*Distributed and Lumped Networks**

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1. Introduction. This paper is concerned with the electrical conductance between terminal surfaces attached to a conducting body. The specific conductance is assumed to be unity. Mathematically the problem is nothing more or less than the Dirichlet problem. In fact, the electrical conductance is equal to the minimum of Dirichlet's integral for functions which have the value unity on one terminal surface and which have the value zero on the other surface.

More generally, the analysis here concerns the system resulting when several terminal surfaces are attached to the conductor. This system may be termed a *distributed network*. As is well known, this network is equivalent to a network in which the conductivity is lumped into "wires" connecting the terminals. This lumped network will be termed the *synthetic network*. The analysis shows that the various properties of the synthetic network can be expressed in terms of the conductance between pairs of terminals.

The numerical treatment of the conductance problem can be reduced to the estimation of Dirichlet's integral D . Various methods have been proposed to make this estimation, but in this paper attention is restricted to developments of COURANT [1] and SYNGE [9, 10, 11]. The method of these authors is to break the region up into cells. Functions are introduced which are linear in the interior of the cells and which satisfy the prescribed boundary conditions. Such functions, when inserted in certain minimum principles, yield upper and lower bounds for D .

When this cell method of COURANT and SYNGE is applied to a given problem, it leads to two systems of linear equations. The point of departure of this paper is the interpretation of these systems of equations as the equations of two electric networks. The first network is termed an *upper network* because the conductance between a pair of its terminals is not less than the conductance between the corresponding pair of terminals of the distributed network. The second network is termed a *lower network* because the conductances are not greater than the corresponding conductances of the distributed network.

The introduction of upper and lower networks is advantageous on several counts: (a) The algebra is systematized with explicit numerical evaluation of coefficients. (b) The networks have geometric and physical interpretation which

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aids in analyzing the algebra. (c) Various theorems and relations from network theory may be applied. (d) The method yields a justification of the usual difference equation approximation to Laplace's equation. (e) *The method is well suited to numerical treatment either by hand or by machine.*

The cell method, as originally proposed, used triangular cells in two dimensions and tetrahedrons in three dimensions. In Section 6 the cell method is extended to rectangular cells. It is then found necessary to use harmonic polynomials of the second degree in the cells rather than linear functions. The upper and lower networks corresponding to rectangular cells are shown to yield more accurate bounds than in a comparable triangular method.

The last two sections of the paper concern special methods available for planar conductors. If a harmonic function U solves a problem for a conductor it was found by RAYLEIGH [19] that the conjugate harmonic function V solves another problem for what is termed the *conjugate conductor*. Conjugate conductors have reciprocal values of conductance. These concepts are related to the theory of extremal length of AHLFORS.

By the use of conjugate conductors both upper and lower bounds for conductance can be obtained from a single upper network. These considerations lead naturally to the concept of conjugate discrete harmonic functions. The notion of *discrete analytic functions* [14] is thereby extended from regular square lattices to various irregular networks.

The mathematics of the flow of electrical current has many interpretations; consequently the present paper has other applications. In particular, upper and lower networks apply to the following fields: heat flow, gravitation, irrotational hydraulic flow, hydraulic flow in sand, diffusion, electric induction and capacity, reluctance of magnetic circuits, and torsional rigidity.

2. Lumped networks. Basic to the study of networks is the system of linear equations

$$(1) \quad w_i = \sum_{j=0}^n g_{ij}(v_i - v_j), \quad i = 0, 1, 2, \dots, n,$$

expressing variables w_i as linear functions of variables v_i . The coefficients g_{ij} are symmetric, that is, $g_{ij} = g_{ji}$. It follows from this symmetry that

$$(2) \quad \sum_{i=0}^n w_i = 0.$$

A quadratic form associated with (1) is

$$(3) \quad Q = \sum_{i < j} \sum g_{ij}(v_i - v_j)^2.$$

It is seen that $\sum_0^n w_i v_i = Q$ because

$$2Q = \sum_0^n \sum_0^n g_{ij}(v_i - v_j)v_i + \sum_0^n \sum_0^n g_{ij}(v_i - v_j)v_j.$$

The quadratic form (3) has the property of vanishing if all the v_i are equal.

Lemma 1. Let $Q^* = \sum_0^n \sum_0^n a_{ij} v_i v_j$ be a semi-definite quadratic form. If $Q^* = 0$ whenever the v_i are equal, then Q^* may be put in the form (3).

Proof. Let v denote the vector with components $(v_0, v_1, v_2, \dots, v_n)$. Let k denote another vector. Then in the usual symbolism for quadratic and bilinear forms

$$Q^*(v + k) = Q^*(v) + 2Q^*(v, k) + Q^*(k).$$

If all the components of k are equal, then $Q^*(k) = 0$. Because Q^* is semi-definite, it follows that $Q^*(v, k) = 0$. Thus $Q^*(v + k) = Q^*(v)$. In particular, let all the components of k have the value v_0 . Then

$$Q^*(v) = \sum_0^n \sum_0^n a_{ij} (v_i - v_0)(v_j - v_0).$$

On the right side employ the identity

$$2(v_i - v_0)(v_j - v_0) = (v_i - v_0)^2 + (v_j - v_0)^2 - (v_i - v_j)^2.$$

This expresses Q^* in the form (3), and the proof is completed.

A quadratic form $Q(v)$ will be termed *potentially definite* if $Q(v) \geq 0$ but $Q(v) = 0$ if and only if all the components of v are equal.

Lemma 2. If the quadratic form associated with equations (1) is potentially definite, then there are constants ρ_{ij} such that if w_i satisfies equations (1),

$$(4) \quad v_i = -\frac{1}{2} \sum_{j=0}^n \rho_{ij} w_j + c$$

where c does not depend on i . Moreover, $\rho_{ii} = \rho_{ji}$, $\rho_{ji} = 0$, and $\rho_{ii} > 0$ for $i \neq j$.

Proof. Let $u_0 = 0$ in equations (1) and delete the first equation. The linear transformation resulting, say $w_i = \sum_1^n a_{ij} v_j$, has quadratic form Q (with $v_0 = 0$). It follows that the linear transformation is non-singular because it is positive definite. Thus the inverse transformation exists, say $v_i = \sum_{j=1}^n b_{ij} w_j$. Since $a_{ij} = a_{ji}$, it follows that $b_{ij} = b_{ji}$. Define ρ_{ij} by

$$(5) \quad \rho_{ii} = -2b_{ii} + b_{ii} + b_{ii}.$$

Here b_{0i} or b_{i0} is given the value zero.

$$\sum_0^n \rho_{ij} w_i = -2 \sum_1^n b_{ii} w_i + b_{ii} \sum_0^n w_i + \sum_0^n b_{ij} w_i = -2v_i + 2c.$$

Here $2c = \sum_0^n b_{ij} w_i$. This proves (4). It is apparent from (5) that $\rho_{ij} = \rho_{ji}$ and that $\rho_{ii} = 0$. To prove the last statement of Lemma 2, let $w_a = J = -w_b \neq 0$ and $w_i = 0$ if $i \neq a$ or b . Then (4) gives $v_a = \frac{1}{2} J \rho_{ab} + c$ and $v_b = -\frac{1}{2} J \rho_{ab} + c$, so

$$(6) \quad Q = \sum w_i v_i = J(v_a - v_b) = J^2 \rho_{ab}.$$

All the v_i cannot be equal, because then equations (1) would give $w_i \equiv 0$. Thus since Q is potentially definite it follows that $Q > 0$. Then $\rho_{ab} > 0$ by (6), and the proof is completed. We now may write in this case

$$(6a) \quad Q = \frac{(v_a - v_b)^2}{\rho_{ab}}.$$

Equations of the form (1) govern the steady flow of current in an electrical network. In this interpretation $v_0, v_1, \dots$ denote the potentials of the $n + 1$ junctions of the network, and $w_0, w_1, \dots$ denote the currents entering these junctions from outside the network. The coefficient g_{ij} is the conductance of the "branch" connecting junction i to junction j . If w_{ij} denotes the current flowing in this branch from i to j , then by the conservation of electricity (Kirchhoff's first law)

$$(7) \quad w_i = \sum_{j=0}^n w_{ij}.$$

On the other hand, by Ohm's law

$$(8) \quad w_{ij} = g_{ij}(v_i - v_j).$$

Substitution of (8) in (7) demonstrates the equations (1).

