*A Note on “The Method of Characteristics for a Perfect Compressible Fluid in General Relativity and Non-Steady Newtonian Mechanics”**

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1. Introduction. In our previous paper [1], we used the terms (1) “order of a quantity” and (2) “order of a quantity as compared to another quantity”. Further, we obtained the characteristic system for a compressible fluid in non-steady Newtonian mechanics from the characteristic system of a compressible fluid in relativity theory under the assumption that a/c is of order zero, where a is the local sound speed and c is the speed of light [1]. In this paper, we shall formulate precise definitions of the terms “order of a quantity” and “order of a quantity as compared to another quantity”. Also we shall show that the characteristic system in the non-steady Newtonian case can be obtained *without use of the assumption that a/c is of order zero*. All of the previously derived equations for the Newtonian case remain valid.

2. Definitions and the Basic Order Assumption. We define the term “order of the quantity Q ” by use of the *maximum range* of values through which the *absolute value of the quantity may vary*. Two types of quantities will be considered: Newtonian quantities; relativistic quantities. *If the absolute value of Q is such that*

$$(2.1) \qquad |Q| \leq 10^k$$

then Q will be said to be of order one. We assume that k is chosen so that 10^k is the upper bound of the magnitude of the velocity in Newtonian mechanics. In the foot-second system, $k = 5$ would appear to suffice. For quantities in relativity theory, it is desirable to compare magnitudes with powers of the

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speed of light c . If the absolute value of Q is such that

$$(2.2) \quad |Q| \leq \left(\frac{10^k}{c}\right)^n \quad \text{or} \quad |Q| \leq (10^k c)^n, \quad n = 1, 2, 3, \dots,$$

then Q will be said to be of order $1/c^n$ or c^n , respectively. When Q is of order $1/c^n$ and $n \geq 2$, then Q will be said to be of order zero. The following relations exist for these orders:

$$(2.3) \quad \frac{1}{c^m} < \frac{1}{c^n} < 1 < c^n < c^m, \quad m > n.$$

That is, the range of order one is larger than that of order $1/c^n$, etc.

The statement " Q is of order one (or $1/c^n$ or c^n) as compared to P " means that " Q/P is of order one (or $1/c^n$ or c^n)". This implies that $0 < |P|$. In the remainder of the theory, the last type of order relation will be used in almost all of our work. In fact, we shall consider quantities Q/P or Q which are at most of order one or which are of order zero. We note that (1) if $(Q/P)^2$ is at most of order one, then Q/P is at most of order one and (2) if Q/P is at most of order one, then $Q/c^2 P$ is of order zero.

As in our previous paper, we make a *basic order assumption*: In any sum or product involving only terms which are at most of order one, a term of order zero may be replaced by zero. This assumption differs slightly from our previous assumption [1] that if P, Q are independent of c , then $P/c^2, Q/c^3$, etc. are of order zero.

3. The Metric Coefficients in the Newtonian Mechanics of a Compressible Fluid. In this section, we shall derive the metric coefficients of Section 7 of the previous paper [1] *without assuming that a/c is of order zero*. The notation will be that of [1] and the starred equations will refer to [1].

In a coordinate system of special relativity, $(\partial\phi/c\partial t, \partial\phi/\partial x^\alpha)$ determine the components of a vector field orthogonal to the relativistic characteristic surfaces, which satisfy (2.18)*. To within terms of order zero, this field determines the Newtonian field $(\partial\phi/\partial t, \partial\phi/\partial x^\alpha)$ which is a solution of (7.18)*. Since the space-like vectors of the normal cone have relativistic magnitude less than zero, it follows that the magnitude of the space components of this field, $[g^{\alpha\beta} (\partial\phi/\partial x^\alpha) (\partial\phi/\partial x^\beta)]$, is positive. *The basic assumption of this paper, which replaces the condition on a/c , is that two vectors of the normal cone can be chosen at each point so that $bN_0, b'N_0$ are of order one, where*

