Put $\sigma_n + \beta/V_n = \Lambda_n < 0$. We show that this sequence has the properties of the sequence in the theorem. First of all,

$$\begin{aligned} \|W_\delta\|^2 &= \int_{-1}^{+1} \int_{-a}^a e^{-2\Lambda_n(x-a)/\mu} b_\delta^2(\mu) f^2(x) dx d\mu > \int_{-1}^{+1} b_\delta^2(\mu) d\mu \int_0^a e^{-2\Lambda_n(x-a)/\mu} dx \\ &= \frac{1}{2|\Lambda_n|} \int_{\delta^2}^{\delta} \mu b_\delta^2(\mu) [1 - e^{2\Lambda_n(a/\mu)}] d\mu > \frac{\text{const.}}{\delta^2} \int_{\delta^2}^{\delta} \mu d\mu = \frac{\text{const.}}{\delta^2} (\delta^2 - \delta^4) \\ &= \text{const.} (1 - \delta^2) > \text{const.} \end{aligned}$$

Hence property (a) is satisfied. Again

$$\begin{aligned} (\lambda - A)W_\delta &= \mu V \frac{\partial W_\delta}{\partial x} + (\sigma V + \lambda I)W_\delta \\ &\quad - \frac{1}{2} VC \int_{-1}^{+1} W_\delta(x, \mu') d\mu' = V(T_1 + T_2) \end{aligned}$$

where it can be verified that

$$\|T_1\|^2 = \int_{-1}^{+1} \int_{-a}^a \mu^2 b_\delta^2(\mu) e^{-2\Lambda_n(x-a)/\mu} |f'(x)|^2 dx d\mu$$

and

$$\|T_2\|^2 = \frac{1}{4} \sum_{i=1}^n C_{in}^2 \int_{-1}^{+1} \int_{-a}^a dx d\mu \left| f(x) \int_{-1}^{+1} e^{-(\sigma_n + \lambda/V_n)(x-a)/\mu'} b_\delta(\mu') d\mu' \right|^2.$$

Now

$$\begin{aligned}
 ||T_1||^2 &= \frac{1}{a^2} \int_{-1}^{+1} \mu^2 b_\delta^2(\mu) d\mu \int_{-a}^a e^{-2\Lambda_n(x-a)/\mu} dx \\
 &= \frac{1}{2a^2 |\Lambda_n|} \int_{-1}^{+1} \mu^3 b_\delta^2(\mu) (1 - e^{4\Lambda_n(a/\mu)}) d\mu \\
 &\leq \frac{1}{2a^2 |\Lambda_n|} \int_{\delta^2}^{\delta} \mu^3 d\mu = \frac{\delta^4 - \delta^8}{8a^2 \Lambda_n \delta^2} \rightarrow 0 \quad \text{as } \delta \rightarrow 0.
 \end{aligned}$$

Also

$$\begin{aligned}
 ||T_2||^2 &\leq 2 \sum_{i=1}^n C_{in}^2 \int_{-a}^a dx \left| \frac{1}{\delta} \int_{\delta^2}^{\delta} e^{-(\sigma_n + \lambda/V_n)(x-a)/\mu'} d\mu' \right|^2 \\
 &\leq \frac{2}{\delta^2} \sum_{i=1}^n C_{in}^2 \int_{-a}^a dx \left(\int_{\delta^2}^{\delta} e^{-\Lambda_n(x-a)/\delta} d\mu' \right)^2 \\
 &= \frac{2(\delta - \delta^2)^2}{\delta^2} \sum_{i=1}^n C_{in}^2 \int_{-a}^a e^{-2\Lambda_n(x-a)/\delta} dx \\
 &= (1 - \delta)^2 \sum_{i=1}^n C_{in}^2 \frac{\delta}{|\Lambda_n|} (1 - e^{4\Lambda_n(a/\delta)}) \\
 &\leq \sum_{i=1}^n C_{in}^2 \frac{\delta}{|\Lambda_n|} (1 - e^{4\Lambda_n(a/\delta)}) \rightarrow 0 \quad \text{as } \delta \rightarrow 0.
 \end{aligned}$$

Hence

$$||(\lambda - A)W_\delta|| \leq ||V|| (||T_1|| + ||T_2||) \rightarrow 0 \quad \text{as } \delta \rightarrow 0,$$

establishing (b). Hence λ with $\text{Re } \lambda < -\sigma_n V_n$ is in σA . Since σA is a closed set, the closed half plane $\text{Re } \lambda \leq -\sigma_n V_n$ is contained in the spectrum. In general we cannot say more about this half plane without restricting the matrix C . The fine structure of this part of σA is not an essential item for representation of solutions of the transport equations, however. We wish to confine our restrictions on C in ways more suitable to the problem at hand. But, for future reference, we have proved

Theorem 1. *Let the components of the transport equation be so arranged that $\sigma_n V_n = \max_i \sigma_i V_i$, $i = 1, 2, \dots, n$. Then the half plane $\text{Re } \lambda \leq -\sigma_n V_n$ is contained in σA .*

4. Interest now centers on the remainder of the spectral plane, $\text{Re } \lambda > -\sigma_n V_n$. Suppose such a λ were in the point spectrum of $A(P\sigma A)$, i.e. $A\psi = \lambda\psi$, $\psi \neq 0$. Then we have $(\psi \in D(A))$

$$(4) \quad \mu \frac{\partial \psi}{\partial x} + (\sigma + \lambda V^{-1})\psi = \frac{1}{2} C \int_{-1}^{+1} \psi d\mu' = C\phi(x)$$

where $\phi \in L$. Considering (4) to be a system of differential equations to be solved for ψ in terms of ϕ , we get

$$(5) \quad \mu \psi(x, \mu) = \begin{cases} \int_{-a}^x \exp \{-(\sigma + \lambda V^{-1})(x - x')/\mu\} C\phi(x') dx', & \mu > 0, \\ -\int_x^a \exp \{-(\sigma + \lambda V^{-1})(x - x')/\mu\} C\phi(x') dx', & \mu < 0, \end{cases}$$

where the exponentiated matrix $\exp \{-(\sigma + \lambda V^{-1})(x - x')/\mu\}$ is diagonal. Since $\phi \in L$, a straightforward application of Schwarz' lemma shows that ϕ has summable components. Then, by definition,

$$\begin{aligned} \phi(x) &= \frac{1}{2} \int_{-1}^{+1} \psi(x, \mu) d\mu = \frac{1}{2} \int_{-1}^0 \psi(x, \mu) d\mu + \frac{1}{2} \int_0^1 \psi(x, \mu) d\mu \\ &= -\frac{1}{2} \int_{-1}^0 \frac{d\mu}{\mu} \int_x^a \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} C\phi(x') dx' \\ &\quad + \frac{1}{2} \int_0^1 \frac{d\mu}{\mu} \int_{-a}^x \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} C\phi(x') dx' \\ &= -\frac{1}{2} \int_x^a \int_{-1}^0 \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} \frac{d\mu}{\mu} C\phi(x') dx' \\ &\quad + \frac{1}{2} \int_{-a}^x \int_0^1 \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} \frac{d\mu}{\mu} C\phi(x') dx' \\ &= \frac{1}{2} \int_x^a \int_1^\infty \exp \{(\sigma + \lambda V^{-1})(x - x')t\} \frac{dt}{t} C\phi(x') dx' \\ &\quad + \frac{1}{2} \int_{-a}^x \int_1^\infty \exp \{-(\sigma + \lambda V^{-1})(x - x')t\} \frac{dt}{t} C\phi(x') dx' \\ &= \frac{1}{2} \int_{-a}^a \int_1^\infty \exp \{(\sigma + \lambda V^{-1}) |x - x'| t\} \frac{dt}{t} C\phi(x') dx'. \end{aligned}$$

Interchange of integrations is justified by a known theorem (Ref. 15, page 390). The matrix

$$\int_1^\infty \exp \{(\sigma + \lambda V^{-1}) |x - x'| t\} \frac{dt}{t}$$

is immediately recognized as a diagonal matrix whose diagonal entries are exponential integral functions. We denote it by $E\{(\sigma + \lambda V^{-1}) |x - x'| \}$. The result is a system of homogeneous integral equations which we write

$$(6) \quad \phi(x) = \frac{1}{2} \int_{-a}^a E\{(\sigma + \lambda V^{-1}) |x - x'| \} C\phi(x') dx'.$$

Unfortunately the kernel matrix EC is in general unsymmetric; this fact causes most of the difficulty in the multi-velocity theory. Let us write the bounded

operator represented by this kernel matrix on the space L as $E_\lambda C$. The above derivation can be carried out in reverse. Hence we have

Theorem 2. *A necessary and sufficient condition that $\lambda \in \text{P}\sigma A$ on H with $\text{Re } \lambda \geq -\sigma_n V_n$, $\lambda \neq -\sigma_n V_n$, is that 1 should belong to the point spectrum of the operator $\frac{1}{2}E_\lambda C$ on L ($\text{P}\sigma \frac{1}{2}E_\lambda C$). An eigenelement ψ corresponding to λ is related to an eigenelement ϕ corresponding to 1 through the projection $\phi(x) = \frac{1}{2} \int_{-1}^{+1} \psi(x, \mu) d\mu$, and by (5).*

Hence we have characterized, for the time being, that part of $\text{P}\sigma A$ for which $\text{Re } \lambda \geq -\sigma_n V_n$, $\lambda \neq -\sigma_n V_n$.

5. Let $\Gamma = \{\lambda \mid \text{Re } \lambda > -\sigma_n V_n, \lambda \notin \text{P}\sigma A\}$. Points $\lambda \in \Gamma$ are either in the continuous ($\text{C}\sigma A$) or the residual ($\text{R}\sigma A$) spectra or in the resolvent set $\rho(A)$. In any of these cases $(\lambda - A)^{-1}$ exists, though its domain $D_\lambda \neq H$ in the case of the first two possibilities. We now prove that for $\lambda \in \Gamma$, $D_\lambda = H$ and $(\lambda - A)^{-1}$ is bounded.

Consider $(\lambda - A)u = g$ with $g \in H$. This is written explicitly as

$$\mu V \frac{\partial u}{\partial x} + (\sigma V + \lambda I)u - \frac{1}{2} VC \int_{-1}^{+1} u(x, \mu') d\mu' = g(x, \mu)$$

whence

$$(7) \quad u(x, \mu) = \begin{cases} \frac{1}{\mu} \int_{-a}^x \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} \\ \quad [\frac{1}{2} C \xi(x') + V^{-1} g(x', \mu)] dx', & \mu > 0, \\ -\frac{1}{\mu} \int_x^a \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} \\ \quad [\frac{1}{2} C \xi(x') + V^{-1} g(x', \mu)] dx', & \mu < 0, \end{cases}$$

where we have employed, as before, the exponentiated matrix notation and where $\xi(x) = \int_{-1}^{+1} u(x, \mu) d\mu$. Integrating over μ and proceeding as in section 4 we obtain the non-homogeneous system

$$\xi(x) = \frac{1}{2} \int_{-a}^a E \{ (\sigma + \lambda V^{-1}) |x - y| \} C \xi(y) dy + G(x)$$

where the matrix E is that appearing in section 4 and

$$G(x) = \int_0^1 \frac{d\mu}{\mu} \int_{-a}^x \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} V^{-1} g(x', \mu) dx' \\ - \int_{-1}^0 \frac{d\mu}{\mu} \int_x^a \exp \left\{ -(\sigma + \lambda V^{-1}) \frac{x - x'}{\mu} \right\} V^{-1} g(x', \mu) dx'.$$