In the quadratic form Q , the term $g_{ij}(v_i - v_j)^2$ is recognized as the heat generated per second in the branch (i, j) . Thus Q represents power dissipated in heat. If $g_{ij} \neq 0$, let $r_{ij} = 1/g_{ij}$; then r_{ij} is the resistance of the branch (i, j) . Let a second quadratic form be

$$(9) \quad H(w) = \sum_{i < j} \sum r_{ij} w_{ij}^2$$

where it is assumed that $w_{ii} = 0$ if $g_{ii} = 0$. If w_{ij} is given by (8), then $H(w) = Q(v)$. A detailed study of various relationships between H and Q is given in [4].

The i^{th} junction is said to be *insulated* if $w_i = 0$. Then (1) gives

$$(10) \quad v_i = \sum_{j=0}^n g_{ji} v_j / \sum_{j=0}^n g_{ji}.$$

This states that v_i is a weighted mean of the other v_j . Note that $\sum_0^n g_{ii} > 0$ for a potentially definite form because it is the value of Q for $v_i = 1$; all other $v_j = 0$.

Equation (10) may be employed to "eliminate" an insulated junction from a network. To make this definite, suppose that $w_0 = 0$, then substituting (10) in (1) gives

$$(11) \quad w_i = \sum_{j=1}^n g_{ij}^*(v_i - v_j)$$

where

$$(12) \quad g_{ij}^* = g_{ij} + g_{i0}g_{0j} / \sum_{i=0}^n g_{0i}.$$

Equations (11) may be thought of as applying to a new network with one less junction; the relationship between the variables is unchanged. In network language, this elimination is termed a *star-mesh transformation*. It is a useful mnemonic for solving the network equations (1).

Relation (6) gives an important network interpretation of the coefficients ρ_{ij} . It is seen that ρ_{ab} gives the *resistance* between junctions a and b , all other junctions being insulated. Likewise, $\gamma_{ab} = \rho_{ab}^{-1}$ is termed the *conductance*. Let c and d denote two other junctions. Then under the same conditions, current J entering at a and leaving at b , it is found that $v_c - v_d = J\rho_t$ where

$$(13) \quad 2\rho_t = \rho_{ad} + \rho_{bc} - \rho_{ac} - \rho_{bd}.$$

The coefficient ρ_t is termed the *transfer resistance*. A related quantity is the *voltage transfer ratio*, defined as

$$(14) \quad \theta = \frac{v_c - v_d}{v_a - v_b} = \frac{\rho_{ad} + \rho_{bc} - \rho_{ac} - \rho_{bd}}{2\rho_{ab}}.$$

In particular, these relations continue to hold if $d = b$.

It is also seen from relation (6) that ρ_{ab} is the value of the quadratic form Q when unit current enters at junction a and leaves at junction b . It is also seen from this same relation that ρ_{ab}^{-1} is equal to the value of Q when there is unit difference of potential between junctions a and b .

Suppose that some of the junctions of a network are at prescribed potentials and that the other junctions are insulated. The potentials of the insulated junctions may be determined by solving equations (10). By analogy with the continuous case this may be called the Dirichlet problem. The prescribed potentials may be called the "boundary conditions."

For given boundary conditions Q is minimized by the solution v of the Dirichlet problem. To see this let s be an arbitrary vector which has zero boundary values. Then

$$Q(v + s) = Q(v) + 2Q(v, s) + Q(s).$$

From equations (1) it is seen that $Q(v, s) = 0$. Thus $Q(v + s) \geq Q(v)$, and this is a strict inequality unless s vanishes identically.

It is of interest to employ the above minimum principle for the particular boundary conditions $v_a = 1$ and $v_b = 0$. This gives the inequality

$$(15) \quad \gamma_{ab} \leq \frac{Q(u)}{(u_a - u_b)^2}.$$

Here u is an arbitrary vector. Relation (15) gives an upper bound for the conductance.

To obtain a lower bound for the conductance it is convenient to introduce the further hypothesis that $g_{ij} \geq 0$. Let w_{ij} denote an antisymmetric matrix which is arbitrary except that $w_{ij} = 0$ if $g_{ij} = 0$. Then a current vector w_i is defined by (7). If v_i is an arbitrary potential vector, we find

$$\sum_0^n w_i v_i = \sum_0^n \sum_0^n w_{ij} v_i = \sum_{i < j} \sum w_{ij} (v_i - v_j).$$

Write $1 = r_{ij}^{\frac{1}{2}} g_{ij}^{\frac{1}{2}}$; then the Cauchy-Schwarz inequality gives

$$\left(\sum_0^n w_i v_i \right)^2 \leq \sum_{i < j} \sum r_{ij} w_{ij}^2 \sum_{i < j} \sum g_{ij} (v_i - v_j)^2.$$

This may be written as

$$(16) \quad Q(v) \geq \frac{(\sum w_i v_i)^2}{H(w)}.$$

In particular, suppose that v solves the Dirichlet problem with $v_a = 1$ and $v_b = 0$. The antisymmetric matrix w_{ij} may be chosen so that $w_i = 0$ except that $w_a = -w_b = J$. Then $Q(v) \geq J^2/H(w)$, and, since $Q(v) = \rho_{ab}^{-1}$, it follows that

$$(17) \quad \rho_{ab} \leq \frac{H(w)}{J^2}.$$

The relations just derived are now put in the form of a theorem.

Theorem O. *Given a network of conductors obeying Ohm's law let ρ_{ab} be the joint resistance of the network as measured between junctions a and b . Then:*

I. *Formula (15) gives a lower bound for ρ_{ab} in terms of any assumed state of junction potentials such that the potential at a differs from the potential at b .*

II. *Formula (17) gives an upper bound for ρ_{ab} in terms of any assumed state of current flow such that the current enters at a and leaves at b .*

Let ρ_{ab}^- denote a lower bound and let ρ_{ab}^+ denote an upper bound, etc. Then:

III. *Lower and upper bounds for the transfer resistance ρ_t between the pair of junctions a, b and the pair of junctions c, d are given by the relation*

$$\rho_{ad}^- + \rho_{bc}^- - \rho_{ac}^+ - \rho_{bd}^+ \leq 2\rho_t \leq \rho_{ad}^+ \rho_{bc}^+ - \rho_{ac}^- - \rho_{bd}^-.$$

IV. *Lower and upper bounds for the voltage transfer ratio θ between the pair of junctions a, b and the pair of junctions c, d are given by the relation*

$$\frac{\rho_t^-}{\rho_{ab}^+} \leq \theta \leq \frac{\rho_t^+}{\rho_{ab}^-}.$$

Of especial interest are the transfer properties between three junctions, say a, b and c . If we identify b and d then $\rho_{bd} = 0$ so (14) becomes

$$(14a) \quad \theta = \frac{V_c - V_b}{V_a - V_b} = \frac{\rho_{ab} + \rho_{bc} - \rho_{ac}}{2\rho_{ab}}.$$

If Ohm's law is satisfied then $g_{ij} \geq 0$ and it is not difficult to show that the voltage transfer ratio satisfies $\theta \geq 0$. It then follows from (14a) that

$$(18) \quad \rho_{ac} \leq \rho_{ab} + \rho_{bc}.$$

Let us regard ρ_{ac} as the "distance" between two "points" a and c . Then (18) is the "triangle inequality". Further properties of the voltage transfer ratio are developed in another paper [13].

The relationships for lumped networks given in this section are introduced principally because of their applicability to distributed networks. Most of these relationships have not previously been discussed in the literature. The numerical evaluation proposed in Theorem O should be of practical value because the arithmetical operations involved are quite simple.

3. Synthesis of distributed networks. Let R be a finite domain of space and let B denote the boundary. It is supposed that B is a regular surface and that B contains disjoint regular surfaces $S_0, S_1, \dots, S_n$. These are called *terminals*. It is assumed that the terminals are closed sets and that there are at least two of them. The complement of the terminals in B is denoted by C ; it may be that C is empty. Here the term "regular surface" may be taken as defined by KELLOGG [6].