$$(3.1) \quad N_0 = \frac{\frac{\partial\phi}{\partial t}}{\left(g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta}\right)^{\frac{1}{2}}}, \quad 'N_0 = \frac{\frac{\partial'\phi}{\partial t}}{\left(g'^{\alpha\beta} \frac{\partial'\phi}{\partial x^\alpha} \frac{\partial'\phi}{\partial x^\beta}\right)^{\frac{1}{2}}},$$

$$b = \frac{9}{\sin \theta}, \quad \cos \theta = N'_\alpha N^\alpha$$

We note that the unit normal vectors in relativity and Newtonian mechanics are, respectively,

$$(3.2) \quad n_i = \frac{\frac{\partial \phi}{\partial x^i}}{\left[-\frac{1}{c^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + {}'g^{\alpha\beta} \frac{\partial \phi}{\partial x^\alpha} \frac{\partial \phi}{\partial x^\beta} \right]^{\frac{1}{2}}}, \quad N_\alpha = \frac{\frac{\partial \phi}{\partial x^\alpha}}{\left({}'g^{\gamma\beta} \frac{\partial \phi}{\partial x^\gamma} \frac{\partial \phi}{\partial x^\beta} \right)^{\frac{1}{2}}}.$$

If we divide the numerator and denominator of n_i by $[{}'g^{\alpha\beta} (\partial\phi/\partial x^\alpha) (\partial\phi/\partial x^\beta)]^{\frac{1}{2}}$ and use the fact that N_0 is of order one, we find, as in our previous paper (see (7.19)*, (7.21)*), that

$$(3.3) \quad n_i \rightarrow N_\alpha, \quad n^i \rightarrow -N^\alpha, \quad \alpha = j.$$

In (7.4)*, we obtained the relation between λ^i and v^α . Thus, by (3.7)*, the Newtonian analogue of (3.6)* is

$$(3.4) \quad \frac{v^\alpha}{c} = T^\alpha - \frac{a}{c} N^\alpha,$$

where T^α is the Newtonian analogue of the bicharacteristic vector. It is well known that (see (7.24)*)

$$(3.5) \quad v^\alpha = \frac{dx^\alpha}{dt} - a N^\alpha.$$

Comparing (3.4) and (3.5), we find

$$(3.6) \quad T^\alpha = \frac{1}{c} \frac{dx^\alpha}{dt}.$$

Since the relativistic bicharacteristic vector $t^i = dx^i/ds$ is equal to T^α to within terms of order zero, it follows from (3.6) that along the time-like vector t^i

$$(3.7) \quad \frac{dx^i}{ds} = \frac{dx^i}{[c^2 dt^2 - {}'g_{\alpha\beta} dx^\alpha dx^\beta]^{\frac{1}{2}}} \rightarrow \frac{dx^\mu}{cdt}, \quad \mu = j.$$

From (3.7), we see that the magnitude of dx^α/dt is at most of order one and hence

$$(3.8) \quad \frac{1}{c^2} \left({}'g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} \right) \rightarrow 0.$$

Now we consider the relativistic vector fields along the bicharacteristics $(c\partial t/\partial u, \partial x^\alpha/\partial u)$, $(c\partial t/\partial v, \partial x^\alpha/\partial v)$ and the vector field $(c\partial t/\partial z, \partial x^\alpha/\partial z)$ which is orthogonal to the bicharacteristics. Since, for time-like vectors, $\partial t/\partial u$, $\partial t/\partial v$ do not vanish, by proper choice of parameters u, v the quantities $\partial t/\partial u$, $\partial t/\partial v$ can be assumed to be larger than one. From (3.8), we see that

$$(3.9) \quad Q^2 = {}'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial u} \frac{\partial x^\beta}{\partial u}$$

is at most of order one compared with

$$(3.10) \quad P^2 = 10^k \left(\frac{\partial t}{\partial u} \right)^2.$$

In particular, $(\partial x^\alpha / \partial u)^2$, for any α , is at most of order one compared to P^2 . Hence $\partial x^\alpha / \partial u$ is at most of order one compared to $\partial t / \partial u$, and $Q^2 / c^2 P^2$, $(\partial x^\alpha / \partial u) / c^2 (\partial t / \partial u)$ are of order zero. Similar results are valid for $\partial x^\alpha / \partial v$, $\partial t / \partial v$. Thus, dividing U^2 , V^2 , M by $(c \partial t / \partial u)^2$, $(c \partial t / \partial v)^2$, $c^2 (\partial t / \partial u) (\partial t / \partial v)$ respectively, the metric coefficients of (6.5)* furnish

