Incidence Matrices and Linear Graphs

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INTRODUCTION

In the study of electrical networks, kinematic mechanisms, traffic control, and lumped physical systems, one of the fundamental tools has been the connection matrix which is associated with the linear graph describing the system. A full and careful discussion of this process is given in [2]. Accordingly, when we are interested in the synthesis of such systems, one of the fundamental questions is the following: When is a given matrix the connection matrix of a linear graph? It is precisely with this question that we will concern ourselves in this paper.

Now a connection matrix may be either of the branch-mesh or the branch-node pair type. It is well known (see [2]), however, that if one of these two types is given the other may be computed with ease. Hence it is sufficient for our purposes to limit the discussion to only one type; henceforth, we shall use only the branch-mesh form of the connection matrix.

The first section of this paper will be used to develop a purely theoretical solution of our problem without any regard to the computability of the problem by this approach. Our theoretical approach to the problem will be to obtain from the branch-mesh matrix a list of all the incidence relations which the branches must satisfy and to show that a graph exists with this matrix as a connection matrix if all these incidence relations are satisfied. The second section will be devoted to developing a body of theorems which may lead to a more reasonable approach to the computations involved in this problem. Our main result is a technique whereby a large matrix can be broken into smaller matrices and the original matrix represents a graph if and only if each of the smaller matrices represents a graph.

1. Theoretical Results. If a matrix $\mathbf{M}$ is the branch-mesh matrix of a linear graph, then, by the discussion in [2], we know that it can be transformed into a matrix of the form

$$\mathbf{M} = \begin{pmatrix} \mathbf{I} \\ \mathbf{T} \end{pmatrix}$$

where $\mathbf{I}$ is the identity $n \times n$ matrix, $\mathbf{T}$ is a matrix with s rows and n columns, and all the entries in $\mathbf{T}$ are zero or one if the branches of the graph are unoriented.

Thus if a given matrix cannot be reduced to this form, then it does not represent a graph. Hence we will, from this point on, assume that our matrix $\mathbf{M}$ is given to us in this form. Now *it is crucial* to realize that the matrix $\mathbf{M}$ is nothing more than a convenient way of listing information about the linear graph. It is implicitly assumed in writing the matrix $\mathbf{M}$ that each column represents an *independent* mesh of the linear graph and that together these n meshes constitute an independent mesh basis for the linear graph. Further, the rows specify which meshes contain a given branch. By this we mean that the matrix $\mathbf{M}$ will have the entry one in the (i, j) place if and only if the i branch is in the j mesh. From the form of our matrix, we see that the last s branches must constitute a tree for the linear graph and hence that the part of the graph containing just the last s branches *must have no closed cycles*. Henceforth, when we talk about branches, we will refer only to one of the meshes corresponding to a column of our matrix. These meshes will sometimes be called the fundamental meshes. It should also be pointed out that we will list a mesh by listing all of its branches in the tree, but not its branch outside the tree. This clearly does not lose any information since the existence of the mesh is equivalent to the existence of the branch outside the tree. This amounts to *considering henceforth only the matrix $\mathbf{T}$ instead of $\mathbf{M}$* .

Before proceeding with our discussion, we will list certain phenomena which may occur in the matrices.

1. *Splitting*. A matrix $\mathbf{T}$ will be said to be splittable if by the interchange of rows and columns the matrix $\mathbf{T}$ can be completely reduced, *i.e.*, if $\mathbf{T}^*$ is $\mathbf{T}$ after these changes then

$$\mathbf{T}^* = \begin{pmatrix} \mathbf{T}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_2 \end{pmatrix}.$$

The matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are called component matrices.

2. *Series in the tree*. The existence of two branches b_1 and b_2 such that b_1 is in a mesh M_i if and only if b_2 is in the mesh M_i .

3. *Parallel branches*. The existence of a mesh with a single element or the existence of two or more identical columns of $\mathbf{T}$.