From the theory of the Dirichlet problem unique functions $f_0, f_1, \dots, f_n$ exist with the following properties:

$$(a) \quad f_i(x, y, z) \text{ is continuous in } R + B.$$

$$(b) \quad \frac{\partial^2 f_i}{\partial x^2} + \frac{\partial^2 f_i}{\partial y^2} + \frac{\partial^2 f_i}{\partial z^2} = 0 \text{ in } R.$$

$$(c) \quad \frac{\partial f_i}{\partial n} = 0 \text{ on } C.$$

$$(d) \quad f_i = 1 \text{ on } S_i \text{ and } f_i = 0 \text{ on } S_k \text{ for } k \neq i.$$

Here $\partial f_i / \partial n$ denotes the normal derivative at points where a normal exists. Let $v_0, v_1, \dots, v_n$ be a set of scalars; then a function F is defined by

$$(1) \quad F = \sum_0^n v_i f_i.$$

Clearly F satisfies conditions (a), (b), and (c). Condition (d) is replaced by

$$(d') \quad F = v_i \text{ on } S_i \text{ for } i = 0, 1, \dots, n.$$

The Dirichlet functional is

$$(2) \quad D(F) = \iiint_R \left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2 \right] dx dy dz.$$

The conditions assumed are sufficient for the existence of this integral. Substituting (1) in (2) gives

$$(3) \quad D(F) = \sum_0^n \sum_0^n c_{ij} v_i v_j = Q(v).$$

The quadratic form Q is potentially definite because if F is not constant in R , then $D > 0$. Also, if $k = v_0 = v_1 = \dots$, the Dirichlet problem has the unique solution $F = k$, which yields $D = 0$.

In (3), c_{ij} is $D(f_i, f_j)$. By Green's theorem this is equal to $\iint_B f_i (\partial f_j / \partial n) dB$. This may be simplified by use of the boundary conditions to give $c_{ij} = \iint_{S_i} (\partial f_j / \partial n) dB$. Let

$$(4) \quad w_i = \sum_0^n c_{ij} v_j = \iint_{S_i} \frac{\partial F}{\partial n} dB.$$

Define $g_{ij} = -c_{ij}$; then by Lemma 1

$$(5) \quad w_i = \sum_0^n g_{ij} (v_i - v_j).$$

These relations may be given the following physical interpretation. F is the electric potential in a body R of unit electrical conductivity. The electric current is $-\text{grad } F$. $S_0, S_1, \dots, S_n$ are perfectly conducting surfaces attached to R ; they serve as electric terminals. The potential of S_i is v_i , and the current entering the body R through S_i is w_i . The boundary C is electrically insulated. With these interpretations it follows from equations (5) that externally this distributed network is equivalent to a lumped network with $n + 1$ junctions. This lumped network is here termed the *synthetic network*.

In Section 2 it was shown that the properties of a network may be simply expressed in terms of the quantities ρ_{ij} . Thus it suffices to confine the study to the ρ_{ij} . To this end it will now be shown that upper and lower bounds for ρ_{ij} of the synthetic network can be obtained from the Dirichlet integral.

Let F be the solution of the Dirichlet problem (a), (b), (c), and (d'). Let U be a continuous function with piecewise continuous derivatives in $R + B$. Moreover, let U satisfy the boundary conditions (d'). Then by Green's theorem

$$D(F, U - F) = \iint_B \frac{\partial F}{\partial n} (U - F) dB = 0.$$

Thus $D(U) = D(F) + D(U - F)$ and

$$(6) \quad D(F) \leq D(U).$$

Substituting (3) and (2-15) in (6) gives

$$(7) \quad \frac{1}{\rho_{ab}} \leq \frac{D(U)}{(u_a - u_b)^2}.$$

This gives the desired upper bound for the conductance. The function U is essentially arbitrary except for being constant on the surfaces S_i ; thus u_a and u_b denote the value of U on S_a and S_b .

Let W be a vector field such that $\text{div } W = 0$. If W and its first derivatives are continuous in $R + B$, then Green's theorem gives

$$(8) \quad \iiint_R W \cdot \nabla F \, dR = \iint_B F W_n \, dB.$$

Here W_n denotes the normal component of W . Following the developments of DIAZ and WEINSTEIN [2, 3], we apply Schwarz' inequality to (8) to obtain

$$(9) \quad D(F) \geq \frac{\left(\iint_B F W_n \, dB \right)^2}{\iiint_R |W|^2 \, dR}.$$

In this relation let F solve the Dirichlet problem (a), (b), (c), and (d') with the particular boundary values such that equation (2-6a) is satisfied with $v_a = 1$ and $v_b = 0$; thus $\rho_{ab}^{-1} = D(F)$. Let $J = - \iint_{S_a} W_n \, dB$ and let $\iint_{S_a} W_n \, dB = 0$ for $i \neq a$ and $i \neq b$. Then (9) gives

$$(10) \quad \rho_{ab} \leq \frac{1}{J^2} \iiint_R |W|^2 \, dR.$$

This is the desired lower bound for the conductance.

It is to be expected that $g_{ij} \geq 0$ for the lumped network. This property is not needed for the present developments, but presumably a rigorous proof could be carried through along the following lines. First suppose that the boundary C is absent. Then the function f_i defined above satisfies $f_i \geq 0$ by the minimum principle for harmonic functions. On the surface S_i the boundary condition is $f_i = 0$, so $\partial f_i / \partial n \leq 0$. Then $g_{ij} = - \iint_{S_i} (\partial f_i / \partial n) \, dB \geq 0$.

The same proof would apply for non-void C , but it must be shown that $f_i \geq 0$. To see that this generalization of the classical minimum principle is valid, let f_i be defined by Dirichlet's principle (minimize D). Then the boundary conditions (c) and (d) are such that $|f|$ is an "admissible function" if f is admissible. Since $D(|f|) = D(f)$, it may be assumed that $f = |f| \geq 0$.

In order to focus attention on essentials the treatment has been restricted here to distributed networks with single-valued potential functions. This would be the general case for thermal problems. However, multiple-valued potentials are of common occurrence in electric and magnetic problems. A simple case of multiple-valued potential will be discussed in Section 9.

A very interesting application of a distributed network has been made to the geometrical problem of dividing a square into smaller squares, no two of which shall be of the same size [12].

KIRCHHOFF's basic paper on lumped networks appeared in 1847. This was followed the next year by a paper on distributed networks [17, pp. 33-49]. The latter paper gives the minimum property of the Kelvin-Dirichlet integral (essentially relation (6) above). A detailed analysis of the distributed network is to be found in Chapters 7, 8, and 9 of MAXWELL's *Treatise* [8]. Chapter 3

treats an equivalent problem concerning coefficients of capacity and induction.

RAYLEIGH was the first to give methods for determining upper and lower bounds for conductance [18]. His work is based on two principles: RAYLEIGH's first principle is precisely inequality (10). "The (second) principle is that if the conductivity of any part of the conductor is increased that of the whole is increased, and if the conductivity of any part be diminished that of the whole is diminished, exception being made of certain very particular cases, where no alteration ensues." This is quoted from *The Theory of Sound* [20].

4. An upper bound network. It has just been shown that the resistances ρ_{ij} of a distributed network are related to certain Dirichlet integrals. The remainder of the paper concerns the approximation of more general Dirichlet integrals by use of certain lumped networks. Consequently these approximations to the Dirichlet integral imply approximate determinations of the quantities ρ_{ij} .

In this section attention is confined to the degenerate case when the region R can be treated as a two-dimensional region. A physical model of this case is furnished by the electric current flow in a conducting plate of constant thickness, the two sides of the plate being insulated. The boundary of the two-dimensional region R will be assumed to be made up of one or more closed polygons.

It is desired to obtain an upper bound for the Dirichlet integral

$$D(F) = \iint_R \left[\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 \right] dx dy$$

assuming conditions (a), (b), (c), and (d'). To this end let the region R be decomposed into triangles in such a manner that each terminal S_i is made up of one or more edges of triangles. Let U be a continuous function which is linear in each triangle and which satisfies the boundary condition (d'). Then, by relation (3-6), $D(U)$ is an upper bound for $D(F)$.

It is of interest to observe that the same procedure applies with the boundary condition (d') replaced by the more general condition

(d'') *On the boundary S_i the function F is a prescribed continuous piecewise linear function.*

To satisfy condition (d'') the triangulation would need to be taken sufficiently "fine."