$$(3.11) \quad \frac{U^2}{c^2} \rightarrow \left(\frac{\partial t}{\partial u} \right)^2, \quad \frac{V^2}{c^2} \rightarrow \left(\frac{\partial t}{\partial v} \right)^2, \quad \frac{M}{c^2} \rightarrow \frac{\partial t}{\partial u} \frac{\partial t}{\partial v}.$$

These are three of the desired relations (7.25)*.

The space-like vector field $(c \partial t / \partial z, \partial x^\alpha / \partial z)$ is such that

$$(3.12) \quad R^2 = 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial z} \frac{\partial x^\beta}{\partial z}$$

does not vanish. Hence, by a proper choice of parameter z , we can assume that $1 \leq R^2$. Now we divide the orthogonality relation

$$(3.13) \quad c^2 \frac{\partial t}{\partial z} \frac{\partial t}{\partial u} - \sum_{\alpha=1}^3 \frac{\partial x^\alpha}{\partial z} \frac{\partial x^\alpha}{\partial u} = 0$$

by $3R(\partial t / \partial u)$ and note that, for all α , $(\partial x^\alpha / \partial u) / (\partial t / \partial u)$ is at most of order one. Since

$$(3.14) \quad \left| \frac{1}{R} \frac{\partial x^\alpha}{\partial z} \right| \leq 1$$

for all α , it follows from (3.13) that $(c^2 \partial t / \partial z) / 3R$ is at most of order one. Hence $(\partial t / \partial z) / R$ is of order zero, and, dividing Z^2 of (6.5)* by R^2 and using our last result, we obtain

$$(3.15) \quad \frac{Z^2}{R^2} \rightarrow 1.$$

If we introduce the parameter $r = cz$, we obtain a relation of the form (3.11)

$$(3.16) \quad \frac{Z^2}{c^2} \rightarrow 'g_{\alpha\beta} \frac{\partial x^\alpha}{\partial r} \frac{\partial x^\beta}{\partial r}.$$

This is another of the desired relations (7.25)*.

The remaining equations of (7.25)* are of the form

$$(3.17) \quad \frac{W^2}{c^2} \rightarrow \left(\frac{\partial t}{\partial w} \right)^2, \quad \frac{N}{c^2} \rightarrow \frac{\partial t}{\partial u} \frac{\partial t}{\partial w}.$$

To verify these relations, we note that the components of the vector p^i are $(c \partial t / \partial w, \partial x^\alpha / \partial w)$ and p^i is orthogonal to $'p^j$, n , $'n^i$. Hence, if $\phi = \text{constant}$,

$\phi = \text{constant}$ denote the characteristic manifolds orthogonal to n^i , $'n^i$ respectively, then

$$(3.18) \quad c^2 \frac{\partial t}{\partial z} \frac{\partial t}{\partial w} - \sum_{\alpha=1}^3 \frac{\partial x^\alpha}{\partial z} \frac{\partial x^\alpha}{\partial w} = 0,$$

$$(3.19) \quad \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial w} + \sum_{\alpha=1}^3 \frac{\partial \phi}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial w} = 0,$$

$$(3.20) \quad \frac{\partial' \phi}{\partial t} \frac{\partial t}{\partial w} + \sum_{\alpha=1}^3 \frac{\partial' \phi}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial w} = 0.$$

If we divide the equation (3.18) by $3R(\partial t/\partial w)$, where R is given by (3.12), we obtain

$$(3.21) \quad \sum_{\alpha=1}^3 \frac{1}{3R} \frac{\partial x^\alpha}{\partial z} \left(\frac{\partial x^\alpha}{\partial w} / \frac{\partial t}{\partial w} \right) = \frac{c^2}{3R} \frac{\partial t}{\partial z}.$$