4. *Series out of the tree*. The existence of a branch in one and only one mesh.

We will be interested only in matrices $\mathbf{M}$ for which none of these phenomena can occur. Such matrices will be called fully reduced matrices.

Lemma 1. *Corresponding to a matrix $\mathbf{T}$ there exists a set $\mathbf{T}_1, \dots, \mathbf{T}_k$ of fully reduced matrices. Further, $\mathbf{T}$ represents a graph if and only if each of the $\mathbf{T}_1, \dots, \mathbf{T}_k$ represents a graph.*

Proof. First split $\mathbf{T}$ into component matrices $\mathbf{T}'_1, \dots, \mathbf{T}'_k$ such that $\mathbf{T}'_i$ is not splittable $i = 1, \dots, k$. Next, in each $\mathbf{T}'_i$, (1) identify any two branches which are in series in the tree and remove one of them, (2) remove any mesh with a single element, (3) remove any branch which is in one and only one mesh, and (4) remove all but one of the identical columns. Each of these removed items

represents a topology; i.e., a non-uniqueness which can be reinserted at will. Call the new matrix $\mathbf{T}_i$. It is then trivial to verify that $\mathbf{T}$ represents a graph if and only if $\mathbf{T}_1, \dots, \mathbf{T}_k$ represent graphs.

In the future we will restrict ourselves to fully reduce matrices unless explicitly stated to the contrary.

With this aside, we can now return to our discussion of incidence relations in a linear graph. These may be formalized as follows:

Incidence relation I. The branches in a mesh must appear head to toe in some order.

Incidence relation II. The collection of branches which appear in two meshes must appear head to toe in exactly the same order in both meshes.

Incidence relation III'. The branches in the tree must not form a closed cycle.

Our first task is to convert the negative statement III' into a pair of positive incidence relations. In order to do this, we must introduce the following concepts.

Definition 1. We will say that two meshes M_1 and M_2 are contiguous across branches $b_1, \dots, b_k, k > 0$, if the meshes M_1 and M_2 have exactly these branches in common and each contains at least one branch which is not in the other.

Definition 2. Let $M_1, \dots, M_r$ be an ordered collection of meshes with M_i contiguous with $M_{i+1}, i = 1, \dots, r-1$, across the set $B_i, i = 1, \dots, r-1$. We will say that $M_1, \dots, M_r$ form a chain of meshes if $B_i \cap B_{i+1} = \emptyset, i = 1, \dots, r-2$. Further, if the chain consists of at least three meshes and the first and the last mesh in the chain are contiguous, we will say that the chain is an apparent cycle and denote it by $[M_1, \dots, M_r]$. If an apparent cycle contains no subchain which is an apparent cycle, we will call it *simple* or a *simple apparent cycle*. The set of branches $B_1, \dots, B_r$ will be called the *star* of the simple apparent cycle $[M_1, \dots, M_r]$, where B_r is the set of branches across which M_1 and M_r are contiguous.

Theorem 1. Let $\mathbf{T}$ be the branch-mesh matrix of a linear graph and let $[M_1, \dots, M_r]$ be a simple apparent cycle with star $B_1, \dots, B_r$. Then

1. M_i and $M_\alpha, \alpha = 1, \dots, r$, have no branches in common except for $\alpha = j \pm 1$, where $M_0 = M_r$ and $M_{r+1} = M_1$.
2. Each of the collections of branches $B_i, j = 1, \dots, r$, must go through a common node.

Proof. Assertion 1 is an immediate consequence of the fact that the apparent cycle $[M_1, \dots, M_r]$ was assumed simple. Hence the essential part is the proof of assertion 2. Before proceeding to the proof of assertion 2, we will prove the following weaker result.

B_α and $B_\beta, 1 \leq \alpha, \beta \leq r$, have at least one node in common.