To evaluate $D(U)$ it is expressed as a sum of integrals over the triangular cells of R , say $D(U) = \sum D_i$. A typical cell is shown in Figure 1. The vertices are denoted by 1, 2, and 3, and the corresponding angles are A , B , and C . Then

$$(1) \quad D_i = \iint_i (\nabla U)^2 dx dy = (\nabla U)^2 A_i$$

where A_i is the area of the triangle. The function U is determined by its values u_1 , u_2 , and u_3 at the vertices 1, 2, and 3. It is seen from (1) and from Lemma

FIGURE 1

1 that D_i is a potentially definite quadratic form. Thus

$$(2) \quad D_i = g_C(u_1 - u_2)^2 + g_A(u_2 - u_3)^2 + g_B(u_3 - u_1)^2.$$

To determine the coefficients g_A , g_B , and g_C , let $u_1 = u_2 = 0$. Equating (1) and (2) gives

$$\frac{1}{2} \left(\frac{u_3}{h} \right)^2 (h \cot A + h \cot B) h = g_A u_3^2 + g_C u_3^2.$$

Here h denotes the length of the altitude to the side (1, 2). Thus $g_A + g_B = \frac{1}{2} (\cot A) + \frac{1}{2} (\cot B)$, and by symmetry two other such equations hold to give

$$(3) \quad g_A = \frac{1}{2}(\cot A), \quad g_B = \frac{1}{2}(\cot B), \quad g_C = \frac{1}{2}(\cot C).$$

The expressions D_i for the various cells are added to yield an expression for $D(U)$. The term involving $(u_i - u_j)^2$ may occur in two contiguous triangles. The coefficients of these two terms are added to give a new coefficient g_{ij} . For example, in Figure 1 this coefficient is $g_{23} = \frac{1}{2} (\cot A + \cot E)$. Thus $D(U)$ is expressed as a quadratic form $Q(u)$ in the discrete values u of U at the vertices of the triangles. This quadratic form is potentially definite and may be regarded as the quadratic form of a network whose branches are the edges of the triangles, these edges being ascribed conductivity g_{ij} . This network is termed an *upper network* because, for any vector u which takes on the right boundary values on the vertices on the boundary pieces S_i ,

$$(4) \quad Q(u) \geq D(F).$$

Here u need only be a virtual state of the upper network; that is, the network equations (2-1) need not be satisfied. This network will be designated by the symbol N_1 .

It is of interest to evaluate the g_{ij} for networks with a periodic structure. If the cells are equilateral triangles, it is seen from (3) that $g_{ij} = 3^{-\frac{1}{2}}$ for interior edges and $g_{ij} = \frac{1}{2} 3^{-\frac{1}{2}}$ for boundary edges. From (3) it is seen that $g_{ij} = 0$ for the hypotenuse of a right triangle. This observation shows that an upper net-

work is given by division of R into squares. The interior edges of the squares have $g_{ij} = 1$ and boundary edges have $g_{ij} = \frac{1}{2}$.

The last network is, of course, the usual regular square lattice approximation. This network will be designated as N' .

5. A lower bound network. A method is now to be developed for obtaining a lower bound for the Dirichlet integral. This method is based on formula (3-9) and SYNGE [11].

Let R again denote the two-dimensional region which has been triangulated. Let W denote a vector field which is constant in each triangle and which has a continuous normal component across edges. It is readily seen that relations (3-8) and (3-9) apply even though the tangential component of W may be discontinuous across edges.

A constant vector field may be regarded as the gradient of a linear function V . Thus let $W = -\nabla V$ in the triangular region t , and hence

$$\iint_t |W|^2 dx dy = D_t(V) = -\int VW_n dB.$$

Here the last integral is over the boundary of the triangle; that is, dB will now denote arc length. Let a, b, c denote the edges of the triangle and let v_a be the potential at the midpoint of a . Let w_a be the total current into the edge, hence $w_a = -\int_a W_n dB$ where the integration is over the edge a . Likewise define w_b and w_c . Then

$$-\int VW_n dB = v_a w_a + v_b w_b + v_c w_c.$$

Clearly w_a, w_b , and w_c can be expressed as linear functions of v_a, v_b , and v_c which vanish if $v_a = v_b = v_c$. Thus

$$w_a = g_{ab}(v_a - v_b) + g_{ac}(v_a - v_c).$$

FIGURE 2

To determine the coefficients g_{ab} , g_{ac} , g_{bc} , let v_1 , v_2 , and v_3 denote the values of the function V on the vertices. It was shown in the previous section that

$$2D_i(V) = \cot A(v_2 - v_3)^2 + \cot B(v_3 - v_1)^2 + \cot C(v_1 - v_2)^2.$$

But if the side a joins vertices 2 and 3, then $v_a = \frac{1}{2}(v_2 + v_3)$ and $v_b = \frac{1}{2}(v_3 + v_1)$, so $2(v_a - v_b) = (v_2 - v_1)$, etc. Thus

$$D_i(V) = 2 \cot A(v_b - v_a)^2 + 2 \cot B(v_c - v_a)^2 + 2 \cot C(v_a - v_b)^2.$$

Thus, in terms of the notation of Figure 2,

$$(1) \quad g_{ab} = 2 \cot C, \quad g_{bc} = 2 \cot A, \quad g_{ca} = 2 \cot B.$$

These coefficients may be interpreted as the conductances of a triangular mesh network joining the midpoints of a , b , and c as shown in Figure 2. Thus the energy dissipation of the flow W in the triangle is equal to the energy dissipated by currents w_a , w_b , and w_c entering the cell network of Figure 2. Note that $w_a + w_b + w_c = 0$.

The network of Figure 2 is equivalent to the network of Figure 3 provided the following three relations hold for branches joining an inner point d of the triangle:

$$(2) \quad g_{ad} = 2 \tan A, \quad g_{bd} = 2 \tan B, \quad g_{cd} = 2 \tan C.$$

The equivalence may be checked by eliminating junction d by the star-mesh transformation.

Let the cell networks of the various triangles in R be connected at the common midpoints to form a network which may be designated as N_2 . It is now seen that there is a one-to-one correspondence between the flows $W(x, y, z)$ in R just considered and conservative flows w_{ij} in N_2 . By a "conservative flow" is meant one which satisfies Kirchhoff's first law; in other words, if i designates a junction not on the boundary of R , then $\sum_j w_{ij} = 0$. It is not assumed that Kirchhoff's second law holds; thus a potential may not exist. In this correspondence

FIGURE 3

$$(3) \quad \iint_R |W|^2 dR = \sum_{i < j} r_{ij} w_{ij}^2 = H(w).$$

Let F denote a function which is continuous in $R + B$ and harmonic in R . Let those junctions of N_2 which are on the boundary of R be denoted by $0, 1, \dots, m$. These boundary points are midpoints of edges of triangles on the boundary. If $\int_i \dots dB$ denotes integration over a boundary edge i , then the average value of F on the edge is

$$f_i = \frac{\int_i F dB}{\int_i dB}.$$

Since W_n is constant on an edge, clearly $\int_i FW_n dB = -w_i f_i$ and

$$(4) \quad \int_B FW_n dB = -\sum w_i f_i.$$

Substituting in (3-9) gives the following lower bound for the Dirichlet integral:

$$(5) \quad D(F) \geq \frac{(\sum w_i f_i)^2}{H(w)}.$$

Now let the flow w in the network N_2 result from a potential v such that $v_i = f_i$ on the boundary. Then we have $H(w) = Q(v) = \sum w_i v_i$, so

$$(6) \quad D(F) \geq Q(v).$$

We term N_2 a *lower network* because of the inequalities (5) and (6).

FIGURE 4

It is equally possible to build the lower network out of the cell networks as shown in Figure 2. However, it is necessary to suppose in relations (3) and (5) that the flow in each cell is governed by a potential. Of course, relation (6) is true without restriction, because then the flow in the whole network is governed by a potential.

An example of a lower network of type N_2 is shown by the dotted lines in Figure 4. The full lines denote the triangulation. As was seen above, the full lines may be regarded as the branches of an upper network of type N_1 . It is apparent that upper and lower networks over the same cells have *dual graphs* in the sense of graph theory [7].

If the region is triangulated with equilateral triangles, then the lower network is made up of regular hexagons. By relation (2) the conductance of branches connecting interior junctions is $\sqrt{3}$.