We show that G , so defined, belongs to the space L , having $g \in H$. We write, componentwise, with $\beta = \text{Re } \lambda$ and $\Lambda_i = \sigma_i + \beta/v_i$,

$$\begin{aligned}
|G^{(i)}(x)| &= \left| \int_0^1 \frac{d\mu}{\mu} \int_{-a}^x e^{-(\sigma_i + \lambda/V_i)(x-x')/\mu} \frac{1}{V_i} g^{(i)}(x', \mu) dx' \right. \\
&\quad \left. - \int_{-1}^0 \frac{d\mu}{\mu} \int_x^a e^{-(\sigma_i + \lambda/V_i)(x-x')/\mu} \frac{1}{V_i} g^{(i)}(x', \mu) dx' \right| \\
&\leq \int_0^1 d\mu \int_{-a}^x \frac{e^{-\Lambda_i(x-x')/\mu}}{\mu} \frac{1}{V_i} |g^{(i)}(x', \mu)| dx' \\
&\quad + \int_{-1}^0 d\mu \int_x^a \frac{e^{-\Lambda_i(x-x')/\mu}}{|\mu|} \frac{1}{V_i} |g^{(i)}(x', \mu)| dx' \\
&= \int_0^1 K_1^{(i)}(x, \mu) d\mu + \int_{-1}^0 K_2^{(i)}(x, \mu) d\mu = G_1^{(i)} + G_2^{(i)}, \quad i = 1, 2, \dots, n.
\end{aligned}$$

Now

$$\begin{aligned}
|K_1^{(i)}(x, \mu)| &= \int_0^{x+a} \frac{e^{-\Lambda_i(t/\mu)}}{\mu} \frac{1}{V_i} |g^{(i)}(x-t, \mu)| dt \\
&\leq \int_0^\infty \frac{e^{-\Lambda_i(t/2\mu)}}{\sqrt{\mu}} \cdot \frac{1}{V_i} |g^{(i)}(x-t, \mu)| \cdot \frac{e^{-\Lambda_i(t/2\mu)}}{\sqrt{\mu}} dt
\end{aligned}$$

with $g^{(i)}(x, \mu) \equiv 0$ outside $|x| \leq a$, $|\mu| \leq 1$. By Schwarz' inequality

$$|K_1^{(i)}(x, \mu)|^2 \leq \frac{1}{|\Lambda_i|} \int_0^\infty \frac{e^{-\Lambda_i(t/\mu)}}{\mu} \frac{1}{V_i^2} |g^{(i)}(x-t, \mu)|^2 dt.$$

Again, by Schwarz' inequality,

$$\begin{aligned}
|G_1^{(i)}(x)|^2 &\leq \int_0^1 d\mu \int_0^1 |K_1^{(i)}(x, \mu')|^2 d\mu' \\
&\leq \frac{1}{|\Lambda_i|} \int_0^1 d\mu \int_0^\infty \frac{e^{-\Lambda_i(t/\mu)}}{\mu V_i^2} |g^{(i)}(x-t, \mu)|^2 dt,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{-a}^a |G_1^{(i)}(x)|^2 dx &\leq \frac{1}{|\Lambda_i|} \int_0^1 d\mu \int_0^\infty \frac{e^{-\Lambda_i(t/\mu)}}{\mu} dt \int_{-a}^a \frac{|g^{(i)}(x-t, \mu)|^2}{V_i^2} dx \\
&\leq \frac{1}{|\Lambda_i|} \int_0^1 d\mu \int_0^\infty \frac{e^{-\Lambda_i(t/\mu)}}{\mu} dt \int_{-\infty}^\infty \frac{|g^{(i)}(x-t, \mu)|^2}{V_i^2} dx \\
&= \frac{1}{|\Lambda_i|} \int_0^1 d\mu \int_0^\infty \frac{e^{-\Lambda_i(t/\mu)}}{\mu} dt \int_{-\infty}^\infty \frac{|g^{(i)}(x, \mu)|^2}{V_i^2} dx \\
&= \frac{1}{|\Lambda_i|^2} \int_0^1 d\mu \int_{-\infty}^\infty \frac{|g^{(i)}(x, \mu)|^2}{V_i^2} dx \\
&= \frac{1}{|\Lambda_i|^2} \int_0^1 d\mu \int_{-a}^a \frac{|g^{(i)}(x, \mu)|^2}{V_i^2} dx, \quad i = 1, 2, \dots, n.
\end{aligned}$$

Working analogously with $K_2^{(i)}$, it is found that

$$\int_{-a}^a |G_2^{(i)}(x)|^2 dx \leq \frac{1}{|\Lambda_i|^2} \int_{-1}^0 \int_{-a}^a \frac{|g^{(i)}(x, \mu)|^2}{V_i^2} dx d\mu, \quad i = 1, 2, \dots, n.$$

Now, since $|G^{(i)}(x)| \leq G_1^{(i)}(x) + G_2^{(i)}(x)$, we get by using Minkowski's inequality

$$\begin{aligned} \left(\int_{-a}^a |G^{(i)}(x)|^2 dx \right)^{\frac{1}{2}} &\leq \left(\int_{-a}^a |G_1^{(i)}(x)|^2 dx \right)^{\frac{1}{2}} + \left(\int_{-a}^a |G_2^{(i)}(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq \frac{1}{|\Lambda_i|} \left[\left(\int_0^1 \int_{-a}^a \frac{|g^{(i)}(x, \mu)|^2}{V_i^2} dx d\mu \right)^{\frac{1}{2}} + \left(\int_{-1}^0 \int_{-a}^a \frac{|g^{(i)}(x, \mu)|^2}{V_i^2} dx d\mu \right)^{\frac{1}{2}} \right] \\ &\leq \frac{2}{|\sigma_i V_i + \beta|} \left(\int_{-1}^{+1} \int_{-a}^a |g^{(i)}(x, \mu)|^2 dx d\mu \right)^{\frac{1}{2}}, \quad i = 1, 2, \dots, n. \end{aligned}$$

Then

$$\begin{aligned} \|G\|^2 &= 2 \sum_{i=1}^n \int_{-a}^a |G^{(i)}(x)|^2 dx \leq 2 \sum_{i=1}^n \frac{4}{|\sigma_i V_i + \beta|^2} \int_{-1}^{+1} \int_{-a}^a |g^{(i)}(x, \mu)|^2 dx d\mu \\ &\leq \frac{8}{|\sigma_n V_n + \beta|^2} \sum_{i=1}^n \int_{-1}^{+1} \int_{-a}^a |g^{(i)}(x, \mu)|^2 dx d\mu = \frac{8}{|\sigma_n V_n + \beta|^2} \|g\|^2 \end{aligned}$$

so that

$$\|G\| \leq \frac{2\sqrt{2}}{|\sigma_n V_n + \beta|} \|g\| < \infty.$$

Hence $G \in L$.

Thus we have the equation

$$\xi = \frac{1}{2} E_\lambda C \xi + G$$

where $G \in L$, and $\frac{1}{2} E_\lambda C$ can be shown to be completely continuous on L . For $\lambda \in \Gamma$, 1 is not an eigenvalue for $\frac{1}{2} E_\lambda C$. By the alternative principle, ξ is uniquely determined for $G \in L$, thus for given $g \in H$. Since $(1 - \frac{1}{2} E_\lambda C)$ is bounded, mapping L onto itself, $(1 - \frac{1}{2} E_\lambda C)^{-1}$ is also bounded. Thus

$$\|\xi\| = \|(1 - \frac{1}{2} E_\lambda C)^{-1} G\| \leq M \|G\| \leq \frac{M_1}{|\sigma_n V_n + \beta|} \|g\|$$

where M_1 , M_2 , etc. are constants depending only on $\lambda \in \Gamma$.

Now put $h = \frac{1}{2} C \xi + V^{-1} g$. Then

$$\begin{aligned} \|h\| &\leq \frac{1}{2} \|C\| \cdot \|\xi\| + \|V^{-1}\| \cdot \|g\| \\ &\leq \frac{1}{2} \frac{\|C\| M_1 \|g\|}{|\sigma_n V_n + \beta|} + \|V^{-1}\| \cdot \|g\| = \frac{M_2 \|g\|}{|\sigma_n V_n + \beta|}, \end{aligned}$$

which shows that $h \in H$. Now from (7) we write componentwise

$$|u^{(i)}(x, \mu)| \leq \begin{cases} \int_{-a}^x \frac{e^{-\Lambda_i(x-x')/\mu}}{\mu} |h^{(i)}(x', \mu)| dx' = J_1^{(i)}(x, \mu), & \mu > 0, \\ \int_x^a \frac{e^{-\Lambda_i(x-x')/\mu}}{|\mu|} |h^{(i)}(x', \mu)| dx' = J_2^{(i)}(x, \mu), & \mu < 0. \end{cases}$$

$J_1^{(i)}$ and $J_2^{(i)}$ are similar to $K_1^{(i)}$ and $K_2^{(i)}$. Defining $h^{(i)}(x, \mu) \equiv 0$ outside $|x| \leq a$, $|\mu| \leq 1$, we have therefore

$$|J_1^{(i)}(x, \mu)|^2 \leq \frac{1}{|\Lambda_i|} \int_0^\infty \frac{e^{-\Lambda_i t/\mu}}{\mu} |h^{(i)}(x - t, \mu)|^2 dt, \quad \mu > 0,$$

$$|J_2^{(i)}(x, \mu)|^2 \leq \frac{1}{|\Lambda_i|} \int_{-\infty}^0 \frac{e^{-\Lambda_i t/\mu}}{|\mu|} |h^{(i)}(x - t, \mu)|^2 dt, \quad \mu < 0.$$

Then

$$\begin{aligned} \int_{-1}^{+1} \int_{-a}^a |u^{(i)}(x, \mu)|^2 d\mu dx &= \int_0^1 \int_{-a}^a |u^{(i)}(x, \mu)|^2 d\mu dx \\ &+ \int_{-1}^0 \int_{-a}^a |u^{(i)}(x, \mu)|^2 d\mu dx \\ &\leq \int_0^1 \int_{-a}^a |J_1^{(i)}(x, \mu)|^2 d\mu dx + \int_{-1}^0 \int_{-a}^a |J_2^{(i)}(x, \mu)|^2 d\mu dx \\ &\leq \frac{1}{|\Lambda_i|} \int_0^1 d\mu \int_0^\infty \frac{e^{-\Lambda_i t/\mu}}{\mu} dt \int_{-a}^a |h^{(i)}(x - t, \mu)|^2 dx \\ &+ \frac{1}{|\Lambda_i|} \int_{-1}^0 d\mu \int_{-\infty}^0 \frac{e^{-\Lambda_i t/\mu}}{|\mu|} dt \int_{-a}^a |h^{(i)}(x - t, \mu)|^2 dx \\ &\leq \frac{1}{|\Lambda_i|} \int_0^1 d\mu \int_0^\infty \frac{e^{-\Lambda_i t/\mu}}{\mu} dt \int_{-\infty}^\infty |h^{(i)}(x, \mu)|^2 dx \\ &+ \frac{1}{|\Lambda_i|} \int_{-1}^0 d\mu \int_{-\infty}^0 \frac{e^{-\Lambda_i t/\mu}}{|\mu|} dt \int_{-\infty}^\infty |h^{(i)}(x, \mu)|^2 dx \\ &\leq \frac{1}{|\Lambda_i|^2} \left[\int_0^1 \int_{-a}^a |h^{(i)}(x, \mu)|^2 dx d\mu + \int_{-1}^0 \int_{-a}^a |h^{(i)}(x, \mu)|^2 dx d\mu \right] \\ &= \frac{1}{|\Lambda_i|^2} \int_{-1}^{+1} \int_{-a}^a |h^{(i)}(x, \mu)|^2 dx d\mu. \end{aligned}$$

Therefore

$$\begin{aligned} \|u\|^2 &= \sum_{i=1}^n \int_{-1}^{+1} \int_{-a}^a |u^{(i)}(x, \mu)|^2 d\mu dx \leq \sum_{i=1}^n \frac{1}{|\Lambda_i|^2} \int_{-1}^{+1} \int_{-a}^a |h^{(i)}(x, \mu)|^2 dx d\mu \\ &\leq \frac{1}{|\Lambda_n|^2} \sum_{i=1}^n \int_{-1}^{+1} \int_{-a}^a |h^{(i)}(x, \mu)|^2 dx d\mu = \frac{V_n^2}{|\sigma_n V_n + \beta|^2} \|h\|^2 \end{aligned}$$

and

$$\|u\| \leq \frac{V_n}{|\sigma_n V_n + \beta|} \|h\| \leq \frac{V_n M_2}{|\sigma_n V_n + \beta|^2} \|g\| = \frac{M_3}{|\sigma_n V_n + \beta|^2} \|g\|$$

where M_3 depends only on λ . Therefore $u \in H$ and $\|u\| = \|(\lambda - A)^{-1}g\| \leq M_4 \|g\|$, so that $(\lambda - A)^{-1}$ is bounded with domain $D_\lambda = H$.