Let us assume that the above statement is false. Then, relabeling if necessary, we may assume, without loss of generality, that B_1 and B_2 have no node in common. Now, in M_i , consider the two sets of branches B_{i-1} and B_i . From the incidence relations I and II, we see that the branches in B_{i-1} and B_i must

be ordered head to toe in M_i . Further, the set of branches B_{i-1} and B_i must be adjacently ordered in M_i . Hence for every branch b in M_i there is a unique ordered chain of branches in M_i from b to elements of B_i . We will say that b , not in B_{i-1} , is on the opposite side of B_{i-1} from B_i , if this chain contains at least one branch of B_{i-1} not in B_i . If in M_i we delete the branches which lie on the opposite side of B_{i-1} from B_i , what remains is a connected chain of branches which we will denote by M'_i .^{*} Now each of the total collections of branches in the meshes $M_2, \dots, M_\alpha$ and $M_{\alpha+1}, \dots, M_r, M_1$ forms a connected set. Further, the total collections of branches in $M'_2, M_3, \dots, M_{\alpha-1}, M_\alpha^*$ and $M'_{\alpha+1}, M_{\alpha+2}, \dots, M_r, M_1^*$ form connected sets. If from these two collections of branches we remove the branches in B_1 and B_α , each of the collections would still be connected, but now they would be disjoint. Let us assume that B_1 and B_α have no vertex in common. Then, since B_1 and B_α can be joined by two disjoint connected sets of branches in the tree, we can conclude that the tree contains a closed cycle. This is impossible. This contradiction shows that B_1 and B_α , $1 \leq \alpha \leq r$, have a node in common.

The proof of assertion 2 can now proceed as follows: Assume $B_1, \dots, B_\alpha$ have a common node, but $B_1, \dots, B_{\alpha+1}$ do not have a common node. Then $B_{\alpha+1}$ meets B_1 and B_β , $1 \leq \beta \leq \alpha$, but not in the node where B_α meets B_1 . This again implies that the tree contains a closed cycle, which is impossible. Hence by induction all the B_α have a common node.

We may restate the results of Theorem 1 in the following form:

Incidence relation III. Let $[M_1, \dots, M_r]$ be simple with star $B_1, \dots, B_s$. Then B_i and B_{i+1} are disjoint adjacently ordered sets of branches in M_i , where B_{s+1} equals B_1 .

Definition 3. The vertex at the center of the star of a simple apparent cycle will be called a branch point.

Lemma 2. Let two simple apparent cycles have stars $B_1, \dots, B_s$ and $B'_1, \dots, B'_t$. Assume $B_i \cap B'_\alpha \neq 0$ and $B_{i+1} \cap B'_{\alpha+1} \neq 0$ for some α, j where $1 \leq j \leq s-1$ and $1 \leq \alpha \leq t-1$. Then the two simple apparent cycles have the same branch point.

The proof of this lemma is straightforward and will be omitted.

We are now left with the task of formulating our fourth and final incidence relation. Before we can proceed with this we will need the following definition.

Geometrically, if we envision our graph as being constructed out of an erector set, a limb may be described as follows: Choose a vertex and pull out the stick entering one hole thus breaking the graph in two connected pieces. The sub-graph attached to the stick we pulled out will be called a limb. We can give an abstract formulation of this geometrical construction as follows:

^{*}If in M_i we delete the branches which lie on the opposite side of B_i from B_{i-1} , we again get a connected chain of branches which we will denote by M_i^* .

