*On the Integrability Conditions in the Problem of Motion in General Relativity**

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Abstract. The problem is investigated of deriving equations of motion from the field equations of general relativity up to an arbitrarily high approximation. The recent work on this problem by INFELD & PLEBANSKI (1956), which presents a modified version of previous methods of attack, is applied to EINSTEIN's field equations of empty space using the EINSTEIN, INFELD & HOFFMANN surface integral technique. A detailed investigation is made of boundary and coordinate conditions, and it is deduced that the EINSTEIN, INFELD & HOFFMANN approximation method does not generally admit (approximate) homogeneous Dirichlet boundary conditions. As a consequence of this global limitation it is difficult at present to prove, by using this approximation method, that the EINSTEIN, INFELD & HOFFMANN equations of motion are unique.

1. Introduction. In the early formulation of the general theory of deriving equations of motion from the field equations a fundamental difficulty was met with in the use of the approximation method. A contradiction appeared between the approximate field equations to be solved and the surface integral conditions following from the structure of the field equations.

The basic idea behind the methods introduced by EINSTEIN, INFELD & HOFFMANN (1938)¹ and EINSTEIN & INFELD (1940, 1949)^{2,3} to overcome this difficulty was to generalize the original EINSTEIN equations so that the motion of the singularities became arbitrary; that is, arbitrary values were attributed to the $\xi^m(x^0)$ (ξ^m the position coordinates of the s^{th} singularity). The surface integral conditions were fulfilled identically, and the procedure for solving the $g_{\mu\nu}$ field approximately in terms of the field equations could go ahead without fundamental difficulty.

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¹A. EINSTEIN, L. INFELD & B. HOFFMANN, *Ann. Math.*, **39**, 66 (1938).

²A. EINSTEIN & L. INFELD, *Ann. Math.*, **41**, 797 (1940).

³A. EINSTEIN & L. INFELD, *Can. J. Math.*, **1**, 209 (1949).

The first method introduced by EINSTEIN, INFELD & HOFFMANN in 1938, to enable one to integrate the field equations, was intimately connected with the coordinate conditions postulated to simplify the field equations and determine the motion. A different version of this was published in 1940,² which did not depend upon any specific choice of coordinate system. However, it contained the previous work as a special case, when the old coordinate conditions were assumed.

Later, in 1949,³ EINSTEIN & INFELD devised a second method, which consisted in adding poles and dipoles to the field to ensure the integrability of the field equations. The surface integral conditions were satisfied identically and the motion of the singularities made arbitrary. But the total field contained unphysical dipoles. These dipoles satisfied a second order differential equation in the time, and diverged in the infinite past and future except for periodic motion. The unphysical dipoles were to be removed after the total field had been calculated, placing a restriction on the motion in the form of 3 p equations of motion. Thus the field was calculated to, say, the $(2n - 2)^{\text{th}}$ order. Then, by equating to zero the *sum* of all the dipoles to the $(2n - 2)^{\text{th}}$ order, this should yield the equations of motion to the $2n^{\text{th}}$ order. This method has been re-examined,^{4,5} and it was found that even though the sum of the dipole moments to each order is equated to zero, this does not necessarily imply that all the unphysical separate order dipoles vanish from the equations of motion to the $2n^{\text{th}}$ order. Any rigorous proof that this method leads to physical equations of motion will depend upon the structure of the field equations in the $2n^{\text{th}}$ order. One can only arrive at a knowledge of the non-linear part of the field equations in the $2n^{\text{th}}$ order by means of an induction hypothesis.

Recently all the previous work on this problem was modified by INFELD & PLEBANSKI⁶ in order to overcome two alleged defects: (1) the procedure was not generally covariant and (2) it is closely related to the approximation procedure used in solving EINSTEIN's field equations. They obtained equations of motion by using an energy-momentum tensor expressed in terms of the Dirac δ -function.^{7,8}

In the following, we shall apply the modified INFELD & PLEBANSKI method to EINSTEIN's field equations of empty space treating matter as singularities of the field. Special attention is paid to the problem of boundary conditions and the related problem of whether the equations of motion are uniquely determined by the EINSTEIN, INFELD & HOFFMANN theory.

⁴J. MOFFAT & R. KERR, On the Dipole Method in the Problem of Gravitational Motion, *unpublished*.

⁵J. MOFFAT, *Il Nuovo Cimento*, **7**, 107 (1958).

⁶L. INFELD & J. PLEBANSKI, *J. Bull. Acad. Pol. Sci.*, **4**, 11, 763 (1956).

⁷L. INFELD & J. PLEBANSKI, *J. Bull. Acad. Pol. Sci.*, **4**, 11, 757 (1956).

⁸L. INFELD, *Rev. Mod. Phys.*, **29**, 398 (1957).