Hence we have the following

Theorem 3. *The open half plane $\operatorname{Re} \lambda > -\sigma_n V_n$, deleting any points $\lambda \in P\sigma A$, is contained in the resolvent set $\rho(A)$.*

6. The remaining significant question about σA is the nature of that part of the point spectrum contained in the half-plane $\operatorname{Re} \lambda \geq -\sigma_n V_n$. We shall call it the restricted point spectrum $P\sigma_r A$. Some general results can be obtained about this set; for the particular results we need, however, it has been necessary to restrict the class of allowable matrices C .

By Theorem 2, any discussion of $P\sigma_r A$ will involve the homogeneous integral equations (6), or in short

$$\phi = \frac{1}{2} E_\lambda C \phi.$$

The matrix $E\{|x|\}$, of course, has exponential integrals $\int_1^\infty t^{-1} e^{-|x|t} dt$ as diagonal elements, with zeros off the diagonal.

Theorem 4. *$P\sigma_r A$ is bounded in the complex λ plane.*

Proof. If $\lambda \in P\sigma_r A$, then $(1 - \frac{1}{2} E_\lambda C)^{-1}$ does not exist. On the other hand,

$$\begin{aligned} \|E_\lambda C\|^2 &\leq \frac{1}{4} \sum_{i,j=1}^n C_{ij}^2 \int_{-a}^a \int_{-a}^a \left| E \left[\left(\sigma_i + \frac{\lambda}{V_i} \right) |x - y| \right] \right|^2 dx dy \\ &< \frac{1}{4} \sum_{i,j=1}^n C_{ij}^2 \int_{-a}^a dx \int_{-\infty}^{\infty} \left| E \left[\left(\sigma_i + \frac{\lambda}{V_i} \right) |y| \right] \right|^2 dy \end{aligned}$$

where $E[|x|]$ stands for the individual exponential integral function. Let $\Lambda_i = \sigma_i + \lambda/V_i$. Now

$$E[\Lambda_i |y|] = \frac{e^{-\Lambda_i |y|}}{\Lambda_i |y|} + \frac{1}{\Lambda_i |y|} \int_1^\infty e^{-\Lambda_i |y|\mu} \frac{d}{d\mu} \left(\frac{1}{\mu} \right) d\mu.$$

Following steps similar to those of J. LEHNER & G. M. WING (Ref. 9, page 132, lemma 5), one finds that $|E[\Lambda_i |y|]| \leq 2/|\Lambda_i| |y|$. For $|\Lambda_i| |y| < 1$ one has the estimate

$$E[\Lambda_i |y|] \leq \text{const.} (\log^2 |\Lambda_i| |y| + \frac{1}{4}\pi^2)^{\frac{1}{2}}.$$

Hence one obtains

$$\int_{-\infty}^{\infty} |E[\Lambda_i |y|]|^2 dy \leq \frac{\text{const.}}{|\Lambda_i|} \leq \frac{\text{const.}}{|\sigma_n + \lambda/V_n|},$$

and

$$\|E_\lambda C\|^2 < \frac{\text{const.}}{|\sigma_n V_n + \lambda|}.$$

Thus for $\operatorname{Re} \lambda \geq -\sigma_n V_n$ and $|\lambda|$ sufficiently large, Neumann's series converges, and $(1 - \frac{1}{2} E_\lambda C)^{-1}$ does exist. This proves the theorem.

Lemma 1. Suppose $P_{\sigma_p} A$ has an accumulation point λ^* . Let $\{\lambda_p\}$ be a sequence such that $\lambda_p \rightarrow \lambda^*$. The i^{th} components of the corresponding sequence of normalized eigenelements $\{\phi_p\}$ form an equicontinuous set.

Proof.

$$\begin{aligned} |\phi_p^{(i)}(x+h) - \phi_p^{(i)}(x)| &< \left\{ \frac{n}{4} \max_{i,i} C_{ii} \int_{-2a}^{2a} \left| E \left[\left(\sigma_i + \frac{\lambda_p}{V_i} \right) |u+h| \right] \right. \right. \\ &\quad \left. \left. - E \left[\left(\sigma_i + \frac{\lambda_p}{V_i} \right) |u| \right] \right|^2 du \right\}^{\frac{1}{2}} \\ &< \frac{\sqrt{n}}{2} \max_{i,i} C_{ii} \left[\int_{-2a}^{2a} \left| \log \frac{|u+h|}{|u|} \right|^2 du \right]^{\frac{1}{2}} \\ &\quad + \frac{\sqrt{n}}{2} \max_{i,i} C_{ii} \left[\int_{-2a}^{2a} |g^{(i)}(u+h, \lambda_p) - g^{(i)}(u, \lambda_p)|^2 du \right]^{\frac{1}{2}} \end{aligned}$$

with

$$g^{(i)}(u, \lambda_p) = \sum_{m=1}^{\infty} \frac{(-1)^m (\sigma_i + \lambda_p/V_i)^m |u|^m}{m \cdot m!}.$$

Straightforward applications of a known result (Ref. 15, page 397, ex. 18) and Dini's theorem give

$$|\phi_p^{(i)}(x+h) - \phi_p^{(i)}(x)| \rightarrow 0 \quad \text{as } h \rightarrow 0$$

independently of x and p , $p = 1, 2, \dots$, $-a \leq x \leq a$. This proves the assertion.

Lemma 2. Under the same hypothesis, the components of the sequence $\{\phi_p\}$ of normalized eigenelements are uniformly bounded.

Proof. If $\lambda^* \neq -\sigma_n V_n$, the proof is straightforward. Let $\Lambda_{ip} = \sigma_i + \lambda_p/V_i$. By Schwarz' inequality

$$\begin{aligned} |\phi_p^{(i)}(x)| &= \frac{1}{2} \left| \int_{-a}^a E[\Lambda_{ip} |x-y|] \sum_{i=1}^n C_{ii} \phi_p^{(i)}(y) dy \right| \\ &\leq \frac{n}{2} \max_{i,i} C_{ii} \left[\int_{-a}^a |E[\Lambda_{ip} |x-y|]|^2 dy \right]^{\frac{1}{2}} \\ &< \frac{n}{2} \max_{i,i} C_{ii} \left[\frac{\max_i V_i}{|\sigma_i V_i + \lambda_p|} \int_{-\infty}^{\infty} |E[u]|^2 du \right]^{\frac{1}{2}} \end{aligned}$$

which, however, is bounded independently of p since, if $\lambda^* \neq -\sigma_n V_n$,

$$|\sigma_i V_i + \lambda_p| > \text{const.} \cdot |\sigma_i V_i + \lambda^*| > 0.$$

If $\lambda^* = -\sigma_n V_n$, the same argument suffices for those components $\phi_p^{(i)}$ such that $\sigma_i V_i > \sigma_n V_n = \min_i \sigma_i V_i$. If, however, $\sigma_i V_i = \sigma_n V_n$, a greatly expanded argument is needed. Consider, in fact, the n^{th} component, with the convention that $\sigma_n V_n = \min_i \sigma_i V_i$, where $\sigma_i V_i \neq \sigma_n V_n$, $i \neq n$.

$$\begin{aligned}
\sum_{i=1}^n C_{ni} \phi_p^{(i)} &= \frac{1}{2} \int_{-a}^a \sum_{k=1}^n C_{nk} E[\Lambda_{kp} |x-y|] \sum_{i=1}^n C_{ki} \phi_p^{(i)} dy \\
&= \frac{1}{2} \int_{-a}^a \sum_{k=1}^n C_{nk} \{-\log |x-y| - \gamma - g^{(n)}(x-y, \lambda_p)\} \cdot \sum_{i=1}^n C_{ki} \phi_p^{(i)} dy \\
&\quad - \frac{1}{2} \sum_{k=1}^n C_{nk} \log \Lambda_{kp} \int_{-a}^a \sum_{i=1}^n C_{ki} \phi_p^{(i)}(y) dy
\end{aligned}$$

where γ is Euler's constant and $g^{(n)}(u, \lambda_p)$ was previously defined. Then

$$\begin{aligned}
&|1 + aC_{nn} \log \Lambda_{np}| \cdot \left| \int_{-a}^a \sum_{i=1}^n C_{ni} \phi_p^{(i)} dx \right| \\
&\leq \frac{1}{2} \int_{-a}^a dx \int_{-a}^a \sum_{k=1}^{n-1} C_{nk} |E[\Lambda_{kp} |x-y|]| \cdot \left| \sum_{i=1}^n C_{ki} \phi_p^{(i)} \right| dy \\
&\quad + \frac{1}{2} \int_{-a}^a dx \int_{-a}^a C_{nn} |\log |x-y| + \gamma + g^{(n)}(x-y, \lambda_p)| \cdot \left| \sum_{i=1}^n C_{ni} \phi_p^{(i)} \right| dy \\
&\cong \sqrt{\frac{a}{2}} \sum_{k=1}^{n-1} nC_{nk} (\max_{i,i} C_{ii}) \left[\int_{-a}^a \int_{-a}^a |E[\Lambda_{kp} |x-y|]|^2 dx dy \right]^{\frac{1}{2}} \\
&\quad + \sqrt{\frac{a}{2}} nC_{nn} (\max_{i,i} C_{ii}) \left[\int_{-a}^a \int_{-a}^a |\log |x-y| + \gamma + g^{(n)}(x-y, \lambda_p)|^2 dx dy \right]^{\frac{1}{2}} \\
&< \sqrt{\frac{a}{2}} n (\max_{i,i} C_{ii})^2 \left\{ \sum_{k=1}^{n-1} \left[\frac{2a \max_k V_k}{|\sigma_k V_k + \beta_p|} \int_{-\infty}^{\infty} |E[|u|]|^2 du \right]^{\frac{1}{2}} \right. \\
&\quad \left. + \left[2a \int_{-2a}^{2a} ||\log |u|| + \gamma + e^{\mathbf{M}|u|/V_n}|^2 du \right]^{\frac{1}{2}} \right\} \\
&< \text{const.}, \quad |\sigma_n V_n + \lambda_p| < M,
\end{aligned}$$

since

$$|\sigma_k V_k + \beta_p| > |\sigma_k V_k - \sigma_n V_n| > 0, \quad k = 1, \dots, n-1, \quad \beta_p = \text{Re } \lambda_p.$$

Hence

$$\left| \int_{-a}^a \sum_{i=1}^n C_{ni} \phi_p^{(i)}(x) dx \right| < \frac{\text{const.}}{|1 + aC_{nn} \log (\sigma_n + \lambda_p/V_n)|}.$$

Now

$$\begin{aligned}
|\phi_p^{(n)}(x)| &\leq \frac{1}{2} \int_{-a}^a |\log |x-y| + \gamma + g^{(n)}(x-y, \lambda_p)| \sum_{i=1}^n C_{ni} |\phi_p^{(i)}(y)| dy \\
&\quad + \frac{1}{2} |\log \Lambda_{np}| \cdot \left| \int_{-a}^a \sum_{i=1}^n C_{ni} \phi_p^{(i)}(y) dy \right| \\
&< \frac{n \max_{i,i} C_{ii}}{2} \left[\int_{-a}^a ||\log |u|| + \gamma + e^{\mathbf{M}|u|/V_n}|^2 du \right]^{\frac{1}{2}} \\
&\quad + \frac{1}{2} \frac{\text{const.} \cdot |\log (\sigma_n + \lambda_p/V_n)|}{|1 + aC_{nn} \log (\sigma_n + \lambda_p/V_n)|} < \text{const.},
\end{aligned}$$

$$\Lambda_{np} = \sigma_n + \lambda_p/V_n, \quad |\sigma_n V_n + \lambda_p| < M,$$

which proves the lemma for the n^{th} component. The argument would be similar, but yet more complicated, if $\sigma_i V_i = \sigma_n V_n$ for more than one index i . Of course, it would be possible in an actual physical problem to choose velocity groups so as to avoid this extra complication.