Definition 4. We will assume that the branches in each mesh have been ordered so as to satisfy incidence relations I, II, III. Let $[M_{i1}, \dots, M_{is_i}]$, $i = 1, \dots, t$, be all the simple apparent cycles with the same branch point. Let $B_{i1}, \dots, B_{is_i}$, $i = 1, \dots, t$, be the star of the simple apparent cycle $[M_{i1}, \dots, M_{is_i}]$. Let B_{ij} and $B_{\alpha\beta}$, $1 \leq \alpha, i \leq t$, $1 \leq j \leq s_i$, $1 \leq \beta \leq s_\alpha$, be given. Then since B_{ij} and $B_{\alpha\beta}$ have a vertex in common, the branches from B_{ij} and $B_{\alpha\beta}$ must be adjacently ordered in any mesh containing branches from both B_{ij} and $B_{\alpha\beta}$. Let $M(i, j)$ be any mesh containing branches from B_{ij} and $B_{\alpha\beta}$. Then, if b is a branch in $M(i, j)$, there is a unique ordered connected sub-collection of branches from b to branches in $B_{\alpha\beta}$. If this collection contains branches in B_{ij} , we will say that b is separated from $B_{\alpha\beta}$ by B_{ij} or b lies on the opposite side of B_{ij} from $B_{\alpha\beta}$.* Let $b(M(i, j))$ denote the set of all branches separated from $B_{\alpha\beta}$ by branches in B_{ij} . Clearly $b(M(i, j))$ contains the branches of B_{ij} which are in $M(i, j)$ as a subset. For each B_{ij} , we may consider $\mathbf{B}(i, j)$ defined as follows: $b \in \mathbf{B}(i, j)$ if $b \in b(M(i, j))$ for some $M(i, j)$ or if b can be joined to $b(M(i, j))$ by a chain of meshes none of which contains a branch of $B_{\alpha\beta}$ for $\alpha \neq i$ and $\beta \neq j$. The set $\mathbf{B}(i, j)$ will be called the limb attached to B_{ij} .

Definition 5. $\mathbf{T}(i, j)$ will be called the matrix of the limb $\mathbf{B}(i, j)$ if the meshes in $\mathbf{T}(i, j)$ are determined as follows: A set of branches in $\mathbf{B}(i, j)$ will determine a fundamental mesh in $\mathbf{T}(i, j)$ if they are all the branches in a fundamental mesh in $\mathbf{T}$ that lie in $\mathbf{B}(i, j)$.

Lemma 3. If the matrix $\mathbf{T}$ represents a graph, then $\mathbf{B}(i, j)$ and $\mathbf{B}(\alpha, \beta)$ are disjoint unless $i = \alpha$ and $j = \beta$. Further, $\mathbf{T}(i, j)$ represents a graph with branches $\mathbf{B}(i, j)$.

The proof of this lemma is again straightforward and will be omitted.

In order to obtain a geometric picture of the graph $\mathbf{T}(i, j)$ we may proceed as follows: Assume our tree is composed out of erector sets and the meshes are formed by tying a piece of string between two vertices. Then, as we explained before, the limb is obtained by pulling one stick out of a hole. Thus, this gives us two connected pieces of erector sets. Cut all the strings that connect these two pieces together and tie the loose ends of the cut strings all to the end of the piece we pulled out. This new graph will be the graph $\mathbf{T}(i, j)$.

Incidence relation IV. For every branch point, the limbs $\mathbf{B}(i, j)$ and $\mathbf{B}(\alpha, \beta)$ are disjoint unless $i = \alpha$ and $j = \beta$.

Before proceeding to the proof of the main existence theorem, we will prove the following lemma.

Lemma 4. Let $\mathbf{T}$ be a matrix with no branch points which satisfies the incidence relations I and II; then $\mathbf{T}$ is the incidence matrix of a graph.

Proof. Since we have assumed that $\mathbf{T}$ is a fully reduced matrix, we can order the fundamental meshes in such a way that M_k has branches in common with $\bigcup_{i=1}^{k-1} M_i$. We may identify M_1 and M_2 together along their common branches.

*Note the slight change in meaning of the expression "on the opposite side of."