2. The field equations. Let the symbol $\eta_{\mu\nu}$ be described by the matrix elements*

$$(2.1) \quad \eta_{\mu\nu} = \begin{Bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{Bmatrix}.$$

Let us write the metric tensor $g_{\mu\nu}$ as

$$(2.2) \quad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where the $h_{\mu\nu}$ are small quantities. In addition, we define the quantities

$$(2.3) \quad \gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h_{\alpha\beta}.$$

The gravitational field equations of empty space are

$$(2.4) \quad R_{\mu\nu} = 0,$$

where

$$(2.5) \quad R_{\mu\nu} = -\left\{\begin{matrix} \rho \\ \mu\nu \end{matrix}\right\}_{|\rho} + \left\{\begin{matrix} \rho \\ \mu\rho \end{matrix}\right\}_{|\nu} + \left\{\begin{matrix} \rho \\ \mu\sigma \end{matrix}\right\}\left\{\begin{matrix} \sigma \\ \rho\nu \end{matrix}\right\} - \left\{\begin{matrix} \rho \\ \mu\nu \end{matrix}\right\}\left\{\begin{matrix} \sigma \\ \rho\sigma \end{matrix}\right\}.$$

The field equations can be written^{2,3}

$$(2.6) \quad \Phi_{00} + 2\Lambda_{00} = 0,$$

$$(2.7) \quad \Phi_{0m} + 2\Lambda_{0m} = 0,$$

$$(2.8) \quad \Phi_{mn} + 2\Lambda_{mn} = 0,$$

where

$$(2.9) \quad \Phi_{00} = -\gamma_{00|ss},$$

$$(2.10) \quad \Phi_{0m} = -\gamma_{0m|ss} + \gamma_{0s|ms},$$

$$(2.11) \quad \Phi_{mn} = -\gamma_{mn|ss} + \gamma_{ms|ns} + \gamma_{ns|ms} - \delta_{mn}\gamma_{rs|rs},$$

and

$$(2.12) \quad 2\Lambda_{00} = \gamma_{sr|sr} + 2\Lambda'_{00},$$

*Greek suffixes range and sum over the values 0, 1, 2, 3, while Latin suffixes range and sum over 1, 2, 3. We shall use the convenient notation of denoting the ordinary derivative of a quantity by means of a line followed by the appropriate suffix:

$$\frac{\partial g_{\mu\nu}}{\partial x_\sigma} \rightarrow g_{\mu\nu|\sigma}; \quad \frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma \partial x_\tau} \rightarrow g_{\mu\nu|\sigma\tau}.$$

We adopt geometric units, in which the gravitational constant and the velocity of light are equal to unity.

$$(2.13) \quad 2\Lambda_{0m} = \gamma_{ms|s0} - \gamma_{00|m0} + 2\Lambda'_{0m},$$

$$(2.14) \quad 2\Lambda_{mn} = -\gamma_{0m|0n} - \gamma_{0n|0m} + 2\delta_{mn}\gamma_{0s|0s} + \gamma_{mn|00} - \delta_{mn}\gamma_{00|00} + 2\Lambda'_{mn}.$$

The $\Lambda'_{\mu\nu}$ are the non-linear terms. EINSTEIN & INFELD write the field equations in the above manner so that all the non-linear terms in $\Lambda'_{\mu\nu}$, and linear terms which behave like non-linear terms, are separated from the linear terms in $\Phi_{\mu\nu}$.

3. The Einstein, Infeld and Hoffman Approximation Method. To solve the gravitational field equations, EINSTEIN, INFELD & HOFFMANN devised a new approximation method specially suited to the case of quasi-static fields.^{1,3}

If we consider the case when a field function ϕ varies quickly in space and slowly in time (quasi-static assumption), we shall have to treat the derivatives with respect to x^0 on a different basis. The derivatives with respect to x^0 will be of a higher order than those with respect to x^s . EINSTEIN, INFELD & HOFFMANN take this into account by introducing an "auxiliary time" τ by means of the *scale transformation*

$$(3.1) \quad \tau = \lambda x^0.$$

We shall adopt the EINSTEIN, INFELD & HOFFMANN notation in which the derivatives with respect to the non-invariant time coordinate τ are treated on the same footing as the derivatives with respect to x^s :

$$(3.2) \quad \begin{cases} \phi_{|0} = \frac{\partial\phi}{\partial x_0} = \lambda \frac{\partial\phi}{\partial\tau} = \lambda\phi_{,0}, \\ \phi_{|s} = \frac{\partial\phi}{\partial x_s} = \phi_{,s}. \end{cases}$$

We assume the γ 's to be represented by the "λ-expansion"

$$(3.3) \quad \begin{cases} \gamma_{00}(\lambda) = \lambda^2 \gamma_{00}^{(2)} + \lambda^4 \gamma_{00}^{(4)} + \dots \\ \gamma_{0m}(\lambda) = \lambda^3 \gamma_{0m}^{(3)} + \lambda^5 \gamma_{0m}^{(5)} + \dots \\ \gamma_{mn}(\lambda) = \lambda^4 \gamma_{mn}^{(4)} + \lambda^6 \gamma_{mn}^{(6)} + \dots \end{cases}$$

The role of the parameter λ is merely to assign a *weight* to the γ 's, and it determines the relative scale of the system. We can eliminate λ at the end of the *iteration* process by taking $(\lambda\gamma)$ as the variables in our equations, or by putting $\lambda = 1$. During the application of the iteration process it is convenient to retain λ as a means of describing the iterations.