Suppose that in Figure 3 angle A is a right angle. Then it follows from (2) that $g_{ad} = \infty$. This means that junctions d and a are to be identified. If the region is triangulated into isosceles right triangles, then it was seen that the upper network N' is a regular square lattice; interior edges have $g = 1$ and boundary edges have $g = \frac{1}{2}$. The lower network is also a square lattice, say N'' , with edges having $g = 1$. The upper and lower networks N' and N'' again have dual graphs.

A simple example of triangulation by isosceles right triangles is shown in Figure 5. The dotted lines denote branches of the lower network N'' and the

FIGURE 5

full lines denote branches of the upper network N' . Suppose that the shading at a and b denotes perfectly conducting terminal surfaces. Then the upper network gives $\gamma_{ab} \leq 13/20 = 0.65$, and the lower network gives $\gamma_{ab} \geq 7/13 = 0.54$. This is about a ten per cent error for the conductance γ_{ab} . The calculations in this example are very simple and hardly require pencil and paper if carried out with the aid of ROSEN's star-mesh transformation, formula (2-12).

6. Interpolating with polynomials. The estimates just obtained for the Dirichlet integral have been obtained by using a linear interpolation function in the cells. In this section a more accurate interpolation is to be developed by employing harmonic polynomials in square cells.

It is supposed that the region R can be divided into square cells all of the same size. Figure 6 shows a typical square s . The vertices are denoted by 0, 1, 2, 3 at points with coordinates $(0, 0)$, $(h, 0)$, (h, h) , $(0, h)$ respectively where h is the length of an edge. It is desired to solve the Dirichlet problem for linear boundary conditions on the edges of s . This problem is solved by a harmonic polynomial of the form $U = a + bx + cy + dxy$. If u_0, u_1, u_2, u_3 are the values of u at the corners of the square, then it is clear that $a = u_0$, $b = (u_1 - u_0)/h$, $c = (u_3 - u_0)/h$, and $d = (u_3 + u_0 - u_1 - u_2)/h^2$. The Dirichlet integral over the square is readily found to be

$$D_s(U) = b^2 h^2 + c^2 h^2 + dch^3 + dbh^3 + \frac{2}{3} d^2 h^4.$$

Substituting the explicit values of b, c , and d leads to

$$6 D_s = (u_3 - u_2)^2 + 2(u_3 - u_1)^2 + (u_3 - u_0)^2 + (u_2 - u_1)^2 + 2(u_2 - u_0)^2 + (u_1 - u_0)^2.$$

FIGURE 6

This may be regarded as the quadratic form Q of a square shaped network; the diagonals having $g = \frac{1}{3}$ and the edges having $g = \frac{1}{6}$.

Let a function u be assigned arbitrarily at the corners of the various squares into which R is decomposed. By use of the harmonic polynomial just considered, u may be extended to be a continuous function inside the square cells. This interpolation is a linear function on the edge of a square; therefore it is uniquely defined on the edge common to two squares. By this method continuous functions can be defined to satisfy a given piecewise linear boundary condition on the boundary of R .

The equivalent networks for the square cells are joined at the corners. This gives a network such as shown in Figure 7 and is designated as N_3 . All branches of N_3 have $g = \frac{1}{3}$ except boundary edges, which have $g = \frac{1}{6}$. Thus N_3 is an upper bound network for the region R because $Q(u) = D(U)$.

A lower bound network with square cells is now obtained by a similar process. Instead of U we now employ the harmonic polynomial

$$V = ax + by + c(x^2 - y^2).$$

The components of current flow are determined by the gradient; thus

$$(1) \quad W_x = -a - 2cx, \quad W_y = -b + 2cy.$$

It is convenient to take the origin at the center of a square. The edges of a square are numbered 1, 2, 3, 4 as shown in Figure 8. By Green's theorem $D_*(V) = \int_* (V \partial V / \partial n) dB$. Since $\partial V / \partial n$ is constant on each edge, we may write

$$\int V \frac{\partial V}{\partial n} dB = w_1 u_1 + w_2 u_2 + w_3 u_3 + w_4 u_4.$$

Here u_1 denotes the average value of V on the first edge, and w_1 denotes $h \partial V / \partial n$ where h is the length of the edge. In terms of a , b , and c these quantities are found to be

FIGURE 7

$$\begin{aligned}
 (2) \quad w_1 &= -ah + ch^2, & u_1 &= -\frac{1}{2}ah + \frac{1}{6}ch^2, \\
 w_2 &= -bh - ch^2, & u_2 &= -\frac{1}{2}bh - \frac{1}{6}ch^2, \\
 w_3 &= ah + ch^2, & u_3 &= \frac{1}{2}ah + \frac{1}{6}ch^2, \\
 w_4 &= bh - ch^2, & u_4 &= \frac{1}{2}bh - \frac{1}{6}ch^2.
 \end{aligned}$$

From these relations it is seen that

$$2w_1 = 3(u_1 - u_2) - (u_1 - u_3) + 3(u_1 - u_4).$$

Similar equations for w_2 , *etc.*, follow by symmetry. These can be interpreted as network equations with $g_{12} = \frac{3}{2}$, $g_{13} = -\frac{1}{2}$, $g_{14} = \frac{3}{2}$, *etc.* This network is shown in Figure 8. The network is seen to satisfy $D(V) = Q(u)$.

If the various cell networks, as shown in Figure 8, are joined at the midpoints of the edges, there results a network N_4 . Consider an equilibrium state of N_4 . In each cell unique values of a , b , c are determined because the expressions (2) for w_1 , w_2 , and w_3 can be solved for a , b , and c . Then relations (1) define a flow W_x , W_y . It is clear that these flows have continuous normal components across edges. Thus a conservative flow is defined in R , and it is seen that $\iint_R (W_x^2 + W_y^2) dx dy = Q(u)$ for N_4 . It results that N_4 is a lower bound network. If the junction i of the network N_4 is on a boundary edge of R , u_i is taken to be the average value of the prescribed boundary values on this edge. This last statement follows from (3-9).

Networks of type N_3 and N_4 are based on subdivision of R into square cells. It was shown in Sections 4 and 5 that the networks of type N' and N'' can also be based on square cell subdivision. However, it is seen from Figure 5 that the cells adjacent to the boundary may not be square. It is of interest to compare these different types of networks as to degree of approximation. First consider an upper bound network. Let s denote a given square cell with linear boundary values on the edges. Clearly an N_3 cell network for s corresponds to the exact

FIGURE 8

solution of the Dirichlet problem. On the other hand, the N' cell network for s would not correspond to the exact solution unless the potential were a linear function in s . An N' network can be converted to an N_3 network by cell-by-cell replacement. A steady improvement results, so N_3 can only be an improvement of N' .

Similar considerations apply to the lower networks N'' and N_4 . It may thereby be shown that the replacement of N'' type cells by N_4 type cells results in an improved lower bound.

The branches crossing at the centers of the squares in Figure 7 and in Figure 8 are not joined at the center points. It results that the networks N_2 and N_4 do not have planar graphs [7].

7. Three-dimensional networks. Most of the concepts just developed for a two-dimensional situation can be extended to three dimensions. This extension will now be indicated, but some of the proofs are omitted because of the similarity to the two-dimensional case.

Of concern are regions R , which can be decomposed into a finite number of irregular tetrahedrons. A continuous function U is introduced which is linear in the tetrahedral cells. It is seen that the Dirichlet integral over a cell can be expressed as a quadratic form $Q(u_1, u_2, u_3, u_4)$ in the values of U at the four vertices, say u_1, u_2, u_3, u_4 , as shown in Figure 9. When this is carried out it is seen that Q is the quadratic form of a network obtained by ascribing the edges of the tetrahedron conductivities $g = \frac{1}{6}h \cot \theta$ where h denotes the length of the "opposite" edge and θ denotes the dihedral angle at the opposite edge. An upper network N_5 results by joining these cell networks at their vertices.

The network N_5 will furnish an upper bound to the Dirichlet problem provided boundary conditions are satisfied on the boundary surface B of R . The boundary conditions permitted are such that, over the tetrahedral faces which are in B , either a linear function is prescribed or $\partial U / \partial n = 0$.