On the basis of Lemmas 1 and 2 and Ascoli's theorem, one can choose a subsequence $\lambda_{p_i} \rightarrow \lambda^*$ such that any sequence of components of the corresponding eigenelements $\{\phi_{p_i}\}$ converges uniformly to a continuous limit. Again, by extracting subsequences only a finite number of times, one can obviously arrive at a sequence of eigenelements for each of which the components form a uniformly convergent sequence. If we consider the correct subsequence already extracted, we may state

Lemma 3. *If λ^* is an accumulation point of $P_{\sigma}A$, then there exists a sequence $\lambda_p \rightarrow \lambda^*$ for which an associated sequence of normalized eigenelements $\{\phi_p\}$, $\phi_p = \frac{1}{2}E_{\lambda}C\phi_p$, possesses uniformly convergent continuous components. The limit element ϕ^* also has continuous components on the interval $[-a, a]$. Moreover, $||\phi^*|| = 1$.*

For the next theorem, it is sufficient to assume that $C_{n_j} \neq 0$, $j = 1, 2, \dots, n$. A more general case could be treated with a continuity argument.

Theorem 5. *Suppose $C_{n_j} \neq 0$, $j = 1, 2, \dots, n$. Then $\lambda = -\sigma_n V_n$ is not an accumulation point of $P_{\sigma}A$.*

Proof. Suppose there were a sequence of points $\lambda_p \in P_{\sigma}A$ such that $\lambda_p \rightarrow -\sigma_n V_n$. By Lemma 3, there would be an associated sequence of elements ϕ_p , $\phi_p = \frac{1}{2}E_{\lambda}C\phi_p$, with uniformly continuous components. Consider the corresponding sequence of eigenelements of A :

$$\psi_p = \begin{cases} \frac{1}{2\mu} \int_{-a}^x e^{-(\sigma + \lambda_p V^{-1})(x-y)/\mu} C\phi_p(y) dy, & \mu > 0, \\ -\frac{1}{2\mu} \int_x^a e^{-(\sigma + \lambda_p V^{-1})(x-y)/\mu} C\phi_p(y) dy, & \mu < 0. \end{cases}$$

If we write these eigenelements componentwise, we get $\psi_p^{(i)} = \sum_{j=1}^n C_{ij} \psi_p^{ij}$ where $\Lambda_i^p = \sigma_i + \lambda_p/V_i$ and

$$(8) \quad \psi_p^{ij} = \begin{cases} \frac{1}{2\mu} \int_{-a}^x e^{-\Lambda_i^p(x-y)/\mu} \phi_p^{(j)}(y) dy, & \mu > 0, \\ -\frac{1}{2\mu} \int_x^a e^{-\Lambda_i^p(x-y)/\mu} \phi_p^{(j)}(y) dy, & \mu < 0. \end{cases}$$

By a standard argument, the sequence $\{\psi_p^{ij}\}$ converges absolutely and uniformly to a limit function continuous in x for each fixed $\mu \neq 0$:

$$|\psi_p^{ij}| \rightarrow |\psi_*^{ij}| \quad \text{as } p \rightarrow \infty.$$

For the n^{th} component we have

$$(9) \quad \psi_{*}^{nj} = \begin{cases} \frac{1}{2\mu} \int_{-a}^x \phi_{*}^{(j)}(y) dy, & \mu > 0, \\ -\frac{1}{2\mu} \int_x^a \phi_{*}^{(j)}(y) dy, & \mu < 0. \end{cases}$$

Since $|\phi_{*}| \neq 0$, $\phi_{*}^{(j)} \equiv 0$ for some $j = J$. Then it is seen that ψ_{*}^{nJ} is not summable over $[-1, 1]$ with respect to μ , for some x in the interval $[-a, a]$.

On the other hand, we have, from (8) with $j = J$, $\beta_p = \text{Re } \lambda_p$,

$$\int_{-1}^{+1} |\psi_p^{nJ}(x, \mu)| d\mu \leq \frac{1}{2} \int_{-a}^a E[(\sigma_n + \beta_p/V_n) |x - y|] |\phi_p^{(J)}(y)| dy,$$

and, through a process similar to that in the proof of Lemma 2, we also get

$$\sum_{i=1}^n C_{ni} \int_{-a}^a |\phi_p^{(i)}(x)| dx < \frac{\text{const.}}{|1 + aC_{nn} \log(\sigma_n + \beta_p/V_n)|}$$

from which we can infer

$$\int_{-a}^a |\phi_p^{(J)}(x)| dx < \frac{\text{const.}}{|1 + aC_{nn} \log(\sigma_n + \beta_p/V_n)|}.$$

Thus

$$\begin{aligned} \int_{-1}^{+1} |\psi_p^{nJ}(x, \mu)| d\mu &\leq \frac{1}{2} \int_{-a}^a |\log |x - y| + \gamma + g^{(n)}(x - y, \beta_p)| \cdot |\phi_p^{(J)}(y)| dy \\ &\quad + \frac{1}{2} |\log(\sigma_n + \beta_p/V_n)| \int_{-a}^a |\phi_p^{(J)}(y)| dy \\ &< \frac{1}{2} \left[2a \int_{-a}^a |\log |u| + \gamma + e^{M|u|/V_n}|^2 du \right]^{\frac{1}{2}} \\ &\quad + \frac{1}{2} \frac{\text{const.} \cdot |\log(\sigma_n + \beta_p/V_n)|}{|1 + aC_{nn} \log(\sigma_n + \beta_p/V_n)|} < \text{const.} \end{aligned}$$

where M is a number such that $|\sigma_n V_n - \lambda_p| < M$, $p = 1, 2, \dots$, γ is Euler's constant and $g^{(n)}(x - y, \beta_p)$ was defined in Lemma 1. Hence we have a sequence $\{\psi_p^{nJ}\}$ such that, for given x , $|\psi_p^{nJ}(x, \mu)| \rightarrow |\psi_{*}^{nJ}(x, \mu)|$ for each $\mu \neq 0$ in $[-1, 1]$. $\int_{-1}^{+1} |\psi_p^{nJ}(x, \mu)| d\mu < \text{const.}$ uniformly in p . By Fatou's lemma, the limit $|\psi_{*}^{nJ}(x, \mu)|$ is summable; this contradicts the previous statement, based on (9), that the limit could not be summable, and proves the theorem.

Theorem 6. $P_{\sigma, A}$ is closed. If λ^* is an accumulation point of $P_{\sigma, A}$ and if $\{\lambda_p\}$ is a sequence of points $\lambda_p \in P_{\sigma, A}$ with $\lambda_p \rightarrow \lambda^*$, and if $\{\phi_p\}$ is a corresponding uniformly convergent sequence of normalized eigenelements, then $\phi_* = \lim_{p \rightarrow \infty} \phi_p$ is an eigenelement corresponding to the eigenvalue λ^* .

Proof. This follows from the continuity of the operator $\frac{1}{2}E_{\lambda}C$ on the basis of Lemma 3 and Theorem 5.

At this point a considerable restriction is placed on the matrix C . We restrict C to be symmetrizable by diagonal matrices, with strictly positive elements. Thus $S = DCD^{-1}$ is symmetric and has positive entries only.

For a matrix C with positive elements, this is no restriction at all for $n = 1, 2$. For $n = 3$, however, it is a severe restriction. The case where all scattering of neutrons by nuclei is elastic, all velocity changes being caused by the fission process, would conform to the requirements. But this case is hardly realistic physically. For $n > 3$, this type of symmetrizability is an awkward restriction from a physicist's viewpoint. The theory is carried on from an n -velocity standpoint, however, for its mathematical content, and in order to suggest the theorems which might be proved generally if we can resolve the commutivity difficulties. Theorems which apparently need this symmetrizability condition on the matrix C will, henceforth, be marked with a dagger †.

Theorem 7†. $P_{\sigma}A$ has no real points of accumulation.

Proof. Suppose that $\beta^* > -\sigma_n V_n$ is a real point of accumulation. Recalling the hypothesis of Theorem 6, for $p = 1, 2, \dots$ there exists a sequence of eigenvalue problems $\phi_p = \frac{1}{2}E_{\lambda_p}C \phi_p$ and, since $\beta^* \in P_{\sigma}A$, a limit problem $\phi_* = \frac{1}{2}E_{\beta^*}C\phi_*$. With $S = DCD^{-1}$ symmetric (D a positive diagonal matrix), let $\psi_p = SD\phi_p$ and $\psi_* = SD\phi_*$. Then for $p = 1, 2, \dots$ we have $S^{-1}\psi_p = \frac{1}{2}E_{\lambda_p}\psi_p$ and in the limit $S^{-1}\psi_* = \frac{1}{2}E_{\beta^*}\psi_*$. Let $\Omega_p = S^{-1}\psi_p - \frac{1}{2}E_{\beta^*}\psi_p$ and consider the relationship $S^{-1}\psi_p - \frac{1}{2}E_{\lambda_p}\psi_p - \Omega_p = -\Omega_p$. Premultiplying by the operator $(\lambda_p - \beta^*)^{-1}S$, $p = 1, 2, \dots$, we have

$$\begin{aligned} \frac{1}{2} S \int_{-a}^a \left[\frac{E\{(\sigma + \lambda_p V^{-1}) |x - y|\} - E\{(\sigma + \beta^* V^{-1}) |x - y|\}}{\lambda_p - \beta^*} \right] \psi_p(y) dy \\ = \frac{1}{\lambda_p - \beta^*} \left[\psi_p(x) - \frac{1}{2} S \int_{-a}^a E\{(\sigma + \beta^* V^{-1}) |x - y|\} \psi_p(y) dy \right], \end{aligned}$$

where we use the matrix kernel notation defined previously (lines above (6)). The expression on the right is an element in the range of $(I - \frac{1}{2}SE_{\beta^*})$ and is annihilated by any member of the null space of the adjoint $(I - \frac{1}{2}E_{\beta^*}S)$. From the limit eigenvalue problem, $S^{-1}\psi_*$ is in this null space. Hence, using inner product notation,

$$\begin{aligned} \left(\frac{1}{2} S \int_{-a}^a \left[\frac{E\{(\sigma + \lambda_p V^{-1}) |x - y|\} - E\{(\sigma + \beta^* V^{-1}) |x - y|\}}{\lambda_p - \beta^*} \right] \psi_p(y) dy, S^{-1}\psi_* \right) \\ = \frac{1}{2} \left(\int_{-a}^a \left[\frac{E\{(\sigma + \lambda_p V^{-1}) |x - y|\} - E\{(\sigma + \beta^* V^{-1}) |x - y|\}}{\lambda_p - \beta^*} \right] \psi_p(y) dy, \psi_* \right) = 0. \end{aligned}$$

Hence the limit as $p \rightarrow \infty$ also vanishes; this can be written

$$\begin{aligned} \lim_{p \rightarrow \infty} \sum_{k=1}^n \int_{-a}^a \int_{-a}^a \frac{E[\Lambda_k^p |x - y|] - E[\Lambda_k^* |x - y|]}{\lambda_p - \beta^*} \psi_p^{(k)} \overline{\psi_*^{(k)}} dx dy \\ = - \sum_{k=1}^n \int_{-a}^a \int_{-a}^a \frac{V_k e^{-\Lambda_k^* |x - y|}}{\Lambda_k^*} \psi_*^{(k)} \overline{\psi_*^{(k)}} dx dy = 0 \end{aligned}$$

where $\Lambda_k^n = (\sigma_k + \lambda_p/V_k)$, $\Lambda_k^* = (\sigma_k + \beta^*/V_k)$, and the Lebesgue bounded convergence theorem is used to justify the interchange of limit and integration. But $e^{-\Lambda |x-y|}$ for $\Lambda > 0$ is known to be the kernel of a positive definite transformation on L_2 . Hence $\psi_* \equiv 0$, or $\phi_* \equiv 0$, which contradicts $\|\phi_*\| = 1$ and proves the result.