This process we will call joining. Let us now assume that the join of M_1 through M_{k-1} has no cycles in the tree. Let us also assume that the join of M_1 through M_k has a cycle in the tree. Let $b_1, \dots, b_s$ be the branches in the cycle and assume $b_1 \in M_k$ and b_1 is not in the join of M_1 through M_{k-1} . Let b_{i_s} be the first branch in the cycle and in the join of $M_1, \dots, M_{k-1}$. Then b_{i_s} must also be in M_k . This follows from the definition of join. Let $M_{i_s} \supset b_{i_s}$. Continue to the first b_{i_s} which is in the join of M_1 through M_{k-1} with M_{i_s} removed. Then b_{i_s} is in M_{i_s} and M_{i_s} with $M_{i_s} \neq M_{i_s}$ and $M_{i_s} \in M_1, \dots, M_{k-1}$. Proceeding in this way we can construct an apparent cycle in the meshes $M_1, \dots, M_k$. Hence we can construct a simple apparent cycle in the meshes $M_1, \dots, M_k$. This contradiction proves the lemma.

Theorem 2 (Existence Theorem). Let $\mathbf{T}$ be a matrix with fundamental meshes $M_1, \dots, M_n$. Assume the ordering of branches in each mesh can be assigned in such a way that the incidence relations I through IV are satisfied. Then there exists a graph G having $M_1, \dots, M_n$ as fundamental mesh basis and incidence matrix $\mathbf{T}$.

Proof. The proof of this theorem will be by induction on the number of branch points. Assume we have no branch points; then Lemma 4 implies that the theorem is true. Assume that the theorem is true for a matrix with $k - 1$ branch points. Let $\mathbf{T}$ have k branch points; and let us choose a branch point and for this branch point the limbs $\mathbf{B}(i, j)$, $1 \leq i \leq t$, $i \leq j \leq s_i$. By hypothesis, these are disjoint sets and have matrices $\mathbf{T}(i, j)$ associated with them. Since any branch point in $\mathbf{T}(i, j)$ is a branch point in $\mathbf{T}$, then each of the $\mathbf{T}(i, j)$ must have less than k branch points.

We will next show that if $\mathbf{T}$ satisfies the incidence relations I through IV, the matrices $\mathbf{T}(i, j)$ will also satisfy them. That this is true for axioms I through III is trivial to verify. Hence we are left with the task of verifying incidence relation IV. But, for each branch point in $\mathbf{T}(i, j)$, the limbs are subsets of the limbs of the same branch point in $\mathbf{T}$. Hence they must again be disjoint, for otherwise they would not be disjoint in $\mathbf{T}$. This proves by induction that $\mathbf{T}(i, j)$ represents a graph. Put the tree to each of these graphs together and then join all these together at the branch point. Since they are all disjoint, there can be no apparent cycles introduced by this procedure. This proves the theorem.

2. Decomposition Theorem. Consider the matrix $\mathbf{T}$ and let b be a branch in $\mathbf{T}$. Then by $\mathbf{T}' = \mathbf{T} \pmod{b}$ we will mean the new matrix $\mathbf{T}'$ obtained from $\mathbf{T}$ by deleting from $\mathbf{T}$ all columns representing meshes containing b and the row representing the branch b itself.

Definition 6. If $\mathbf{T}'$ splits into $\mathbf{T}_1$ and $\mathbf{T}_2$ we will say that the branches in $\mathbf{T}_1$ are separated mod b from the branches in $\mathbf{T}_2$.

Of course, $\mathbf{T}'$ may split into many pieces or such a split may not occur at all. There is, however, still another phenomenon which may occur. It may happen that in $\mathbf{T}'$ the row representing the element b_1 may have zero entries. In other words, when we removed b , we also removed b_1 .

Definition 7. $\mathbf{T}(\text{mod } b)$ will be called complete if $\mathbf{T}'$ contains all the other elements in $\mathbf{T}$ except b . If $\mathbf{T}(\text{mod } b)$ splits and is complete we will call this a *clean break*.