FIGURE 9

Turning to the problem of an upper bound, let V be a linear function defined in a tetrahedral cell with faces numbered 1, 2, 3, 4. Let u_1, u_2, u_3, u_4 denote the value of V at the centroids of the faces. (This is equivalent to saying that u_1 is the average value of V on face 1, *etc.*) The value of the Dirichlet integral is found to be the same as the quadratic form $Q(u_1, u_2, u_3, u_4)$ of a network obtained by ascribing conductivities $g = \frac{3}{2}h \cot \theta$ to the lines connecting the centroids, as shown in Figure 10. Here h denotes the length of the edge which is common to the two faces joined, and θ is the dihedral angle formed by these faces. If these cell networks are joined at the centroids of common faces, a lower network N_6 results.

The network N_6 just defined furnishes a lower bound for the Dirichlet integral. The value of u_i , if i denotes a junction on the boundary B , is taken to be the average of the ascribed boundary value over the face.

Now a more refined polynomial interpolation is to be described. It is supposed that the region R can be divided into cubic cells of edge length h . Let coordinate axes be taken parallel to the edges of a cube and consider the harmonic polynomial

$$U = a + bx + cy + dxy + cz + fxz + gyz + Hxyz.$$

It is seen that U is uniquely defined by its value at the eight corners of the cubic cell. An equivalent network for the cell results by joining all corners by diagonal branches of conductivity $\frac{1}{2}h$. There are 12 short diagonals and 4 long diagonals; edges are omitted.

An upper network N_7 results when the cell networks are joined at the corners. This follows because the interpolation is continuous across faces of the cubic cells.

It is of interest to compare the network N_7 with the usual approximation which amounts to ascribing conductivity to the edges only. This network will be designated as N' as in the two-dimensional case. Thus N' is constructed of cubic cells with each edge having $g = \frac{1}{4}h$.

FIGURE 10

Let $Z = (\partial U / \partial z)^2$ and set

$$D_z = \int_0^h \int_0^h \int_0^h Z \, dx \, dy \, dz = h \int_0^h \int_0^h Z \, dx \, dy.$$

But Z is a convex function of both x and y , so

$$\begin{aligned} D_z &\leq \frac{1}{2}h^2 \int_0^h [Z(x, h) + Z(x, 0)] \, dx \\ &\leq \frac{1}{4}h^3 [Z(h, h) + Z(h, 0) + Z(0, h) + Z(0, 0)]. \end{aligned}$$

For Z we may substitute $[U(x, y, h) - U(x, y, 0)]^2 / h^2$ to obtain

$$\frac{4}{h} D_z \leq (u_4 - u_0)^2 + (u_5 - u_1)^2 + (u_6 - u_2)^2 + (u_7 - u_3)^2.$$

Here the subscripts denote the numbering of the corners of the cube as shown in Figure 11; thus $u_1 = U(h, 0, 0)$, etc. Now, by adding similar inequalities relating to $\partial U / \partial y$ and $\partial U / \partial x$, an inequality is obtained for $D(U)$ over the cube, say $D(U) \leq Q$. It is obvious that Q is the quadratic form of the network given by the twelve edges of the cube; each branch having $g = \frac{1}{4}h$. When the cell networks are connected, there results the network N' . Internal edges have $g = h$, and boundary edges have $g = \frac{1}{4}h, \frac{1}{2}h$, or $\frac{3}{4}h$. The inequality just proved shows that N' is an upper network. It also shows that N' does not give as good an estimate of D as the previous network N_7 .

A lower bound network for the cubical subdivision may be obtained from the function

$$V = ax + by + cz + Ax^2 + By^2 + Cz^2.$$

FIGURE 11

It is assumed that $A + B + C = 0$, so V is harmonic. It is convenient to take the origin of coordinates at the center of a cubical cell as shown in Figure 12. The faces of the cube are designated as 1, 2, 3, 4, 5, 6; the x axis connects faces 1 and 2, *etc.* The x component of current flow is $W_x = -a - 2Ax$, and it is seen that this flow is constant over face 1 and over face 2. Let u_1 denote the average value of V over face 1, *etc.* Clearly $u_1 = -\frac{1}{2}ax^2 + \frac{1}{6}Ah^2$. If w_1 denotes the total current entering the cube through face 1, then it is readily checked that

$$\frac{w_1}{h} = 3u_1 + u_2 - u_3 - u_4 - u_5 - u_6 .$$

Thus the equivalent network consists of branches connecting mid-points of faces. The branches connecting adjacent faces have $g = h$ and branches connecting opposite faces have $g = -h$.

A lower bound network N_s is obtained by connecting the cell networks together at the face centers. This is shown by an argument analogous to that given in the two-dimensional case. The potentials of the boundary junctions of N_s are taken to be the average of the assumed boundary values on the surface of the corresponding square.

A simpler lower bound network N'' may be derived from N_s . Referring to Figure 12, let the dotted lines denote a cell network of N'' . Thus a junction 0 has been added at the center and is connected to the faces by six branches of conductivity $2h$. (The analogy to the two-dimensional case is apparent.) By (2-1) we have

$$6u_0 = u_1 + u_2 + u_3 + u_4 + u_5 + u_6 .$$

Without loss of generality it may be assumed that $u_0 = 0$. Then the quadratic form of the cell network of N'' is

$$Q'' = 2h(u_1^2 + u_2^2 + u_3^2 + u_4^2 + u_5^2 + u_6^2) .$$

FIGURE 12

The quadratic form of the cell network of N_s is

$$Q_s = 4h(u_1^2 + u_1u_2 + u_2^2 + u_3^2 + u_3u_4 + u_4^2 + u_5^2 + u_5u_6 + u_6^2).$$

Thus

$$Q_s - Q'' = 2h(u_1 + u_2)^2 + 2h(u_3 + u_4)^2 + 2h(u_5 + u_6)^2.$$

Thus $Q_s \geq Q''$.

Now let $\tilde{Q}_s$ and $\tilde{Q}''$ be the quadratic forms for the complete networks N_s and N'' . Under the prescribed boundary conditions let u denote the solution of N_s . Then $\tilde{Q}_s(u) \geq \tilde{Q}''(u)$, because $Q_s \geq Q''$. Let v denote the solution of N'' , then $\tilde{Q}''(u) \geq \tilde{Q}''(v)$. Thus $\tilde{Q}_s(u) \geq \tilde{Q}''(v)$, and it follows that N'' is a lower network. It also follows that N_s gives a better lower bound.

It has now been shown that the cubic lattice network can be employed both as upper bound network N' and as lower bound network N'' . This duality relationship between N' and N'' can be visualized from Figure 12; the full lines denote N' and the dotted lines denote N'' .

Networks having a repeated pattern, such as the cubic lattice networks, can be brought under the theory of difference equations. Thereby various analytical devices are made available; see [5] and the references cited there.

The use of lattice networks to simulate continuous conductors is due to MAXWELL [8, §324]. MAXWELL divided the region up into cubic cells but chose as cell network an inscribed tetrahedron whose edges are face diagonals.

8. Transformations. Let $U(x, y)$ be a harmonic function in a plane region R . Under a conformal transformation, then, U goes into a harmonic function U' in the transformed region R' . As is well known, the Dirichlet integral is a conformal invariant. Thus $D(U)$ evaluated over R is equal to $D(U')$ evaluated over R' .

Now consider a distributed network defined by R together with an assignment of terminal and insulating parts of the boundary of R . A conformal transformation gives rise to a new distributed network with the same number of terminals. For given potentials of the terminals, the quadratic form Q is seen to be invariant because $Q = D$. This shows that the synthetic network is a conformal invariant. An upper network furnishes upper bounds to the terminal conductivities of the synthetic network, so an upper network is a conformal invariant. Likewise a lower network is a conformal invariant.

If the cells used to define upper and lower networks are subjected to a conformal transformation, it results that the straight line boundaries become curvilinear. Moreover, the interpolating functions no longer satisfy piecewise linear boundary conditions. In some cases it is of interest to obtain the explicit form of the cells. For example, let the s plane and the s' plane be conformally related by $s = \log s'$. If $s = x + iy$ and $s' = re^{i\theta}$, the following relations can be assumed: $x = \log r$ and $y = \theta$. Thus the lattice of square cells is transformed

into a polar coordinate lattice of rays and circles. The applicability of these notions is clear.