Theorem 8†. $P_{\sigma,A}$ is real.

Proof. If $\lambda \in P_{\sigma,A}$, then $\phi = \frac{1}{2}E_\lambda C\phi$ with $\phi \equiv 0$. With $S = DCD^{-1}$ symmetric, let $\psi = D\phi$. Then $\psi = \frac{1}{2}E_\lambda S\psi$. Now put $\theta = S\psi$. Then

$$(\psi, S\psi) = \overline{(\psi, S\psi)} = \frac{1}{2}(E_\lambda S\psi, S\psi) = \frac{1}{2}(E_\lambda \theta, \theta).$$

Hence

$$(E_\lambda \theta, \theta) = \sum_{i=1}^n \int_{-a}^a \int_{-a}^a E \left[\left(\sigma_i + \frac{\lambda}{V_i} \right) |x - y| \right] \overline{\theta^{(i)}(x)} \theta^{(i)}(y) dx dy$$

is real. Proceeding from here in the manner of LEHNER & WING (Ref. 8, page 222) one shows that reality of $(E_\lambda \theta, \theta)$ implies reality of λ , which proves the assertion.

We have shown that $P_{\sigma,A}$ is a bounded set in the half plane $\text{Re } \lambda \geq -\sigma_n V_n$, that it is closed and, provided $S = DCD^{-1}$ is symmetric, without limit points. Hence, we have proved

Theorem 9†. $P_{\sigma,A}$ consists of finitely many isolated points.

Can $P_{\sigma,A}$ be empty? If a result proved by K. JORGENS (Ref. 7, Theorem 6.3) for proper inelastic scattering in the case of a finite geometry can be extended to the slab case, the answer is yes. There is a case, however, for which we now show that $P_{\sigma,A}$ is not empty. Suppose $DCD^{-1} = P$, a positive definite matrix. Results which we obtain under this restriction we shall denote by a double dagger ††. For a symmetrizable matrix C this implies that the diagonal elements are numerically larger than the off-diagonal elements, a condition which is observed in a great many physical transport matrices.

Theorem 10††. $P_{\sigma,A}$ is not empty.

Proof. Consider the problem $\phi = \frac{1}{2}E_\beta C\phi$ where $\lambda = \beta$ is real. Let $\psi = D\phi$; then $\psi = \frac{1}{2}E_\beta P\psi$. P has the unique square root $P^{\frac{1}{2}} > 0$. Putting $\theta = P^{\frac{1}{2}}\psi$ we get $\theta = \frac{1}{2}P^{\frac{1}{2}}E_\beta P^{\frac{1}{2}}\theta$, where now the operator is positive definite: $(P^{\frac{1}{2}}E_\beta P^{\frac{1}{2}}f, f) = (E_\beta[P^{\frac{1}{2}}f], [P^{\frac{1}{2}}f]) > 0$. This is an immediate generalization of a result of LEHNER & WING (Ref. 8, page 224). Then for each fixed $\beta > -\sigma_n V_n$ there is a finite or denumerable set of positive characteristic values of $\frac{1}{2}P^{\frac{1}{2}}E_\beta P^{\frac{1}{2}}$: $\rho_1 \geq \rho_2 \geq \dots$. By a known theorem (Ref. 1, page 151), $\rho_\nu(\beta)$ is continuous as a function of β . Moreover, the transformation $\Gamma = P^{\frac{1}{2}}(E_\beta - E_{\beta'})P^{\frac{1}{2}}$, $\beta < \beta'$, has the property $(\Gamma f, f) > 0$ because of the positivity of $E_\beta - E_{\beta'}$; this, in turn, is based on the fact that $E_\beta - E_{\beta'}$ has a diagonal matrix of kernels each of which generates a positive transformation on L_2 (Ref. 8, page 225). Since $P^{\frac{1}{2}}E_\beta P^{\frac{1}{2}} = P^{\frac{1}{2}}E_{\beta'} P^{\frac{1}{2}} + \Gamma$,

we use Weyl's theorem (Ref. 12, page 239) to show that $\rho_r(\beta) \geq \rho_r(\beta')$. Thus $\rho_r(\beta)$ is a continuous non-increasing function of $\beta > -\sigma_n V_n$.

Now let ϕ_1 be a normalized eigenelement associated with ρ_1 , which we know exists for a positive definite operator. Then

$$\begin{aligned} \rho_1^2 &= (P^\dagger E_\beta P^\dagger \phi_1, \phi_1) \leq \|P\|^2 \|E_\beta\|^2 \\ &< \text{const.} \cdot \sum_{i=1}^n \int_a^a dx \int_{-\infty}^{\infty} E^2[(\sigma_i + \beta/V_i) |u|] du \\ &< \frac{\text{const.}}{\sigma_n + \beta/V_n} \sum_{i=1}^n \int_{-\infty}^{\infty} E^2[|u|] du \rightarrow 0 \quad \text{as } \beta \rightarrow \infty. \end{aligned}$$

Further, by the maximum principle,

$$\rho_1 = \max_{\epsilon} \frac{(P^\dagger E_\beta P^\dagger \epsilon, \epsilon)}{(\epsilon, \epsilon)} \geq \frac{(E_\beta g, g)}{(P^{-1}g, g)} \geq \text{const.} \cdot (E_\beta g, g)$$

where $g \in L$ is a vector element with null components, $i = 1, 2, \dots, n-1$, $g^{(n)} = 1/\sqrt{2a}$. Hence

$$\rho_1 \geq \text{const.} \cdot \int_{-a}^a \int_{-a}^a E \left[\left(\sigma_n + \frac{\beta}{V_n} \right) |x - y| \right] dx dy \rightarrow \infty \quad \text{as } \beta \rightarrow -\sigma_n V_n.$$

Hence there exists a value β_1 such that $\rho_1(\beta_1) = 1$, so that $\phi_1 = \frac{1}{2} E_{\beta_1} C \phi_1$ with $\phi_1 \neq 0$, and $\beta_1 \in P\sigma_r A$.

We summarize the results obtained about the decomposition of the spectral plane of A :

- (a) The half plane $\text{Re } \lambda \leq -\sigma_n V_n$ is contained in σA .
- (b) Those points in the half plane $\text{Re } \lambda > -\sigma_n V_n$ not contained in $P\sigma_r A$ belong to $\rho(A)$.
- (c) Points $\lambda \in P\sigma_r A$ for which $\text{Re } \lambda \geq -\sigma_n V_n$ form a set which we have called $P\sigma_r A$. This set is closed and bounded and does not accumulate at $-\sigma_n V_n$.
- (d) If DCD^{-1} is symmetric, where D is diagonal, then $P\sigma_r A$ contains only a finite number of points, all of which lie on the real axis, $\beta > -\sigma_n V_n$.
- (e) If DCD^{-1} is positive definite, $P\sigma_r A$ contains at least one point.

7. We wish now to apply our knowledge about the spectrum of A to the problem of representing the solution of the transport equation

$$\frac{\partial N}{\partial t} = AN, \quad N(x, \mu, 0) = f(x, \mu), \quad f \in D(A).$$

The domain of A was defined as the linear manifold of n -tuples in H with components absolutely continuous in x for each fixed μ in $|\mu| \leq 1$ and which satisfy the homogeneous boundary conditions. Such components are dense in the component space H_1 since they include all continuously differentiable functions which vanish outside a compact subset of the region $|x| \leq a$, $|\mu| \leq 1$. Hence, under the norm of H , $D(A)$ is dense in H .

In section 5 we proved that $R(\lambda, A) = (\lambda - A)^{-1}$ is bounded for $\lambda \in \Gamma$, with domain $D_\lambda = H$. Thus $(\lambda - A)^{-1}$, $\lambda - A$ and also A are closed operators.

We can employ the existence theorem (Ref. 11, pages 201, 203) of Hille-Yosida for this problem, provided the proper estimate on the resolvent $(\lambda - A)^{-1}$ can be obtained. To this end, let $A = -VD - \sigma V + VCJ$, where $D = \mu \partial \cdot / \partial x$ and $J = \frac{1}{2} \int_{-1}^{+1} d\mu$. If $g \in D(A)$, we have

$$|(VCJg, g)| \leq \|V\| \cdot \|C\| \cdot \|J\| \cdot \|g\|^2 \leq n^2 v M \|g\|^2$$

since $\|J\| = 1$ and $\|C\| \leq nM$, where $M = \max_{i,i} C_{ii}$, $v = \max_i V_i$. Also

$$\begin{aligned} 2 \operatorname{Re} (VDg, g) &= \sum_{i=1}^n \int_{-1}^{+1} \int_{-a}^a \left(\mu V_i \frac{\partial g_i}{\partial x} \bar{g}_i + \mu V_i \frac{\partial \bar{g}_i}{\partial x} g_i \right) dx d\mu \\ &= \sum_{i=1}^n V_i \int_{-1}^{+1} d\mu [\mu g_i \bar{g}_i] \Big|_{-a}^a \\ &= \sum_{i=1}^n V_i \int_0^1 \mu |g_i(a, \mu)|^2 d\mu - \sum_{i=1}^n V_i \int_{-1}^0 |g_i(-a, \mu)|^2 d\mu \geq 0 \end{aligned}$$

and

$$2 \operatorname{Re} (\sigma Vg, g) = 2 \sum_{i=1}^n \sigma_i V_i \int_{-a}^a \int_{-1}^{+1} |g_i|^2 dx d\mu \geq 0.$$

Hence, for $\operatorname{Re} \lambda = \beta$ and $g \in D(A)$,

$$\begin{aligned} \operatorname{Re} ((\lambda - A)g, g) &= \beta \|g\|^2 + \operatorname{Re} (VDg, g) + \operatorname{Re} (\sigma Vg, g) - \operatorname{Re} (VCJg, g) \\ &\geq \beta \|g\|^2 - n^2 v M \|g\|^2 = (\beta - n^2 v M) \|g\|^2, \quad \beta > n^2 v M. \end{aligned}$$

Now put $(\lambda - A)g = f \in H$. Then

$$\|f\| \cdot \|g\| \geq |(f, g)| \geq \operatorname{Re} (f, g) = \operatorname{Re} ((\lambda - A)g, g) \geq (\beta - n^2 v M) \|g\|^2,$$

so that

$$\|g\| = \|R(\lambda, A)f\| \leq \frac{\|f\|}{\beta - n^2 v M}$$

and

$$\|R(\lambda, A)\| \leq \frac{1}{\beta - n^2 v M} \quad \text{with} \quad v = \max_i V_i, \quad M = \max_{i,i} C_{ii}.$$

With this estimate on the resolvent, the Hille-Yosida theorem assures the existence of a one parameter semi-group $T(t)$ of bounded solution operators, strongly continuous for $t \geq 0$, with $T(0) = I$. This family of operators satisfies the differential equation

$$\frac{d}{dt}T(t)f = AT(t)f, \quad f \in D(A),$$

and thus solves the initial value problem for the transport equation in the strong sense. That this solution is unique in H follows from a theorem of R. S. PHILLIPS (Ref. 11, page 216). Hence our problem is well set. We have, moreover, a representation of the solution:

$$(10) \quad N(x, \mu, t) = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{\alpha - i\omega}^{\alpha + i\omega} e^{\lambda t} R(\lambda, A) f \, d\lambda, \quad t > 0, \quad f \in D(A), \quad \alpha > n^2 v M,$$

which follows from a known result (Ref. 6, page 232). The integral is to be considered the strong limit of Riemann sums in the topology of H .