4. The Surface Integral Conditions. We shall treat particles as point singularities and consider a system of p particles. The position coordinates of each particle are described by $\xi^s(x^0)$; $s = 1, 2, \dots, p$. The spatial coordinates of

each particle will be functions of x^0 only and the index written on top will always label a particular singularity.

The field equations are

$$(4.1) \quad \Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = 0.$$

Introducing into these equations the series expansions for the γ 's, we get for the $_{00}$ equations

$$(4.2) \quad \sum_l \lambda^{2l} (\Phi_{00} + 2\Lambda_{00}) = 0,$$

and similar equations for $_{0m}$ and $_{mn}$. Equating to zero the coefficients of the various powers of λ , we get the system of equations

$$(4.3) \quad \Phi_{00} + 2\Lambda_{00} = 0,$$

$$(4.4) \quad \Phi_{0m} + 2\Lambda_{0m} = 0,$$

$$(4.5) \quad \Phi_{mn} + 2\Lambda_{mn} = 0,$$

where

$$(4.6) \quad \Phi_{00} = -\gamma_{00,rr},$$

$$(4.7) \quad \Phi_{0m} = -\gamma_{0m,rr} + \gamma_{0r,mr},$$

$$(4.8) \quad \Phi_{mn} = -\gamma_{mn,rr} + \gamma_{mr,nr} + \gamma_{nr,mr} - \delta_{mn}\gamma_{rs,rs},$$

and

$$(4.9) \quad 2\Lambda_{00} = \gamma_{rs,rs} + 2\Lambda'_{00},$$

$$(4.10) \quad 2\Lambda_{0m} = -\gamma_{00,0m} + \gamma_{mr,0r} + 2\Lambda'_{0m},$$

$$(4.11) \quad 2\Lambda_{mn} = -\gamma_{0m,0n} - \gamma_{0n,0m} + 2\delta_{mn}\gamma_{0r,0r} + \gamma_{mn,00} - \delta_{mn}\gamma_{00,00} + 2\Lambda'_{mn}.$$

From the structure of the above expressions, we see that at each stage of approximation the unknown functions are contained in Φ , and the known functions in Λ .

We now consider the question of the integrability of the above equations at each stage of approximation. Let us begin by looking at (4.3). We have

$$(4.12) \quad \gamma_{00,rr} = 2\Lambda_{00}.$$

There is no difficulty in solving this equation, since it is just a Poisson equation for *known* Λ_{00} . The next equation we look at is (4.4). From (4.7), we easily check that

$$(4.13) \quad \Phi_{0m, m} = 0.$$

Consequently, we can only integrate (4.4) when

$$(4.14) \quad \Lambda_{0m, m} = 0,$$

where Λ_{0m} is known. Next we consider (4.5). Because of the structure of (4.8)

$$(4.15) \quad \Phi_{mn, n} = 0.$$

Thus the six equations (4.5) can only be integrated if

$$(4.16) \quad \Lambda_{mn, n} = 0.$$

We now turn to the exact surface integral conditions which follow from an application of the EINSTEIN, INFELD & HOFFMANN surface integral lemma (footnote 3, page 212, sec. 3) and from the structure of Φ_{0m} and Φ_{mn} :

$$(4.17) \quad \int^s \Phi_{mk} n_k dS = 0, \quad \int^s \Phi_{0m} n_m dS = 0,$$

where $\int^s$ indicates that the surface S encloses only the s^{th} singularity and $n_k = \cos(x^k, \mathbf{n})$ are the components of the normal unit vector to the surface.

From these equations and (2.7), (2.8) it follows that

$$(4.18) \quad \int^s 2\Lambda_{mn} n_n dS = 0, \quad \int^s 2\Lambda_{0m} n_m dS = 0.$$

Equations (4.18) indicate that the surface integral around the closed surface S is independent of the shape of the surface. Therefore since the surface is completely arbitrary, it can only provide us with information about the coordinates of the s^{th} singularity and their time derivatives.

Let us develop the first of (4.18) into a power series in the weighting parameter λ :

$$(4.19) \quad \sum_{l=1}^{\infty} \lambda^{2l} \int^s 2\Lambda_{mn} n_n dS = 0,$$

or, using the notation

$$(4.20) \quad \int^s 2\Lambda_{mn} n_n dS = 4\pi \dot{C}_m,$$

we get

$$(4.21) \quad \sum_{l=1}^{\infty} \lambda^{2l} \dot{C}_m = 0,$$

where $\dot{C}_m$ is a function of τ and $\dot{\xi}^m$ ($\dot{\xi}^m = d\xi^m/d\tau$). (4.21) is a system of $3p$ ordinary

differential equations in λ . According to conventional methods, we should solve (4.21) by equating to zero the coefficients of the various powers of λ . For instance,

$$(4.22) \quad \frac{1}{4\pi} \int 2\Lambda_{mn} n_n dS = C_m = 0.$$

But this is not permitted. It would give, in the general case, an infinite number of equations for the three functions $\xi^m(\tau)$. The field equations and the surface integral conditions, solved in terms of the EINSTEIN, INFELD & HOFFMANN approximation method, lead to inconsistent equations of motion at each stage of approximation. It is at this stage that EINSTEIN, INFELD & HOFFMANN reject the field equations and the approximation method which give the system (4.3)–(4.5) and (4.22).¹