Again, if we introduce the unit vector n_i of (3.2) and the similar vector $'n_i$, the equations (3.19), (3.20) become

$$(3.22) \quad \sum_{\alpha=1}^3 n_\alpha \left(\frac{\partial x^\alpha}{\partial w} / \frac{\partial t}{\partial w} \right) = -n_0,$$

$$(3.23) \quad \sum_{\alpha=1}^3 'n_\alpha \left(\frac{\partial x^\alpha}{\partial w} / \frac{\partial t}{\partial w} \right) = -'n_0.$$

By use of (3.1), (3.3), and the fact that

$$n_0 \rightarrow N_0, \quad 'n_0 \rightarrow 'N_0,$$

we find that (3.22), (3.23) become—to within terms of order zero—

$$(3.24) \quad \sum_{\alpha=1}^3 N_\alpha \left(\frac{\partial x^\alpha}{\partial w} / \frac{\partial t}{\partial w} \right) = -N_0,$$

$$(3.25) \quad \sum_{\alpha=1}^3 'N_\alpha \left(\frac{\partial x^\alpha}{\partial w} / \frac{\partial t}{\partial w} \right) = -'N_0.$$

The terms N_0 , $'N_0$, $c^2 (\partial t/\partial z)/3R$ are at most of order one (see (3.1) and the discussion following (3.14)). Further, by use of (3.12), (3.2), we see that $(\partial x^\alpha/\partial z)/R$, N_α , $'N_\alpha$ are unit vectors. Since the vectors $'n_i$, n_i are orthogonal to $'p_i$, we see that

$$(3.26) \quad \frac{\partial' \phi}{\partial t} \frac{\partial t}{\partial z} + \sum_{\alpha=1}^3 \frac{\partial' \phi}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial z} = 0,$$

$$(3.27) \quad \frac{\partial \phi}{\partial t} \frac{\partial t}{\partial z} + \sum_{\alpha=1}^3 \frac{\partial \phi}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial z} = 0.$$

If we divide (3.26) by $R[g^{\alpha\beta} (\partial' \phi/\partial x^\alpha) (\partial' \phi/\partial x^\beta)]^{\frac{1}{2}}$ and (3.27) by a similar expression in ϕ , and note that $(\partial t/\partial z)/R$ is of order zero and N_0 , $'N_0$ are of order

one, then from (3.26), (3.27) we see that N_α , $'N_\alpha$ are orthogonal to $\partial x^\alpha/\partial z$. From geometry, we see that the absolute value of the determinant of (3.21), (3.24), (3.25) is $\frac{1}{3} \sin \theta$ (the volume of the parallelopiped formed by N_α , $'N_\alpha$, and a vector of magnitude one-third which is orthogonal to both of them). Further, the minors of the above determinant are equal in absolute value to the projections on the coordinate planes of the areas of the faces of the parallelopiped and hence their values are less than one. Hence the cofactors divided by the determinant have absolute values less than or equal to $3/\sin \theta$. Hence the solutions

$$\frac{3}{e} \frac{\partial x^\alpha}{\partial w} \bigg/ \frac{\partial t}{\partial w}, \quad e = 11,$$

of the system (3.21), (3.24), (3.25) are *at most of order one*. By dividing W^2 , N of (6.5)* by $(ce\partial t/\partial w)^2$, $c^2 e^2 (\partial t/\partial w)(\partial t/\partial u)$, respectively, and using our last result, the formulas (3.17) follow.

Thus we have obtained the equations (7.25)* by use of the single assumption that bN_0 , $b'N_0$ are of order one (see 3.1). This implies some restriction on the choice of the two families of characteristic surfaces. However, we have not used the assumption of our previous paper that a/c is of order zero, which restricts the magnitude of a .

REFERENCE

- [1] COBURN, N., The Method of Characteristics for a Perfect Compressible Fluid in General Relativity and Non-steady Newtonian Mechanics, *Journal of Mathematics and Mechanics*, **7** (1958), pp. 449-482 (see Section 7).

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