8. In the case of nonsymmetrizable C , in the sense of section 6, we can say little more about the solution at the present state of our technique. We can say considerably more if there exists a diagonal matrix D such that $D^{-1}CD$ is symmetric. Hence our further results will carry daggers.

If $D^{-1}CD = S$, we have shown that $P_{\sigma, A}$ consists only of a finite number of real points which are therefore poles of the resolvent $R(\lambda, A)$, a function defined and analytic, in terms of the strong topology, for $\lambda \in \rho(A)$. By a known theorem, $P_{\sigma, A}$ also constitutes P_{σ, A^*} , where A^* is the adjoint of A :

$$(11) \quad A^*f = \mu V \frac{\partial f}{\partial x} - \sigma V f + \frac{1}{2} C' V \int_{-1}^{+1} f \, d\mu', \quad f \in D(A^*),$$

where C' is C transposed and $D(A^*)$ is the linear manifold of elements $f \in H$ such that $A^*f \in H$, which satisfy adjoint boundary conditions $f(a, \mu) = 0$, $0 \leq \mu \leq 1$, $f(-a, \mu) = 0$, $-1 \leq \mu < 0$. Indeed, all of the foregoing results could have been proved just as well for A^* . In particular, it follows that $\rho(A^*) = \rho(A)$.

Let $\beta_1, \dots, \beta_m$ be the real eigenvalues, of multiplicity s_i , $j = 1, \dots, m$. Let $\psi_{i,1}, \psi_{i,2}, \dots, \psi_{i,s_i}$ be linearly independent eigenelements corresponding to β_i . Also denote the corresponding eigenelements of A^* by $\psi_{i,1}^*, \dots, \psi_{i,s_i}^*$. Suppose D symmetrizes C : $S = DCD^{-1} = D^{-1}C'D$. It is easily shown that

$$\psi_{i,1}^* = \overline{V^{-1}D^2\psi_{i,1}(-x, \mu)}, \dots, \psi_{i,s_i}^* = \overline{V^{-1}D^2\psi_{i,s_i}(-x, \mu)}.$$

Lemma 4†. If $\beta \in P_{\sigma, A} = P_{\sigma, A^*}$ and if $A\psi = \beta\psi$, $A^*\psi^* = \beta\psi^*$ with $\psi^* = V^1D^2\psi(-x, \mu)$, then $(\psi, \psi^*) \neq 0$.

Proof. ψ can be assumed to have real components. Further, $\phi = \int_{-1}^{+1} \psi \, d\mu$ can be assumed either even or odd on $(-a, a)$, since $\phi_1 = \frac{1}{2}(\phi(x) + \phi(-x))$ and $\phi_2 = \frac{1}{2}(\phi(x) - \phi(-x))$ satisfy $\phi = \frac{1}{2}E_\beta C\phi$ separately. From Theorem 2 we then obtain $\psi(x, \mu) = \pm\psi(-x, -\mu)$, the $+$ sign going with even ϕ . Now for $i = 1, 2, \dots, n$

$$\begin{aligned}
& \pm \int_{-1}^{+1} \psi^{(i)}(x, \mu) \psi^{*(i)}(x, \mu) d\mu = \frac{d_i^2}{V_i} \int_{-1}^{+1} \psi^{(i)}(x, \mu) \psi^{(i)}(x, -\mu) d\mu \\
& = \frac{2d_i^2}{V_i} \int_0^1 \psi^{(i)}(x, \mu) \psi^{(i)}(x, -\mu) d\mu \\
& = \frac{1}{2} \frac{d_i^2}{V_i} \int_0^1 \frac{1}{\mu} \int_{-a}^x e^{-\Lambda_i(x-s)/\mu} \sum_{j=1}^n C_{ij} \phi^{(j)}(s) ds \cdot \frac{1}{\mu} \int_x^a e^{-\Lambda_i(x-t)/\mu} \sum_{j=1}^n C_{ij} \phi^{(j)}(t) dt d\mu \\
& = \frac{1}{2} \frac{d_i^2}{V_i} \int_{-a}^x \int_x^a \sum_{j=1}^n C_{ij} \phi^{(j)}(s) \sum_{k=1}^n C_{ik} \phi^{(k)}(t) dt ds \int_0^1 e^{-\Lambda_i(t-s)/\mu} \frac{d\mu}{\mu^2} \\
& = \frac{1}{2} \frac{d_i^2}{\sigma_i V_i + \beta} \int_{-a}^x \int_x^a \frac{e^{-\Lambda_i(t-s)}}{t-s} \sum_{j=1}^n C_{ij} \phi^{(j)}(s) \sum_{k=1}^n C_{ik} \phi^{(k)}(t) dt ds
\end{aligned}$$

where d_i , V_i are i^{th} elements of diagonal matrices, $\Lambda_i = \sigma_i + \beta/V_i$, and interchange of integrations can be justified by a known result (Ref. 15, page 390). Then

$$\begin{aligned}
& \pm \int_{-a}^a \int_{-1}^{+1} \psi^{(i)}(x, \mu) \overline{\psi^{*(i)}(x, \mu)} d\mu dx \\
& = \frac{1}{2} \frac{d_i^2}{\sigma_i V_i + \beta} \int_{-a}^a \int_{-a}^x \int_x^a \frac{e^{-\Lambda_i(t-s)}}{t-s} \sum_{j=1}^n C_{ij} \phi^{(j)}(s) \sum_{k=1}^n C_{ik} \phi^{(k)}(t) dt ds dx \\
& = \frac{1}{2} \frac{d_i^2}{\sigma_i V_i + \beta} \int_{-a}^a \int_s^a \frac{e^{-\Lambda_i(t-s)}}{t-s} \sum_{j=1}^n C_{ij} \phi^{(j)}(s) \sum_{k=1}^n C_{ik} \phi^{(k)}(t) dt ds \int_s^t dx \\
& = \frac{1}{2} \frac{d_i^2}{\sigma_i V_i + \beta} \int_{-a}^a \int_s^a e^{-\Lambda_i|t-s|} \sum_{j=1}^n C_{ij} \phi^{(j)}(s) \sum_{k=1}^n C_{ik} \phi^{(k)}(t) dt ds \\
& = \frac{1}{2} \frac{d_i^2}{\sigma_i V_i + \beta} \int_{-a}^a \int_{-a}^t e^{-\Lambda_i|t-s|} \sum_{j=1}^n C_{ij} \phi^{(j)}(s) \sum_{k=1}^n C_{ik} \phi^{(k)}(t) ds dt
\end{aligned}$$

where we have merely interchanged the order of integration. Hence

$$\begin{aligned}
& \pm(\psi, \psi^*) = \pm \sum_{i=1}^n \int_{-a}^a \int_{-1}^{+1} \psi^{(i)}(x, \mu) \overline{\psi^{*(i)}(x, \mu)} d\mu dx \\
& = \frac{1}{4} \sum_{i=1}^n \frac{d_i^2}{\sigma_i V_i + \beta} \int_{-a}^a \int_{-a}^a e^{-\Lambda_i|t-s|} \sum_{j=1}^n C_{ij} \phi^{(j)}(s) \sum_{k=1}^n C_{ik} \phi^{(k)}(s) ds dt > 0
\end{aligned}$$

since the kernel $e^{-\Lambda_i|x|}$ is known to generate a positive definite transformation on L_2 space. This proves the lemma.

The lemma shows that the eigenfunctions $\psi_{i,k}$, $j = 1, \dots, m$; $k = 1, \dots, s_j$, of A and those of A^* , $\psi_{j,k}^*$, supposed real and either even or odd, form a bi-orthonormal set corresponding to the points $\beta_i \in P\sigma_r A$.

Lemma 5†. $R(\lambda, A) = (\lambda - A)^{-1}$ has a simple pole at $\lambda = \beta_i$, $j = 1, \dots, m$.

Proof. Since A on H , the product space of components, is closed, the theorems of A. E. TAYLOR are employed (Ref. 14, pages 189–223). The method of proof

is formally identical with that of LEHNER & WING (Ref. 9, page 131) and we omit details.

Since $R(\lambda, A)$ has simple poles at $\lambda = \beta_j$, $j = 1, 2, \dots, m$, it is of interest to be able to calculate the residue of $R(\lambda, A)f$, $f \in D(A)$ at β_j .

Lemma 6†.

$$\text{Residue}_{\lambda=\beta_j} R(\lambda, A)f = \sum_{k=1}^{s_j} (f, \psi_{j,k}^*) \psi_{j,k}.$$

Proof. The proof is formally the same as the proof of LEHNER & WING for the equivalent statement in one-velocity theory (Ref. 9, page 131).

9. We wish now to shift the path of integration of the Laplace integral expression for the solution (10) so that a residue term can be exhibited for each $\beta_j \in P_{\sigma,A}$. (We continue to assume that $S = D^{-1}CD$ is symmetric.) This will require estimates on $R(\lambda, A)f$ as a function of $\lambda \in \rho(A)$. To begin with, let us confine $\lambda = \beta + i\tau$ to the closed region

$$\begin{aligned} -\sigma_n V_n &\leq \beta \leq M < \infty, \\ E: \quad |\lambda - \beta_j| &\geq \epsilon_j > 0, \quad j = 1, \dots, m, \\ |\lambda + \sigma_n V_n| &\geq \epsilon_0 > 0, \end{aligned}$$

where M is large enough to include all of $P_{\sigma,A}$ and $\epsilon_0 < 1$ is a positive quantity to be determined. Let $S_\lambda = (1 - \frac{1}{2}E_\lambda C)^{-1}$.

Lemma 7. $\|S_\lambda\|$ is continuous on E .

Proof. Again the argument is the same as that for the one-velocity case (Ref. 9, page 132).

Lemma 8. $\|E_\lambda C\| \leq \text{const.}/\sqrt{1 + |\tau|}$ uniformly in E .

Proof. An estimate for $E_\lambda C$ resulted in the proof of Theorem 4, namely

$$\|E_\lambda C\| \leq \frac{\text{const.}}{\sqrt{|\sigma_n V_n + \lambda|}}.$$

Then since $\sigma_n V_n$ is real, $\|E_\lambda C\| \leq \text{const.}/\sqrt{|\tau|}$. For $|\tau| > 1$, $1 + 1/|\tau| = (1 + |\tau|)/|\tau|$ is bounded, so that $1/|\tau| < \text{const.}/(1 + |\tau|)$, so that

$$\|E_\lambda C\| \leq \frac{\text{const.}}{\sqrt{1 + |\tau|}}.$$

For $|\tau| \leq 1$ and $\lambda \in E$ we have $(1 + |\tau|)^{\frac{1}{2}} \|E_\lambda C\|$ continuous on a closed bounded set. This proves the lemma.

Lemma 9. $\|S_\lambda\| \leq \text{const.}$, $\lambda \in E$, $\lambda = \beta + i\tau$.