Now we turn to a three-dimensional problem with axial symmetry. Then the Laplace equation can be written as

$$\frac{\partial^2 U}{\partial z^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) = 0$$

where $r^2 = x^2 + y^2$. The Dirichlet problem is thereby reduced to a two-dimensional problem for a region R in the z, r plane. To obtain upper and lower networks the region R is divided into rectangular cells by lattice lines parallel to the z and r axes. To get an upper network use is made of the harmonic function

$$U = a + bz + c \log r + dz \log r.$$

This function is used in a rectangular cell to interpolate the potentials at the corners. To obtain a lower network use is made of the harmonic function

$$V = az + b \log r + c(r^2 - 2z^2).$$

This function has the property that $\partial V / \partial z$ does not depend on r and that $\partial V / \partial r$ does not depend on z . It is thereby seen that the gradient of V is suitable for interpolating a flow through the sides of the rectangular cell. The explicit evaluation for these upper and lower networks will be omitted.

The above transformations are useful for certain problems in which the boundary surfaces are not planes. Another method for treating such problems is by approximating the region by inscribed and circumscribed polyhedral regions. It is seen that for fixed terminals, the inscribed region would have a smaller conductivity and the circumscribed region would have a greater conductivity.

It has been assumed that the conductivity is constant; however, there is one type of variable conductivity to which the methods apply almost without change. This case concerns piecewise constant conductivity. For example, the region R might be formed of a material of conductivity c_1 embedded in a material of conductivity c_2 . To obtain upper and lower networks it is assumed that the division into cells can be carried out so that in a cell the conductivity is constant, say c_1 . The procedure is the same as before, except that the previous values for the conductivities of the cell network are multiplied by c_1 . In this way composite conductors (of common occurrence in heat transfer and in magnetism) can be analyzed by use of upper and lower networks.

9. Conjugate conductors. In this section attention is confined to two-dimensional conductors because properties of conjugate harmonic functions are to be employed. Consider first a simply connected region R with boundary arcs a and b as terminals. The complementary part of the boundary consists of two arcs c and d ; these arcs are to be insulated. Such a conductor, say C , is shown in Figure 13. We define the *conjugate conductor*, say C^* , to have a region of the same

FIGURE 13

shape as R , but now arcs c and d are to be the terminals and arcs a and b are to be insulated. (In order to preserve perfect symmetry between conductors C and C^* it may be supposed that the arcs a , b , c , and d are open at their ends.)

The dotted lines sketched in R denote streamlines of a flow of current in C . The solid lines denote the equipotentials of the harmonic function $U(x, y)$ corresponding to this flow of current. Of course, the streamlines and the equipotentials form an orthogonal system of curves. No current flows through the arcs c or d , so these arcs may be considered to be streamlines.

Let $V(x, y)$ be a harmonic function conjugate to $U(x, y)$. Then the equipotentials of V are the streamlines for U , and the streamlines for V are the equipotentials of U . In particular, c and d are equipotentials of V and a and b are streamlines for V . Thus V is a harmonic function corresponding to a flow of current in the conjugate conductor C^* . These considerations lead to the following result of RAYLEIGH [19].

Theorem A. *Conjugate conductors have reciprocal resistances.*

To complete the proof let ρ denote the resistance of C . Then we can define ρ as

$$(1) \quad \rho = \frac{\int_A^B \left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy \right)}{\int_c^d \left(-\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy \right)}.$$

The line integral in the numerator is along a curve connecting a point A on arc a with a point B on arc b . Thus the numerator gives the potential difference. The line integral in the denominator is along a curve connecting a point C on

c with a point D on d . The denominator gives the total current flow. The function U is now eliminated by the use of the Cauchy-Riemann equations. Thus the right side of (1) becomes

$$(2) \quad \frac{\int_A^B \left(\frac{\partial V}{\partial y} dx - \frac{\partial V}{\partial x} dy \right)}{\int_c^D \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \right)} = \frac{1}{\rho^*}.$$

The right side of (2) follows by definition of the resistance ρ^* of the conjugate conductor C^* and the proof is completed.

R. C. BUCK pointed out to the writer that Theorem A is a basic result in the theory of "extremal length." The extremal length of a region is a geometrically defined quantity but it turns out that it is identical with the resistance ρ . The theory of extremal length was introduced by AHLFORS and GRÖTZSCH. It has been further developed in unpublished work of AHLFORS and BEURLING.

We say that a conductor C is *self-conjugate* if the region R is a reflection of itself in the straight line connecting the initial point of terminal a with the initial point of terminal b . A self-conjugate conductor is shown in Figure 14. Thus the arc c is a reflection of the arc b , and the arc d is a reflection of the arc a . By symmetry it is obvious that $\rho = \rho^*$. This proves the following.

Corollary. *A self-conjugate conductor has unit resistance.*

Since reflection in a straight line is a special case of inversion in a circle it is of interest to know if the corollary holds for this more general situation. By a

FIGURE 14

special conformal transformation, inversion in a circle can be reduced to reflection in a straight line. On the other hand, it was seen in Section 8 that resistance is a conformal invariant, so the corollary remains true for inversion in a circle.

The conductor shown in Figure 13 may be described as a two-terminal distributed network over a simply-connected region. It is of interest to indicate the extension of the above concepts to general distributed networks. For example, Figure 15 shows a four-terminal distributed network N over a simply-connected region R . The terminals are the arcs a, b, c, d . The arcs e, f, g, h are insulated. Let ρ be the resistance between terminals a and c when terminals b and d are free. Again let U be the harmonic function corresponding to a flow in such a case and let V be a harmonic function conjugate to U . Since no net current can leave terminal b

$$(3) \quad 0 = \int_E^F \left(-\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy \right).$$

This line integral is taken along a curve connecting a point E of arc e to a point F of arc f . Transforming (3) by the Cauchy-Riemann equations gives

$$(4) \quad 0 = \int_E^F \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy \right).$$

FIGURE 15

In other words, V has the same constant value on arcs e and f . In a similar manner, it follows that V has another constant value on arcs g and h .

We define the *conjugate distributed network* N^* to be over the same region R but with e, f, g, h as terminals and a, b, c, d as insulated arcs. Let terminals e and f of N^* be connected together by a wire and let terminals g and h be connected together by a wire. Let ρ^* be the resistance which would be measured between these two sets of "shorted" terminals. It then follows by the same proof as given before that $\rho\rho^* = 1$. In other words, Theorem A still holds if properly interpreted.

Networks over multiply-connected regions are of especial interest. The simplest multiply-connected case is the ring-shaped conductor C as shown in Figure 16. Here the terminals are the inner boundary a and the outer boundary b . The conjugate conductor C^* is defined to be the closed circuit around the ring. Conductor C^* has no terminals.

Let U be a harmonic function corresponding to a flow in C . The conjugate function V is multiple-valued, but the problem can be reduced to a single-valued case by use of a branch cut along a streamline of U . Such a branch cut is shown as a dotted line in Figure 16. The resistance ρ^* of C^* can be defined by use of terminals c and d on each side of the branch cut. The proof that $\rho\rho^* = 1$ is then the same as given for Theorem A.

The introduction of the branch cut does not change the value of ρ . Moreover, the introduction of terminals c and d yields the same value of ρ^* as would be found by Faraday magnetic induction of currents in the ring.

The problem just treated has an application to the propagation of electric waves along coaxial-type cables. Consider a cable whose cross section is not necessarily two concentric circles but, more generally, is of the type such as shown in Figure 16. Now a denotes the inner conducting tube and b denotes the outer conducting tube. These tubes are assumed to be perfect conductors.

FIGURE 16

From fundamental definitions it is found that the capacitance per unit length of the cable is given by

$$(5) \quad C_1 = \frac{1}{4\pi\rho} \quad \text{absolute electrostatic units.}$$

Also from fundamental definitions it is found that the inductance per unit length of the cable is

$$(6) \quad L_1 = \frac{4\pi}{\rho^*} \quad \text{absolute electromagnetic units.}$$

These quantities enter into the equation for electric waves along the cable, but only as the product $L_1 C_1$. From Theorem A and (5) and (6) we have

$$(7) \quad L_1 C_1 = 1.$$

From this it follows directly that *the velocity of waves along the cable is independent of the shape of the cross section*. Of course, because of the relation between electrostatic and electromagnetic units it results that the velocity is equal to the velocity of light. (It is worth noting that the above considerations do not employ Maxwell's equations and it is conceivable that the stated theorem might have been deduced before MAXWELL had introduced the displacement current. See KIRCHHOFF [17, pp. 131–168].)