5. The Equations of Motion. The EINSTEIN field equations are of the form

$$(5.1) \quad \Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = 0.$$

As generalized equations, we choose²

$$(5.2) \quad \Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = C_{\mu\nu},$$

where $C_{\mu\nu}$ is a non-localized source-tensor and a function of x^s and τ . $C_{\mu\nu}$ is chosen so that a development into powers of λ is possible, and the EINSTEIN, INFELD & HOFFMANN method is applicable, without encountering the difficulty met with in the previous section.

Let us consider whether the field equations (5.2) are solvable at each stage of approximation. We suppose that $C_{\mu\nu}$ is a known function of τ , x^s and λ ; and we develop $C_{\mu\nu}$ into a power series in the weighting parameter λ :

$$(5.3) \quad C_{mn} = \sum_{l=1}^{\infty} \lambda^{2l} C_{mn}^{2l},$$

$$(5.4) \quad C_{0m} = \sum_{l=1}^{\infty} \lambda^{2l-1} C_{0m}^{2l-1},$$

$$(5.5) \quad C_{00} = \sum_{l=1}^{\infty} \lambda^{2l-2} C_{00}^{2l-2}.$$

On substituting the expansions for γ 's and C 's into (5.2) and equating the various powers of λ , we obtain the system

$$(5.6) \quad \Phi_{00}^{2l-2} + 2\Lambda_{00}^{2l-2} = C_{00}^{2l-2},$$

$$(5.7) \quad \Phi_{0m}^{2l-1} + 2\Lambda_{0m}^{2l-1} = C_{0m}^{2l-1},$$

$$(5.8) \quad \Phi_{mn}^{2l} + 2\Lambda_{mn}^{2l} = C_{mn}^{2l}.$$

The equations (5.6) have the form

$$(5.9) \quad \gamma_{00,ss}^{2l-2} = 2\Lambda_{00}^{2l-2} - C_{00}^{2l-2}.$$

There is no difficulty in solving this Poisson equation for known Λ_{00}^{2l-2} and C_{00}^{2l-2} .

Next we investigate (5.7). We know that

$$(5.10) \quad \Phi_{0m,m}^{2l-1} = 0.$$

Therefore we cannot solve (5.7) unless

$$(5.11) \quad 2\Lambda_{0m,m}^{2l-1} = C_{0m,m}^{2l-1},$$

where $2\Lambda_{0m}^{2l-1}$ and C_{0m}^{2l-1} are *known*. Finally, we look at the equations (5.8). In view of the fact that

$$(5.12) \quad \Phi_{mn,n}^{2l} = 0,$$

(5.8) must satisfy

$$(5.13) \quad 2\Lambda_{mn,n}^{2l} = C_{mn,n}^{2l}.$$

The field equations (5.6)–(5.8) cannot be solved unless the integrability conditions (5.11) and (5.13) are satisfied at every stage of approximation.

The tensor

$$(5.14) \quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

satisfies the Bianchi identity

$$(5.15) \quad G_{\nu|\mu}^{\mu} + \left\{ \begin{matrix} \alpha \\ \alpha\beta \end{matrix} \right\} G_{\nu}^{\beta} - \left\{ \begin{matrix} \beta \\ \nu\alpha \end{matrix} \right\} G_{\beta}^{\alpha} = 0.$$

We find that

$$(5.16) \quad \Phi_{\mu\nu} + 2\Lambda_{\mu\nu} = -2(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}R_{\alpha\beta}).$$

From (5.15) and (5.2) it follows that

$$(5.17) \quad C_{\nu|\mu}^{\mu} + \left\{ \begin{matrix} \alpha \\ \alpha\beta \end{matrix} \right\} C_{\nu}^{\beta} - \left\{ \begin{matrix} \beta \\ \nu\alpha \end{matrix} \right\} C_{\beta}^{\alpha} = 0.$$

It is important that our choice of $C_{\mu\nu}$ in (5.2) satisfies (5.17). It can be proved, by a straightforward generalization to this case of the proof given by EINSTEIN & INFELD (footnote 3, page 217, sec. 7), that (5.11) and (5.13) are satisfied if the field equations are satisfied in all the previous orders of approximation, and if (5.17) holds. Thus the integrability conditions are satisfied in each approximation, though not identically but because $C_{\mu\nu}$ satisfies (5.17) and the field equations are satisfied in all the previous orders.