Lemma 5. Let $\mathbf{T}' = \mathbf{T}(\text{mod } b_0)$ and let $b_1, \dots, b_k$ not appear in $\mathbf{T}'$. Then $\mathbf{T}(\text{mod } b_i)$ for some $1 \leq i \leq k$ will be complete.

Proof. Consider $\mathbf{T}(\text{mod } b_1)$. This is either complete or it is not. Assume it is not complete. Then b_1 cannot appear in exactly those meshes that contain b_0 . But b_1 disappears mod b_0 . Hence every mesh containing b_1 contains b_0 . Hence b_1 appears in a proper subset of the meshes containing b_0 . Now let us reduce mod b_1 . Then b_0 will remain and no more branches than $b_1, \dots, b_k$ can be removed. Hence by an elementary induction we see that the theorem is true.

Lemma 6. Let $\mathbf{T}$ represent a graph and let $\mathbf{T}(\text{mod } b_0)$ be a clean break. Then the branches in each of the component matrices appear contiguously in every mesh of $\mathbf{T}$.

Proof. Any two branches in the same component matrix can be joined by a chain of meshes in the component matrix. If the lemma were false, then this chain of meshes and the mesh containing the separated branches from the same component matrix would form a cycle in the tree. This contradiction proves the lemma.

Remark. We will always consider b_0 itself as constituting a component.

Lemma 7. Let $\mathbf{T}$ represent a graph and let $\mathbf{T}(\text{mod } b_0)$ be a clean break. Let M_1 and M_2 be meshes and let $\mathbf{T}'_i, i = 1, \dots, k, k > 1$, be the component matrices of $\mathbf{T}(\text{mod } b_0)$ with branches in both meshes M_1 and M_2 . Then it is possible to order the component matrices common to $\mathbf{T}'_i, i = 1, \dots, k$, in such a way that at most the first and last have different branches in M_1 and M_2 .

Proof. Let b_1 and $b_2 \in \mathbf{T}'_i$ and let b_1 be in M_1 and b_2 be in M_2 and either b_1 or b_2 not in M_1 or M_2 respectively. Since b_1 and $b_2 \in \mathbf{T}'_i$ there exists a chain of meshes $M_3, \dots, M_s$ from $\mathbf{T}'_i$ joining b_1 and b_2 which contain no subcycles. Now M_1 and M_2 have b_0 in common. Hence $M_1, M_2, \dots, M_s$ is a simple apparent cycle. Hence the branches common to M_1 and M_2 and the branches in $\mathbf{T}'_i \cap M_1$ and $\mathbf{T}'_i \cap M_2$ are part of a star of the simple apparent cycle $[M_1, \dots, M_s]$. Hence they must be adjacently ordered in all meshes containing them. This proves the lemma.

Reduced incidence relation. Let $\mathbf{T}$ be a matrix. We will say that $\mathbf{T}$ satisfies the reduced incidence relation relative to b_0 if $\mathbf{T}(\text{mod } b_0)$ is a clean break with the following property: If $\mathbf{T}'_i, i = 1, \dots, k$, are the component matrices of $\mathbf{T}(\text{mod } b_0)$ and M_1 and M_2 are any meshes in $\mathbf{T}$, then it is possible to order the component matrices with branches appearing in both M_1 and M_2 in such a way that at most the first and last have different branches in M_1 and M_2 .

Definition 8. Let $\mathbf{T}$ be a matrix which satisfies the reduced incidence relation relative to b_0 . In the meshes containing b_0 identify the branches in the same component matrix with one letter. These meshes then form an incidence matrix

called the reduced incidence matrix mod b_0 and denoted by $R(b_0)$. We will say that $R(b_0)$ represents a graph if it represents a graph where each mesh satisfies the following additional order relation: If M_1 and M_2 are any two meshes in $\mathbf{T}$, the component matrices with branches appearing in both M_1 and M_2 have been ordered in such a way that at most the first and last have different branches in M_1 and M_2 .