Proof. Let $E = E_1 + E_2$, where $E_2 = \{\lambda \mid \lambda \in E, |\tau| > \tau_0\}$. τ_0 can be taken

sufficiently large to ensure $\|E_\lambda C\| < 1$ in E_2 . Since $\|S_\lambda\| = \|(1 - \frac{1}{2}E_\lambda C)^{-1}\| \leq (1 - \frac{1}{2}\|E_\lambda C\|)^{-1}$, we have $\|S_\lambda\| \leq 2$ for $\lambda \in E_2$. Since $\|S_\lambda\|$ is continuous on $E_1 = \{\lambda \mid \lambda \in E, |\tau| \leq \tau_0\}$, the lemma is proved.

In the proof of Theorem 3, we obtained an estimate for the resolvent, namely

$$\|R(\lambda, A)f\| \leq \frac{\text{const.}}{|\sigma_n V_n + \beta|^2} \|f\|.$$

Lemma 9 shows that this estimate is uniform for $\lambda \in E$.

A well known identity for the resolvent is written $\lambda R(\lambda, A)f = f + R(\lambda, A)Af$. We modify this to obtain

$$(\lambda + \sigma_n V_n)R(\lambda, A)f = f + \sigma_n V_n R(\lambda, A)f + R(\lambda, A)Af.$$

Then

$$\begin{aligned} \|R(\lambda, A)f\| &\leq \frac{\|f\| + |\sigma_n V_n| \cdot \|R(\lambda, A)f\| + \|R(\lambda, A)Af\|}{|\lambda + \sigma_n V_n|} \\ &= \frac{M(\beta) \cdot \|f\|}{|\lambda + \sigma_n V_n|}, \quad \lambda \in E, \quad f \in D(A), \end{aligned}$$

where $M(\beta)$ is a constant depending only on $\beta = \text{Re } \lambda$. With this estimate, we can shift the vertical path of integration, collect residues, and prove the following

Theorem 11†. *The solution of the initial value problem (1) and (2), which we have set for the transport equation in the slab, has the representation*

$$\begin{aligned} N(x, \mu, t) &= \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{\alpha - i\omega}^{\alpha + i\omega} e^{\lambda t} R(\lambda, A)f d\lambda = \sum_{j=1}^m e^{\beta_j t} \sum_{k=1}^{s_j} (f, \psi_{i,k}^*) \psi_{i,k} \\ &\quad + \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{-\sigma_n V_n + \delta + i\omega}^{-\sigma_n V_n + \delta + i\omega} e^{\lambda t} R(\lambda, A)f d\lambda, \\ &\quad f \in D(A), \quad \alpha > n^2 v M, \quad \delta > 0, \\ &\quad -\sigma_n V_n < -\sigma_n V_n + \delta < \beta_m, \quad \beta_m \leq \beta_{n-1} \leq \dots \leq \beta_1. \end{aligned}$$

Hence the solution appears as a finite expansion in terms of eigenelements of A and a residual term. We now propose to estimate the residual term. For this, we shall need the following lemma, the proof of which requires $DCD^{-1} = P > 0$ at present.

Lemma 10††. $\|S_\lambda\| \leq \text{const.} (\log |\lambda + \sigma_n V_n|)^2$, where $S_\lambda = (1 - \frac{1}{2}E_\lambda C)^{-1}$, $\text{Re } \lambda \geq -\sigma_n V_n$, $|\lambda + \sigma_n V_n| < \epsilon_0$, with $\epsilon_0 > 0$ to be determined, and $C \notin \mathcal{C}$, where $\mathcal{C}$ is an exceptional class of transfer matrices.

Remark. $\mathcal{C}$ is that class of matrices which might produce the constant eigenvalue $\rho_r(q) \equiv 2$, as indicated below. We cannot yet handle this case. In practical physical application, the difficulty can be avoided by choosing cross sections and velocity groups appropriately.

Proof. By the symmetrizability assumption, we have $\|S_\lambda\| \leq \text{const.} \cdot \|(1 - \frac{1}{2}BE_\lambda B)^{-1}\|$, where $B^2 = P$. Consider now the diagonal matrix of kernels whose entries are

$$E\left[\left(\sigma_i - \frac{\sigma_n V_n}{V_i}\right) |x - y|\right], \quad i = 1, \dots, n-1,$$

with n^{th} diagonal entry

$$-\log bq |x - y|, \quad q = \sigma_n + \frac{\lambda}{V_n}, \quad \text{Re } \lambda > -\sigma_n V_n,$$

and $b = e^\gamma$ where $\gamma = \text{Euler's constant}$. We shall call the operator induced on L by this kernel matrix K_q . It is clear that

$$BK_q B \sim BE_\lambda B \quad \text{as } q \rightarrow 0, \quad \text{Re } q > 0.$$

Let q be real. Then $BK_q B$ is symmetric and compact. Hence the positive eigenvalues form continuous curves $\rho_\nu(q)$, $\nu = 1, 2, \dots$, in a ρ, q plane. Now supposing that $C \notin \mathbb{C}$ so that $\rho_\nu(q) \equiv 2$ for any ν , it is possible to find a value $q_0 > 0$ such that $\rho_\nu(q_0) \neq 2$, $\nu = 1, 2, \dots$. Then we may write $BK_q B = BK_{q_0} B + BN_q B$, where N_q is an operator of rank one. In fact, $N_q f = (f, u)W$ where

$$u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad W = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k \end{bmatrix} \quad \text{with } k = -\log \frac{q}{q_0}.$$

N_q has the single eigenvalue

$$(W, u) = -\int_{-a}^a \log \frac{q}{q_0} dx = 2ak > 0 \quad \text{for } q < q_0.$$

Hence $BN_q B$ is also of rank one, with eigenvalue $(B^2 W, u)$.

Now consider the operator identity

$$\begin{aligned} (1 - \tfrac{1}{2}BK_q B) &= (1 - \tfrac{1}{2}BK_{q_0} B - \tfrac{1}{2}BN_q B) \\ &= (1 - \tfrac{1}{2}BK_{q_0} B)[1 - \tfrac{1}{2}(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}BN_q B] = (1 - \tfrac{1}{2}BK_{q_0} B)(1 - \tfrac{1}{2}Q_q) \end{aligned}$$

where Q_q is of rank one with the single eigenvalue

$$\omega_0 = (B(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}BW, u) = k(B(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}Bu, u) = kM, \quad q < q_0.$$

Then we have

$$(1 - \tfrac{1}{2}BK_q B)^{-1} = (1 - \tfrac{1}{2}Q_q)^{-1}(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}.$$

Using the identity $Q_q^2 = \omega_0 Q_q$, valid for operators of rank one, we find that

$$(1 - \tfrac{1}{2}Q_q)^{-1} = I + \frac{\frac{1}{2}Q_q}{1 - \frac{1}{2}\omega_0}.$$

Hence

$$\begin{aligned} ||(1 - \tfrac{1}{2}BK_q B)^{-1}|| &\leq \left| \left| \left(1 + \frac{Q_q}{2 - \omega_0}\right) \right| \right| \cdot ||(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}|| \\ &\leq \left(1 + \text{const.} \cdot \frac{||N_q|| \cdot ||(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}||}{|2 - \omega_0|}\right) ||(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}||. \end{aligned}$$

Now since $\rho_r(q_0) \neq 2$, $\nu = 1, 2, \dots$,

$$||(1 - \tfrac{1}{2}BK_{q_0} B)^{-1}|| < \max_{\nu} |(1 - \tfrac{1}{2}\rho_{\nu}(q_0))^{-1}| < \infty.$$

Moreover, $||N_q|| = -2a \log(q/q_0) \rightarrow \infty$ as $q \rightarrow 0$. Hence

$$\frac{||N_q||}{|2 - \omega_0|} \sim \frac{2ak}{\omega_0} = \frac{2a}{M} \quad \text{as } q \rightarrow 0,$$

provided $M \neq 0$. If $M = 0$, we should have

$$\frac{||N_q||}{|2 - \omega_0|} = \tfrac{1}{2} ||N_q|| = ak = -a \log \frac{q}{q_0} \rightarrow \infty \quad \text{as } q \rightarrow 0.$$

In any case,

$$||(1 - \tfrac{1}{2}BK_q B)^{-1}|| = O(-\log q) \quad \text{as } q \rightarrow 0,$$

or as $B \rightarrow -\sigma_n V_n$.

For complex $q = \sigma_n + \lambda/V_n = re^{i\theta}$, $r > 0$, we have $K_q = K_r + S_{\theta}$ where S_{θ} is an operator of rank one: $S_{\theta}f = (f, u)W_1$, where

$$u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -i\theta \end{bmatrix}.$$

S_{θ} has the single eigenvalue $-2ia\theta$. $BS_{\theta}B$ is also of rank one with eigenvalue (B^2W_1, u) . By a process exactly analogous to that used above for real q , we get

$$||(1 - \tfrac{1}{2}BK_q B)^{-1}|| \leq \left\{1 + \frac{\text{const.} \cdot ||S_{\theta}|| \cdot ||(1 - \tfrac{1}{2}BK_r B)^{-1}||}{|2 - \omega'_0|}\right\} ||(1 - \tfrac{1}{2}BK_r B)^{-1}||$$

where $\omega'_0 = (B(1 - \tfrac{1}{2}BK_r B)^{-1}BW_1, u)$ is the eigenvalue of the operator $Q_{\theta} = (1 - \tfrac{1}{2}BK_r B)^{-1}BS_{\theta}B$ which is of rank one. ω'_0 turns out to be imaginary so that $|2 - \omega'_0| > 2$. Also it is readily seen that $||S_{\theta}|| \leq 2a|\theta|$. Using the previous estimate for $(1 - \tfrac{1}{2}BK_r B)^{-1}$, we have

$$||(1 - \tfrac{1}{2}BK_q B)^{-1}|| = O((\log r)^2) \quad \text{as } r = |q| \rightarrow 0,$$

or as $\lambda \rightarrow -\sigma_n V_n$, $\text{Re } \lambda \geq -\sigma_n V_n$.

Let us now try to compare K_q with E_{λ} . We let $\varepsilon_q = E_{\lambda} - K_q$. Now the

kernel element

$$E\left[\left(\sigma_i - \frac{\sigma_n V_n}{V_i} + \frac{V_n}{V_i} q\right) |x - y|\right] - E\left[\left(\sigma_i - \frac{\sigma_n V_n}{V_i}\right) |x - y|\right],$$

$i = 1, 2, \dots, n - 1$, is analytic in q within and on some circle $|q| \leq R < \sigma_i V_i / V_n - \sigma_n$. Hence, by use of the maximum modulus theorem and Schwartz' lemma, we arrive at an estimate for these components;

$$\left| E\left[\left(\sigma_i - \frac{\sigma_n V_n}{V_i} + \frac{V_n}{V_i} q\right) |x - y|\right] - E\left[\left(\sigma_i - \frac{\sigma_n V_n}{V_i}\right) |x - y|\right] \right| \leq \frac{M(|x - y|) |q|}{R},$$

where $M(|x - y|)$ is square integrable, $-a \leq x \leq a$, $-a \leq y \leq a$. Thus

$$\begin{aligned} \left| \int_{-a}^a \left(E\left[\left(\sigma_i - \frac{\sigma_n V_n}{V_i} + \frac{V_n}{V_i} q\right) |x - y|\right] - E\left[\left(\sigma_i - \frac{\sigma_n V_n}{V_i}\right) |x - y|\right] \right) f_i dy \right| \\ \leq \int_{-a}^a \frac{M(|x - y|) |q|}{R} |f_i| dy \leq \text{const.} \cdot \|f_i\| \cdot |q| \end{aligned}$$

for $i = 1, 2, \dots, n - 1$. Also, since $b = e^\gamma$,

$$E[q |x - y|] + \log bq |x - y| = - \sum_{m=1}^{\infty} \frac{(-q |x - y|)^m}{m \cdot m!},$$

so that for the n^{th} component of the diagonal kernel of $\mathcal{E}_q$ we have

$$|E[q |x - y|] + \log bq |x - y| \leq \sum_{m=1}^{\infty} \frac{|q|^m |x - y|^m}{m!} \leq |q| e^{2a}, \quad |q| \leq 1.$$

Hence

$$\left| \int_{-a}^a (E[q |x - y|] + \log bq |x - y|) f_n dy \right| \leq \int_{-a}^a |q| e^{2a} f_n dy \leq \text{const.} \cdot |q| \cdot \|f_n\|.$$

For $|q| \leq \min(R, 1)$ we have

$$\|\mathcal{E}_q f\|^2 = \|(E_\lambda - K_q) f\|^2 \leq \text{const.} \cdot |q|^2 \sum_{i=1}^n \|f_i\|^2 = \text{const.} \cdot |q|^2 \|f\|^2,$$

and $\|\mathcal{E}_q\| \leq \text{const.} \cdot |q|$. Hence also $\|B\mathcal{E}_q B\| \leq \text{const.} \cdot |q|$.