We are now in a position to make a specific choice of the $C_{\mu\nu}$ functions (5.2). In dealing with this problem, EINSTEIN & INFELD made the following choice:²

$$(5.18) \quad C_{mn} = - \sum_{s=1}^n \{ (\overset{s}{C}_m/\overset{s}{r})_{|n} + (\overset{s}{C}_n/\overset{s}{r})_{|m} - \delta_{mn}(\overset{s}{C}_a/\overset{s}{r})_{|a} \},$$

$$(5.19) \quad C_{0m} = - \sum_{s=1}^n \{ (\overset{s}{C}_0/\overset{s}{r})_{|m} + (\overset{s}{C}_m/\overset{s}{r})_{|0} \},$$

$$(5.20) \quad C_{00} = - \sum_{s=1}^n \{ (\overset{s}{C}_a/\overset{s}{r})_{|a} \},$$

where

$$(5.21) \quad \overset{s}{r} = [(x^k - \overset{s}{\xi}^k)(x^k - \overset{s}{\xi}^k)]^{\frac{1}{2}}.$$

The $\overset{s}{C}_0$, $\overset{s}{C}_m$ are functions of τ only and can be developed into power series in λ .

As can be seen on inspection, (5.18) implies that $C_{mn|n} = 0$, since $\sum_s \overset{s}{C}_m/\overset{s}{r}$ is a harmonic function. Instead of the above, we shall use in the following work the (locally) *Lorentz invariant* form of the $C_{\mu\nu}$ used by INFELD & PLEBANSKI:⁶

$$(5.22) \quad C_{\mu\nu} = - \sum_{s=1}^n \{ \overset{s}{D}_{\mu| \nu} + \overset{s}{D}_{\nu| \mu} - \eta_{\mu\nu} \eta^{\alpha\beta} \overset{s}{D}_{\alpha| \beta} \}.$$

This can be written

$$(5.23) \quad C_{mn} = - \sum_{s=1}^n \{ \overset{s}{D}_{m|n} + \overset{s}{D}_{n|m} + \delta_{mn} \overset{s}{D}_{0|0} - \delta_{mn} \overset{s}{D}_{a|a} \},$$

$$(5.24) \quad C_{0m} = - \sum_{s=1}^n \{ \overset{s}{D}_{0|m} + \overset{s}{D}_{m|0} \},$$

$$(5.25) \quad C_{00} = - \sum_{s=1}^n \{ \overset{s}{D}_{a|a} + \overset{s}{D}_{0|0} \}.$$

We represent the D 's by the following expansion in λ :

$$(5.26) \quad \overset{s}{D}_0 = \sum_{l=1}^{\infty} \lambda^{2l-1} \overset{s}{D}_{0, 2l-1},$$

$$(5.27) \quad \overset{s}{D}_m = \sum_{l=1}^{\infty} \lambda^{2l} \overset{s}{D}_{m, 2l}.$$

On substituting these expansions into (5.23)–(5.25), we obtain the system of equations

$$(5.28) \quad \overset{s}{\Phi}_{mn, 2l} + 2\overset{s}{\Lambda}_{mn, 2l} = - \sum_{s=1}^n \{ \overset{s}{D}_{m, n, 2l} + \overset{s}{D}_{n, m, 2l} + \delta_{mn} \overset{s}{D}_{0, 0, 2l-1} - \delta_{mn} \overset{s}{D}_{a, a, 2l} \},$$

$$(5.29) \quad \overset{s}{\Phi}_{0m, 2l-1} + 2\overset{s}{\Lambda}_{0m, 2l-1} = - \sum_{s=1}^n \{ \overset{s}{D}_{0, m, 2l-1} + \overset{s}{D}_{m, 0, 2l-2} \},$$

$$(5.30) \quad \overset{s}{\Phi}_{00, 2l-2} + 2\overset{s}{\Lambda}_{00, 2l-2} = - \sum_{s=1}^n \{ \overset{s}{D}_{a, a, 2l-2} + \overset{s}{D}_{0, 0, 2l-3} \}.$$

In view of the Bianchi identity, $C_{\mu\nu}$ must satisfy (5.17). Then the integrability conditions (5.11) and (5.13) will hold at every stage of approximation *provided the field equations are satisfied in all the previous orders*. Thus the field equations are solvable up to an arbitrarily high approximation. We shall see in a moment that the development of the field equations in the manner (5.28)–(5.30) does not lead to the wrong surface integrals at each stage of approximation; the surface integral conditions are identically fulfilled at each step and do not place restrictions on the motion.

The choice of $C_{\mu\nu}$ in (5.22) is not a covariant one. However, we shall find it most convenient to choose a particular frame of reference to solve the system of equations (5.28)–(5.30). The original method used by EINSTEIN, INFELD & HOFFMANN in 1938 was connected with a particular coordinate system. In the 1940 version, nothing was assumed in advance about the coordinate system except that it was Galilean at infinity.