Definition 9. Consider the component matrices $\mathbf{T}'_i$ of $\mathbf{T}(\text{mod } b_0)$. We will modify these matrices by the addition of fundamental cycles of two types:

1. Let M be a mesh in $\mathbf{T}$. Then adjoin to $\mathbf{T}'_i$ as a mesh the collection of branches common to M and $\mathbf{T}'_i$.
2. Let M_1 and M_2 be any two meshes of $\mathbf{T}$. Assume that $\mathbf{T}'$ does not have identical branches in common with M_1 and M_2 . Then adjoin one mesh to $\mathbf{T}'_i$ consisting of those branches in $\mathbf{T}'$ which are in M_1 and not in M_2 and those branches in $\mathbf{T}$ which are in M_2 and not in M_1 .*

Let $\mathbf{T}_i$ represent this new matrix. It will be called the adjoined component matrix.

Theorem 3. *A necessary and sufficient condition for $\mathbf{T}$ to be the incidence matrix for a graph is that if $\mathbf{T}(\text{mod } b_0)$ is a clean break, then the reduced matrix and the adjoined component matrices represent graphs.*

Proof. The necessity of this theorem is obvious. All that remains, therefore, is to verify that the hypothesis implies the existence of an ordering of the branches in all meshes of $\mathbf{T}$ which satisfies the incidence relations I through IV.

We will first state how the branches are ordered in the meshes and then verify that these satisfy the required incidence relations. First, order the branches in the meshes of each adjoined component matrix as for the graph of this matrix. Next, order the branches in the meshes containing b_0 in two stages: first, group together all the branches from an adjoined component matrix and order these as in the adjoined component matrix graph; second, order the branches from the adjoined components as ordered in the reduced graph. There is then an ordering for every mesh. We have to prove that this ordering satisfies incidence relations I through IV. But, from the very definition, it is clear that incidence relations I and II are satisfied. It remains, therefore, to verify incidence relations III and IV.

We observe first that there are simple apparent cycles of two types in $\mathbf{T}$. The first type of simple apparent cycle, called type I, occurs in the component matrices $\mathbf{T}'_i$. Clearly the stars of simple apparent cycles of type I are all in the component matrices $\mathbf{T}'_i$ and hence in the adjoined component matrix $\mathbf{T}_i$, and have hence been properly ordered to satisfy incidence relation III by the existence of the graph for $\mathbf{T}_i$. To see that incidence relation IV is also satisfied, we have merely to notice that in all fundamental meshes of $\mathbf{T}$ joining a branch

*This operation can introduce meshes not defined in the matrix of the original graph.

in $\mathbf{T}_i$ to a branch in $\mathbf{T}_j$ the chain of branches between these two branches must be exactly the same for all meshes. This follows easily from the reduced incidence relation. Hence, if incidence relation IV is to be violated, it must be violated in an adjoined component matrix. But this cannot happen, since we have assumed that each adjoined component matrix represents a graph.

The second type of simple apparent cycle, called type II, involves meshes containing b_0 and meshes in a component matrix $\mathbf{T}'_i$. But, by the definition of the adjoined component matrix $\mathbf{T}_i$, each simple apparent cycle of the second type determines a simple apparent cycle in adjoined component matrix $\mathbf{T}_i$. Hence the star $B_1, \dots, B_k$ of the simple apparent cycle of type II gives rise to a star $B_1, \dots, T_{k-1}$ in $\mathbf{T}_i$, and B_k contains a set of branches common to two meshes M_1 and M_2 containing distinct branches from the component matrix $\mathbf{T}'_i$. Hence, by the reduced incidence relation, $B_1, \dots, B_k$ have been properly ordered in $\mathbf{T}$. This proves incidence relation III for simple apparent cycles of type II. The proof that incidence relation IV is satisfied for simple apparent cycles of type II is then identical to the proof for simple apparent cycles of type I. This proves the theorem.

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