Now choose $\epsilon_0 < \min(R, 1)$ so small that

$$\|B\mathcal{E}_q B\| \cdot \|(1 - \tfrac{1}{2}BK_q B)^{-1}\| < \tfrac{1}{2} \quad \text{for } |q| < \frac{\epsilon_0}{V_n},$$

as is possible since $\|(1 - \tfrac{1}{2}BK_q B)^{-1}\| = O(\log^2 |q|)$. Since

$$\begin{aligned} (1 - \tfrac{1}{2}BE_\lambda B)^{-1} &= (1 - \tfrac{1}{2}BK_q B - \tfrac{1}{2}B\mathcal{E}_q B)^{-1} \\ &= (1 - \tfrac{1}{2}(1 - \tfrac{1}{2}BK_q B)^{-1}B\mathcal{E}_q B)^{-1}(1 - \tfrac{1}{2}BK_q B)^{-1} \end{aligned}$$

we have

$$\begin{aligned} \|(1 - \tfrac{1}{2}BE_\lambda B)^{-1}\| &\leq \|(1 - \tfrac{1}{2}BK_q B)^{-1}\| \sum_{m=0}^{\infty} \|(1 - \tfrac{1}{2}BK_q B)^{-1}\|^m \cdot \|B\varepsilon_q B\|^m \\ &\leq 2 \|(1 - \tfrac{1}{2}BK_q B)^{-1}\| = O(\log^2 |q|) \quad \text{for } |q| < \frac{\epsilon_0}{V_n}, \quad q = \sigma_n + \frac{\lambda}{V_n}. \end{aligned}$$

Then

$$\|S_\lambda\| \leq \text{const.} \cdot \|(1 - \tfrac{1}{2}BE_\lambda B)^{-1}\| < \text{const.} \cdot \log^2 |\lambda + \sigma_n V_n|, \quad |\lambda + \sigma_n V_n| < \epsilon_0,$$

and the lemma is proved.

We now estimate the resolvent in the small half circle $|\lambda + \sigma_n V_n| < \epsilon_0$, $\text{Re } \lambda \geq -\sigma_n V_n$. In fact, referring to the proof of Theorem 3, we see that

$$\|R(\lambda, A)f\| \leq \text{const.} \frac{\log^2 |\lambda + \sigma_n V_n|}{|\sigma_n V_n + \beta|^2} \|f\|$$

where the constant is uniform in λ . Then evidently the estimate previously obtained for $\lambda \in E$ is good for our small half circle as well; namely, with suitable readjustment of constants,

$$\|R(\lambda, A)f\| \leq \frac{\|f\| + \frac{\text{const.} |\sigma_n V_n|}{|\beta + \sigma_n V_n|^2} \|f\| + \frac{\text{const.}}{|\beta + \sigma_n V_n|^2} \|Af\|}{|\lambda + \sigma_n V_n|}, \quad f \in D(A).$$

The residual term of the eigenelement expansion is recalled to be

$$(12) \quad \zeta(x, \mu, t) = \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{-\sigma_n V_n + \delta - i\omega}^{-\sigma_n V_n + \delta + i\omega} e^{\lambda t} R(\lambda, A) f d\lambda,$$

$$-\sigma_n V_n < -\sigma_n V_n + \delta < \beta_m.$$

Now, using the first identity for the resolvent,

$$\begin{aligned} &\left\| \frac{1}{2\pi i} \int_{-\sigma_n V_n + \delta - i\omega}^{-\sigma_n V_n + \delta + i\omega} e^{\lambda t} R(\lambda, A) f d\lambda \right\| \\ &\leq \left\| \frac{1}{2\pi i} \int_{-\sigma_n V_n + \delta - i\omega}^{-\sigma_n V_n + \delta + i\omega} e^{\lambda t} f \frac{d\lambda}{\lambda + \sigma_n V_n} \right\| \\ &\quad + \left\| \frac{\sigma_n V_n}{2\pi i} \int_{-\sigma_n V_n + \delta - i\omega}^{-\sigma_n V_n + \delta + i\omega} e^{\lambda t} R(\lambda, A) f \frac{d\lambda}{\lambda + \sigma_n V_n} \right\| \\ &\quad + \left\| \frac{1}{2\pi i} \int_{-\sigma_n V_n + \delta - i\omega}^{-\sigma_n V_n + \delta + i\omega} e^{\lambda t} R(\lambda, A) A f \frac{d\lambda}{\lambda + \sigma_n V_n} \right\| \\ &\leq \left| \frac{e^{-\sigma_n V_n t}}{2\pi i} \int_{-i\omega}^{i\omega} e^{(z+\delta)t} \frac{dz}{z+\delta} \right| \cdot \|f\| \\ &\quad + \frac{e^{(-\sigma_n V_n + \delta)t}}{2\pi} \left\{ \sigma_n V_n \int_{-\omega}^{\omega} \|R(\lambda, A)f\| \frac{d\tau}{|\delta + i\tau|} \right. \\ &\quad \left. + \int_{-\omega}^{\omega} \|R(\lambda, A)Af\| \frac{d\tau}{|\delta + i\tau|} \right\} \end{aligned}$$

$$\begin{aligned}
&< e^{-\sigma_n V_n t} \left\{ 2 \|f\| + e^{\delta t} \frac{\sigma_n V_n}{2\pi} \left[\|f\| + \frac{\text{const.} |\sigma_n V_n|}{\delta^2} \|f\| \right. \right. \\
&\quad \left. \left. + \frac{\text{const.}}{\delta^2} \|Af\| \right] \int_{-\omega}^{\omega} \frac{d\tau}{\delta^2 + \tau^2} \right. \\
&\quad \left. + \frac{e^{\delta t}}{2\pi} \left[\|Af\| + \frac{\text{const.} |\sigma_n V_n|}{\delta^2} \|Af\| + \frac{\text{const.}}{\delta^2} \|A^2 f\| \right] \int_{-\omega}^{\omega} \frac{d\tau}{\delta^2 + \tau^2} \right. \\
&< e^{-\sigma_n V_n t} \left\{ 2 \|f\| + \frac{e^{\delta t}}{2\delta} \left[(\sigma_n V_n \|f\| + \|Af\|) + \frac{\text{const.}}{\delta^2} \|f\| + \frac{\text{const.}}{\delta^2} \|Af\| \right. \right. \\
&\quad \left. \left. + \frac{\text{const.}}{\delta^2} \|A^2 f\| \right] \right\}, \quad f \in D(A), \quad Af \in D(A).
\end{aligned}$$

Here we have used the fact that

$$\left| \frac{1}{2\pi i} \int_{-i}^i e^{(z+\delta)t} \frac{dz}{z+\delta} \right| \leq 1 + \frac{e^{\delta t}}{2\pi} \int_{\pi/2}^{3\pi/2} \frac{e^{\omega t \cos \theta}}{|1 - \delta/\omega|} d\theta < 2$$

for sufficiently large ω . If we let $\delta = 1/t$, as we may for large t , then

$$\begin{aligned}
\|\zeta(x, \mu, t)\| &= \left\| \lim_{\omega \rightarrow \infty} \frac{1}{2\pi i} \int_{-\sigma_n V_n + \delta - i\omega}^{-\sigma_n V_n + \delta + i\omega} e^{\lambda t} R(\lambda, A) f d\lambda \right\| \\
&\leq e^{-\sigma_n V_n t} \cdot (\text{a 3}^{\text{rd}} \text{ degree polynomial in } t).
\end{aligned}$$

Hence we can now write the final result under the restrictions imposed by Lemma 10.

Theorem 12††. *The solution in the strong sense of the initial value problem for the n -velocity transport equation is given by*

$$\begin{aligned}
N(x, \mu, t) &= \sum_{j=1}^m e^{\beta_j t} \sum_{k=1}^{s_j} (f, \psi_{j,k}^*) \psi_{j,k} + \zeta(x, \mu, t) \\
&= e^{-\sigma_n V_n t} \left\{ \sum_{j=1}^m e^{(\beta_j + \sigma_n V_n) t} \sum_{k=1}^{s_j} (f, \psi_{j,k}^*) \psi_{j,k} + n(x, \mu, t) \right\}.
\end{aligned}$$

Here

$$\|n(x, \mu, t)\| = e^{\sigma_n V_n t} \|\zeta(x, \mu, t)\| \leq \text{const.} \cdot t^3.$$

Thus, after a time lapse, the residual term is unimportant compared with the exponential behavior of the finite expansion in terms of eigenelements.

BIBLIOGRAPHY

1. COURANT, R. & D. HILBERT, *Methods of Mathematical Physics*, Vol. 1, Interscience Publishers, New York, 1953.
2. DAVISON, B., *Transport Theory of Neutrons*, National Research Council of Canada, Atomic Energy Project, LT-18, 1947.
3. DAVIDSON, B., *Neutron Transport Theory*, Oxford University Press, London, 1957.

4. GRAVES, L. M., *The Theory of Functions of Real Variables*, McGraw-Hill, New York, 1946.
5. GRÖBNER, W. & N. HOFREITER, *Integraltafel*, Part 2, Springer Verlag, Vienna, 1949.
6. HILLE, E., *Functional Analysis and Semi-groups*, American Mathematical Society Colloquium, Vol. 31, New York, 1948.
7. JÖRGENS, K., An Asymptotic Expansion in the Theory of Neutron Transport, *Communications on Pure and Applied Mathematics*, **11** (1958), pages 219–242.
8. LEHNER, J. & G. M. WING, On the Spectrum of an Unsymmetric Operator Arising in the Transport Theory of Neutrons, *Communications on Pure and Applied Mathematics*, **8** (1955), pages 217–234.
9. LEHNER, J. & G. M. WING, Solution of the Linearized Boltzman Equation for the Slab Geometry, *Duke Mathematical Journal*, **23** (1956), pages 125–142.
10. PIMBLEY, G. H., *Solution of the Initial Value Problem for the Linearized Multi-Velocity Transport Equation with a Slab Geometry*, Dissertation, New York University, 1957. See also Los Alamos Scientific Laboratory Report LAMS-2116.
11. PHILLIPS, R. S., Perturbation Theory for Semi-groups of Linear Operators, *Transactions of the American Mathematical Society*, **74** (1953), pages 199–221.
12. RIESZ, F. & B. SZ.-NAGY, *Functional Analysis*, Translated from the 2nd French Edition by LEO F. BORON, Frederick Ungar, New York, 1955.
13. STONE, M. H., *Linear Transformations in Hilbert Space*, American Mathematical Society Colloquium, Vol. 15, New York, 1932.
14. TAYLOR, A. E., Spectral Theory of Closed Distributive Operators, *Acta Mathematica*, **84** (1951), pages 189–223.
15. TITCHMARSH, E. C., *The Theory of Functions*, Oxford University Press, London, 1939.

Los Alamos Scientific Laboratory of the
University of California
Los Alamos, New Mexico