Let us consider whether the field equations (5.28)–(5.30) can be simplified by normalizing the γ -field in a suitable fashion. We assume that

$$(5.31) \quad \tilde{\gamma}_{00}^{2l-2}, \quad \tilde{\gamma}_{0m}^{2l-1}, \quad \tilde{\gamma}_{mn}^{2l}$$

are solutions of the system of equations (5.28)–(5.30). We then introduce new solutions given by

$$(5.32) \quad \gamma_{00}^{2l-2} = \tilde{\gamma}_{00}^{2l-2},$$

$$(5.33) \quad \gamma_{0m}^{2l-1} = \tilde{\gamma}_{0m}^{2l-1} + \Pi_{0,m}^{2l-1},$$

$$(5.34) \quad \gamma_{mn}^{2l} = \tilde{\gamma}_{mn}^{2l} + \Pi_{n,m}^{2l} + \Pi_{m,n}^{2l} - \delta_{mn} \Pi_{a,a}^{2l} + \delta_{mn} \Pi_{0,0}^{2l-1},$$

where the Π_0 , Π_m constitute four arbitrary functions. A direct substitution of (5.32)–(5.34) into the field equations (5.28)–(5.30) verifies that the new γ 's are in fact solutions.

We choose Π_0 , Π_m to be solutions of the equations

$$(5.35) \quad \Pi_{0,ss}^{2l-1} = \tilde{\gamma}_{00,0}^{2l-2} - \tilde{\gamma}_{0s,s}^{2l-1} - \sum_s^s D_0^{2l-1},$$

$$(5.36) \quad \Pi_{m,ss}^{2l} = \tilde{\gamma}_{0m,0}^{2l-1} - \tilde{\gamma}_{ms,s}^{2l} - \sum_s^s D_m^{2l}.$$

With this choice the γ 's satisfy the four coordinate conditions

$$(5.37) \quad \gamma_{00,0}^{2l-2} - \gamma_{0s,s}^{2l-1} = \sum_s^s D_0^{2l-1},$$

$$(5.38) \quad \gamma_{0m,0}^{2l-1} - \gamma_{ms,s}^{2l} = \sum_s^s D_m^{2l}.$$

Our field equations (5.28)–(5.30) now reduce to the system

$$(5.39) \quad \gamma_{00,ss} = \gamma_{00,00} + 2\Lambda'_{00},$$

$$(5.40) \quad \gamma_{0m,ss} = \gamma_{0m,00} + 2\Lambda'_{0m},$$

$$(5.41) \quad \gamma_{mn,ss} = \gamma_{mn,00} + 2\Lambda'_{mn}.$$

Differentiating (5.37) with respect to X^m and adding the result to (5.40), we get

$$(5.42) \quad (\gamma_{0m,s} - \gamma_{0s,m})_{,s} = 2\Lambda'_{0m} + \gamma_{0m,00} - \gamma_{00,0m} + \sum_s \dot{D}_{0,m}.$$

Differentiating (5.38) with respect to x^n and adding the result to (5.41), we obtain

$$(5.43) \quad (\gamma_{mn,s} - \gamma_{ms,n})_{,s} = 2\Lambda'_{mn} + \gamma_{mn,00} - \gamma_{0m,0n} + \sum_s \dot{D}_{m,n}.$$

We take the surface integral of both sides of (5.42) and (5.43) around a small surface S , enclosing the s^{th} singularity and it alone, and by applying the EINSTEIN, INFELD & HOFFMANN surface integral lemma obtain

$$(5.44) \quad \int^s (2\Lambda'_{0m} + \gamma_{0m,00} - \gamma_{00,0m})n_m dS = - \int^s D_{0,m}n_m dS$$

and

$$(5.45) \quad \int^s (2\Lambda'_{mn} + \gamma_{mn,00} - \gamma_{0m,0n})n_n dS = - \int^s D_{m,n}n_n dS.$$

If we apply Gauss' theorem to (5.44) and (5.45), we have

$$(5.46) \quad \int^s (2\Lambda'_{0m} + \gamma_{0m,00} - \gamma_{00,0m})_{,m} dV = - \int^s D_{0,aa} dV$$

and

$$(5.47) \quad \int^s (2\Lambda'_{mn} + \gamma_{mn,00} - \gamma_{0m,0n})_{,n} dV = - \int^s D_{m,aa} dV,$$

where the integrals have been taken only over the interiors of the surfaces. From (5.47) we get

$$(5.48) \quad \dot{D}_{m,aa} = -2\Lambda^*_{mn,n},$$

where

$$(5.49) \quad 2\Lambda^*_{mn} = 2\Lambda'_{mn} + \gamma_{mn,00} - \gamma_{0m,0n}.$$

A solution of the Poisson equation (5.48) is

$$(5.50) \quad D_{2l}^s(x) = \frac{1}{4\pi} \int^s \frac{2\Lambda_{mn,n}^*(x')}{r(x, x')} dV' + \Psi_{2l}^s(x),$$

where the integral is taken only over the interior of the surface surrounding the s^{th} singularity and

$$(5.51) \quad r(x, x') = [(x^s - x'^s)(x^s - x'^s)]^{\frac{1}{2}}.$$

The additional term in (5.50) is any solution of the homogeneous equation

$$D_{2l,aa}^s = 0.$$

The kind of $\Psi_{2l}^s(x)$ we add to the solution of (5.48) will of course depend upon the boundary conditions imposed on the problem. If we are dealing with a closed system of p particles in a spatially unbounded universe the natural boundary condition to choose is to let $\Lambda_{mn,n}^*$ go to zero at infinity. If $\Lambda_{mn,n}^*$ is a reasonably behaved function, in this sense, we can also demand that $D_{2l}^s(x)$ should vanish at infinity. Then $\Psi_{2l}^s(x)$, in (5.50), must be zero everywhere, because no solution of Laplace's equation can have a maximum or minimum anywhere, and the requirement that $\Psi_{2l}^s(x)$ should vanish at infinity results in it being zero everywhere. Therefore a unique solution of (5.48) is