Theorem A leads to the following direct method for bounding the conductance γ of a conductor C . The upper bound can be obtained from Dirichlet's principle. Thus an arbitrary function U , constant on the terminals of C , gives an upper bound when substituted in inequality (3–7). Likewise, another arbitrary function V , constant on the terminals of the conjugate conductor C^* , furnishes an upper bound to γ^* . Since $\gamma\gamma^* = 1$, the upper bound to γ^* automatically gives a lower bound to γ . Thus

$$(8) \quad \frac{(V_c - V_a)^2}{D(V)} \leq \gamma \leq \frac{D(U)}{(U_a - U_b)^2}.$$

Conjugate harmonic functions in three space can be defined by use of equations of Cauchy-Riemann type [15]. However, the counter-example of ELKINS [16] shows that *it is impossible, in general, to have a harmonic function whose flow lines are orthogonal to the flow lines of a given harmonic function*. Thus no direct generalization of conjugate conductors is possible in three dimensions.

The theory of current flow on curved conducting surfaces was given by KIRCHHOFF [17, 56–66]. Further references are given by WEBSTER [21, pp. 196–202, 349–350]. In the light of this theory it can be seen that Theorem A and its corollary also apply to such curved shells. However, these questions will not be treated here.

10. Conjugate discrete harmonic functions. In this section it is desired to relate upper and lower networks with the properties of conjugate conductors.

First it is clear that an upper (lower) network for a plane conductor C is also an upper (lower) network for the conjugate conductor C^* . It is now seen that an upper network N furnishes a direct method for obtaining both upper and lower bounds for the conductance γ of a polygonal conductor C . Let $Q(u)$ be the quadratic form of the network N which gives the heat dissipated as a function of junction potentials u . Then

$$(1) \quad \frac{(v_c - v_d)^2}{Q(v)} \leq \gamma \leq \frac{Q(u)}{(u_a - u_b)^2}.$$

Here u and v are arbitrary assignments of junction potentials except that u is constant on a and b , say u_a and u_b , and v is constant on c and d . Manifestly, relation (1) gives a simple practical method for bounding conductance.

As an example, consider the conductor shown in Figure 5. By solving the upper network between the conjugate terminals c and d it is readily found that the conductance has the value $13/7$. Thus $\gamma^* \leq 13/7$ and consequently $\gamma \geq 7/13$. It is remarkable that this is exactly the same bound as was obtained before by use of the lower network shown in Figure 5.

The equivalence of the two bounding methods is not a coincidence and we are led to assert the following analogue of Theorem A.

Theorem B. *The resistance of a network N_1 for a conductor is the reciprocal of the resistance of the dual network N_2 for the conjugate conductor.*

Proof. We shall confine attention to the case that C is a two-terminal conductor over a simply-connected region R . The network N_1 is the upper network based on division of R into triangular cells. Attention will be confined to the case of acute-angled triangles. The network N_2 is the lower network based on the division of R into triangular cells. N_1 and N_2 are said to be *dual* if they are based on the same cell division. Dual networks N_1 and N_2 are shown in Figure 4. It follows from (4-3) and (5-2) that *crossing branches of dual networks have reciprocal values of conductance*.

Consider a flow in the network N_1 satisfying the boundary conditions of C . The junction potentials defining this flow are denoted by u . For the sake of analogy we term u a *discrete harmonic function*.

We define a function v on the network N_2 by the relation

$$(2) \quad \delta v = -I.$$

Here I denotes the current through a branch B of N_1 . Then δv is the jump in v across the branch B' of N_2 which crosses B . Any direction assigned to B induces a direction in B' by the following rule: The relative orientation of B and B' is taken to be the same as of the x and y axes respectively. This rule determines the sign in (2).

Let $\sum_2$ denote a sum directed around a simple closed chain of branches of N_2 . Then by (2) we have $\sum_2 \delta v = -\sum_2 I$. Clearly $\sum_2 I$ is the total current flowing out of the portion of the network N_1 bounded by the chain. This vanishes by

the conservation of electricity (Kirchhoff's first law). Consequently

$$(3) \quad \sum_2 \delta v = 0.$$

This holds for any chain so it follows that v is a single-valued function on N_2 . Relation (3) may be regarded as a statement of Kirchhoff's second law for the network N_2 .

Multiply relation (2) by r , the resistance of branch B . This gives

$$(4) \quad J = -\delta u$$

where $J = -r\delta v$. Since r is also the conductance of branch B' we say that J is the current through B' . Let $\sum_1$ denote a sum directed around a simple closed chain of branches of N_1 . Then, by (4), $\sum_1 J = -\sum_1 \delta u$. By Kirchhoff's second law for the network N_1 we have $\sum_1 \delta u = 0$. This gives

$$(5) \quad \sum_1 J = 0.$$

This is Kirchhoff's first law for the network N_2 . Relations (3) and (5) show that v is a discrete harmonic function for the network N_2 . We say that u and v are *conjugate discrete harmonic functions*.

It follows from (2) that v is constant on the boundary segments c and d . It follows from (4) that $J = 0$ on the boundary segments a and b . Thus v satisfies the boundary conditions for the conjugate conductor C^* .

Let $\sum_A^B$ denote a sum directed along a simple chain of branches of N_1 connecting a and b . Let $\sum_c^d$ denote a sum directed along a simple chain of branches of N_2 connecting c and d . Then the resistance ρ_1 of the network N_1 , under the assumed boundary conditions, may be defined as

$$(6) \quad \rho_1 = \frac{\sum_A^B \delta u}{\sum_c^d - I}.$$

Substituting relations (2) and (4) in the right side of (6) gives

$$(7) \quad \frac{\sum_A^B - J}{\sum_c^d \delta v} = \frac{1}{\rho_2}.$$

Here it is seen that ρ_2 is the resistance of N_2 and the proof is completed. (A proof of different type can be based on explicit resistance formulæ given by the writer in reference [13]; see formulæ (2a) and (2b) of that paper.)

It is apparent from the proof just given that Theorem B holds for a much more general class of dual networks. Thus the cells of N_1 do not have to be of the same type. Moreover, N_1 and N_2 do not have to be upper and lower networks.

Returning to the original problem it is seen that Theorem B applies to dual

square cell networks such as shown in Figure 5. Conjugate discrete harmonic functions on square cell networks have already been introduced in connection with the theory of "discrete analytic functions." This theory has been developed by RUFUS ISAACS, JACQUELINE FERRAND (LELONG), and the writer [14]. However this previous work does not touch on the concepts involved in Theorem B.

The remarks in the last paragraph raise the question of discrete analytic functions defined on irregular networks such as shown in Figure 4. To get at this question it is necessary to make precise the geometry of the diagram for the network N_2 . We suppose that the dotted lines are the perpendicular bisectors for the triangles. Thus junctions of N_2 are located at the circumcenters of the triangles. Let δx be the length of a branch of N_1 in the network diagram and let δy be the length of the perpendicular crossing branch of N_2 . Simple geometry shows that

$$(8) \quad \frac{\delta y}{\delta x} = \frac{1}{2}(\cot A + \cot E)$$

where A and E are the angles opposite the branch of length δx (see Figure 1). On the other hand, by relation (4-3) the right side of (8) is the conductance of this branch of N_1 . Thus (8) and (2) give

$$(9) \quad \frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}.$$

This may be regarded as the discrete analogue of the Cauchy-Riemann equations.

Let u and v be one pair of conjugate functions and let u' and v' be another pair. Let the function f have the value $u + iu'$ on N_1 and the value $-v' + iv$ on N_2 . Then (7) becomes

$$(10) \quad \frac{\delta f}{\delta x} = \frac{\delta f}{i \delta y}.$$

Of course, the left side of (10) refers to N_1 and the right side to N_2 . Any function f which satisfies (10) for all branches is termed a *discrete analytic function*.

To carry the theory further it is desirable to combine the junctions of N_1 and N_2 in a single network diagram which we term a "lattice." The branches of the lattice are the lines obtained by joining the vertices of each triangle with its circumcenter. For example, the networks of Figure 4 give the lattice shown in Figure 17. The lattice is regarded as being in the complex z -plane and it is then seen that the following analogue of Cauchy's theorem holds:

$$(11) \quad \int f(z) dz = 0.$$

This contour integral is along any closed path in the lattice. The value of $f(z)$ at intermediate points of the lattice branches is determined by linear interpolation. The proof of the Cauchy theorem (11) is straightforward application of (10) and will be omitted.

FIGURE 17

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