$$(5.52) \quad D_{2l}^s(x) = \frac{1}{4\pi} \int^s \frac{2\Lambda_{mn,n}^*(x')}{r(x, x')} dV'$$

if the boundary condition requires vanishing at infinity. This choice of homogeneous Dirichlet boundary conditions corresponds to the coordinates being Galilean at infinity. With this choice of boundary conditions a similar unique solution can be found for D_{2l-1} by considering (5.46). The Λ 's can always be found

from the previous approximation steps, and therefore the functions D_{2l-1} and D_{2l} are known if the $\gamma_{l'}^s$ are known for $l' < 2l$. The motion of the particles is now arbitrary, as in the case of electrodynamics, and the field is determined in each of the approximation steps by the equations (5.39)–(5.41), in which the right-hand sides are known. Thus we can calculate the γ 's without encountering the difficulty with the surface integrals.

In making these generalizations, we have gone beyond EINSTEIN's gravitational equations to others which contain them as a particular case. The γ 's will not be solutions of EINSTEIN's equations, but more general ones. We must impose conditions on the motion of the singularities in order that our solutions will be those of the gravitational equations (5.1).

Let us suppose that we have found the γ 's in all the successive approximations,

and the D_0 , D_m are represented by the expansions (5.26), (5.27). To restrict the motion, we postulate the following conditions:

$$(5.53) \quad \overset{s}{D}_0 = 0, \quad \overset{s}{D}_m = 0.$$

By imposing these conditions on the motion, our solutions will be those of EINSTEIN's equations. From (5.44) and (5.45), we see that the conditions (5.53) are equivalent to the surface integral conditions

$$(5.54) \quad \sum_{l=1}^{\infty} \lambda^{2l-1} \int^s (2\overset{s}{\Lambda}'_{0m} + \underset{2l-3}{\gamma}_{0m,00} - \underset{2l-2}{\gamma}_{00,0m}) n_m dS = 0,$$

$$(5.55) \quad \sum_{l=1}^{\infty} \lambda^{2l} \int^s (2\overset{s}{\Lambda}'_{mn} + \underset{2l-2}{\gamma}_{mn,00} - \underset{2l-1}{\gamma}_{0m,0n}) n_n dS = 0.$$

Let us suppose that we have been able to solve the approximations up to the n^{th} order. Then, by setting $\lambda = 1$, we derive

$$(5.56) \quad \sum_{l=1}^n \overset{s}{D}_0 = 0, \quad \sum_{l=1}^n \overset{s}{D}_m = 0,$$

and these are the equations of motion correct up to the order $2n$.

The second set of conditions (5.56) corresponds in the *unique* solution (5.52) to $\sum_{l=1}^n \overset{s}{\Lambda}_{ma,a}^*$ vanishing everywhere except at the s^{th} singularity. In order to derive unique and physically acceptable equations of motion from the conditions (5.56), we must obtain homogeneous Dirichlet boundary conditions for the $\overset{s}{D}_m(x)$; this corresponds to choosing a coordinate system in which the $g_{\mu\nu}$ tend to the Galilean values $\eta_{\mu\nu}$.

To obtain knowledge about the behaviour of Λ at infinity, we shall investigate the form of the solutions obtained for the γ 's from the field equations. In the n^{th} order the field equations have the form (here we disregard for the moment the D -functions)

$$(5.57) \quad \overset{s}{\Phi}_{\mu\nu} + 2\overset{s}{\Lambda}_{\mu\nu} = L(\underset{n}{\gamma}_{\mu\nu,mn}, \underset{n-1}{\dot{\gamma}}_{\mu\nu,m}, \underset{n-2}{\ddot{\gamma}}_{\mu\nu}) + N(\underset{n-2}{\gamma}_{\mu\nu}, \dots, \underset{2}{\gamma}_{\mu\nu}) = 0,$$

where L is a linear function and N a non-linear function of the arguments. A solution of $\underset{n}{\gamma}_{\mu\nu}$ will contain expressions arising from both the linear and non-linear parts of (5.57). As an example, we can consider solutions of the field equations (5.39)–(5.41) with the coordinate conditions (5.37) and (5.38). The equations for $\underset{n}{\gamma}_{00}$ up to the order $l = 3$ are

$$(5.58) \quad \underset{2}{\gamma}_{00,ss} = 0; \quad \underset{4}{\gamma}_{00,ss} = \underset{2}{\gamma}_{00,00} - \underset{2}{\frac{3}{8}} \underset{2}{\gamma}_{00,s} \underset{2}{\gamma}_{00,s}.$$

If, for simplicity, we consider only one particle, a solution of $\underset{2}{\gamma}_{00}$ is

$$(5.59) \quad \underset{2}{\gamma}_{00} = -\frac{4m}{r},$$

where m is a constant corresponding to the mass of the particle.³ Then the solution for γ_{00} , in terms of (5.58), is

$$(5.60) \quad \gamma_{00} = -\frac{3m^2}{r^2} - \frac{mr_{,00}}{r^2}.$$

The contributions to the solution γ_{00} from the *linear* part of (5.39) will have the form

$$(5.61) \quad \text{linear contribution to } \gamma_{00} = -\frac{2m}{(2l-4)!} \frac{d^{2l-4}}{d\tau^{2l-4}} (r^{2l-5}).$$

INFELD has shown, by analogy with electrodynamics, that the sum of these linear terms corresponds to the standing wave solution.⁹ The magnitudes of the expressions in (5.61) will not tend to zero at large distances for $l > 2$. Thus for large values of r the linear part of $\gamma_{00} = O(r^{l-3})$ for $l \geq 2$. In general, we cannot postulate, in the framework of the theory, that $\gamma_{\mu\nu}$ vanishes at infinity.

Therefore, in general, *we cannot conclude that Λ_{mn}^* or $\Lambda_{mn,n}^*$ vanish at infinity.*

This does not exclude the possibility that $\gamma_{\mu\nu} = \lim_{n \rightarrow \infty} \sum_n \gamma_{\mu\nu}$ will vanish at infinity if the system of equations is convergent. If d is the characteristic wavelength of the field the EINSTEIN, INFELD & HOFFMANN approximation method is only applicable when $r \ll d$;¹⁰ this is a limitation on the theory additional to the quasi-static one $v \ll c$ (c the velocity of light). Thus we do not obtain homogeneous boundary conditions for the $D_m(x)$ on account of the behavior of $\Lambda_{mn,n}^*$ at large distances, and $\Psi_m(x)$ in (5.50) is not identically zero. As the orders of approximation increase the integral in (5.50) will not behave reasonably at infinity.

We now observe that the $C_{\mu\nu}$ in (5.18), (5.19) and (5.20) is not in general compatible with the expanded form of the field equations (5.6), (5.7) and (5.8), since the components of $C_{\mu\nu}$ in (5.18), (5.19) and (5.20) tend to zero at infinity, whereas the left-hand sides of (5.6), (5.7) and (5.8) do not for $l > 2$. To see how this choice of $C_{\mu\nu}$ results, we have, for example, only to let $\Lambda_{mn,n}^*$ in (5.52) tend to a product of Dirac δ -functions:

$$(5.62) \quad 2\Lambda_{mn,n}^* \rightarrow 4\pi C_m(\tau) \delta(x^1 - \xi^1) \delta(x^2 - \xi^2) \delta(x^3 - \xi^3).$$

We can conclude from the above analysis that the boundary conditions play

⁹L. INFELD, *Phys. Rev.*, **53**, 836 (1938).

¹⁰A. TRAUTMAN, *Lectures on General Relativity*, King's College, London (May-June 1958).

a peculiar role in the EINSTEIN, INFELD & HOFFMANN method, owing to the global limitations placed on the theory by the nature of the approximation procedure. These global limitations make it difficult at present to answer the fundamental question of whether the EINSTEIN, INFELD & HOFFMANN equations of motion are unique.

It is possible that other coordinate conditions exist for which the expanded γ 's vanish at infinity. In place of the coordinate conditions used here, we can consider as alternative ones those used by EINSTEIN, INFELD & HOFFMANN¹ in their calculations (we disregard for the moment the D -functions):

$$(5.63) \quad \gamma_{2l-2,0}^{00,0} - \gamma_{2l-1,s}^{0s,s} = 0,$$

$$(5.64) \quad \gamma_{2l}^{mn,n} = 0.$$

The equations then become

$$(5.65) \quad \gamma_{2l-2}^{00,ss} = 2\Lambda_{2l-2}'^{00},$$

$$(5.66) \quad \gamma_{2l-1}^{0m,ss} = 2\Lambda_{2l-1}'^{0m},$$

$$(5.67) \quad \gamma_{2l}^{mn,ss} = -\gamma_{2l-1,1}^{0m,0n} - \gamma_{2l-1,1}^{0n,0m} + \delta_{mn}\gamma_{2l-2,2}^{00,00} + \gamma_{2l-2,2}^{mn,00} + 2\Lambda_{2l}'^{mn}.$$

We observe that even though the linear expressions have disappeared from the right-hand sides of (5.65) and (5.66) they still appear on the right-hand side of (5.67). Even though γ_{2l-2}^{00} and γ_{2l-1}^{0m} now vanish at infinity γ_{2l}^{mn} will not generally do so.

Although the coordinate conditions which have been used until now in the EINSTEIN, INFELD & HOFFMANN procedure do not admit (approximate) homogeneous boundary conditions, it is possible that coordinate conditions do exist which lead to satisfactory boundary conditions and unique equations of motion. As far as the author is aware, the form of these coordinate conditions is at present unknown.

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