*A Strong Maximum Principle for Weakly L-Subharmonic Functions**

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1. Introduction. E. HOPF's maximum principle for a second order elliptic operator [2] states that if

$$L = a_{ij}(x) D_{ij} + a_i(x) D_i$$

is an elliptic operator** defined on $C^2(\mathfrak{D})$, with $\mathfrak{D}$ a domain in n -dimensional Euclidean space, and if

$$(1) \quad Lu \geq 0$$

(u belongs to $C^2(\mathfrak{D})$), then u cannot possess an interior maximum point unless u is identically equal to a constant in $\mathfrak{D}$. In the proof the only requirements made on the coefficients of L are local boundedness and local uniform ellipticity. A function u in C^2 satisfying (1) is frequently called L -subharmonic, by analogy with the case where $a_{ij} \equiv \delta_{ij}$ and $a_i \equiv 0$. It is our aim to extend the notion of L -subharmonicity to as wide a class of functions as seems reasonable and to generalize the maximum principle to this class of functions.

The concept of L -subharmonicity has been extended to continuous functions in various ways. (See, for example, [4], [1].) In case the coefficients satisfy a (locally uniform) Hölder condition it is relatively easy to show that the Dirichlet problem for $Lu = 0$ is uniquely solvable for small spheres (that this is also the case for more or less arbitrary bounded domains is true, but more difficult to show, and is not needed in the following definition). If $u \in C(\mathfrak{D})$ and γ is a solid open sphere contained in $\mathfrak{D}$, then let u_γ denote the unique continuous function in $\mathfrak{D}$, equal to u in $\mathfrak{D} - \gamma$ and satisfying $Lu = 0$ in γ . One may then define u to be L -subharmonic if, for all (sufficiently small) spheres γ , $u_\gamma \geq u$. The ex-

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** $D_i = \partial/\partial x_i$, $D_{ij} = D_i D_j$ and we use the summation convention. Also all numbers are real. By "elliptic" we mean here that the form $a_{ij}\xi_i\xi_j$ is positive definite. We also assume that $a_{ij} = a_{ji}$.

tension of HOPF's maximum principle to such functions is almost immediate. For this treatment see [4].

In a recent paper E. CALABI [1] has extended the notion of L -subharmonicity in another way and has proved the maximum principle for functions satisfying his definition. (Although in his definition the functions are only required to be upper semicontinuous, it is easy to show that if a function satisfies the definition it must also be lower semicontinuous, hence continuous. See Section A1 of the appendix.)

2. Definition of weak L -subharmonicity and statement of results. To extend the notion of L -subharmonicity to as large a class of functions as seems reasonable, we must pay a price, namely we shall assume more smoothness for the coefficients of the operator L :

$$a_{ij} \in C^2(\mathfrak{D}) \quad \text{and} \quad a_i \in C^1(\mathfrak{D}).$$

(Of course we keep the ellipticity assumption.) Under these conditions the formal adjoint L^* of L exists and we write it as

$$L^* = b_{ij} D_{ij} + b_i D_i + b \quad (b_{ij} = a_{ij}).$$

Now suppose u is measurable and essentially bounded from above (i.e., bounded from above except for a possible set of measure zero) in $\mathfrak{D}$ and satisfies the inequality

$$\int_{\mathfrak{D}} u(x) L^* v(x) \, dx \geq 0$$

for all non-negative v in $C^2(\mathfrak{D})$ having compact support in $\mathfrak{D}$. Then we say that u is *weakly L -subharmonic* in $\mathfrak{D}$. There is, of course, an analogous definition for *weakly L -superharmonic*. It is easy to see that, for twice continuously differentiable functions u , $Lu \geq 0$ is equivalent to weak L -subharmonicity.

We can now state our main result.

Theorem 1: *If u is weakly L -subharmonic in $\mathfrak{D}$ and assumes its essential supremum M at a point of continuity of u (in $\mathfrak{D}$) then $u = M$ almost everywhere in $\mathfrak{D}$.*

Theorem 1 is an immediate consequence of Theorems 2 and 3 which we shall prove in later sections.

Theorem 2: *Let u be weakly L -subharmonic in $\mathfrak{D}$ and assume its essential supremum M (over $\mathfrak{D}$) almost everywhere in an open set in $\mathfrak{D}$. Then $u = M$ almost everywhere in $\mathfrak{D}$.*

Theorem 3: *Let u be weakly L -subharmonic in a domain $\mathfrak{D}$. If u assumes its essential supremum M (over $\mathfrak{D}$) at a point of continuity y of u then $u = M$ almost everywhere in a neighborhood of y .*

Note 1. Actually, in Theorems 1 and 3 it suffices for u to be continuous at y after being redefined on a set of measure zero in a deleted neighborhood of y .

3. Proof of Theorem 2. Let S be the set of points x in $\mathfrak{D}$ having the property that $u = M$ almost everywhere in an open set containing x . From the hypothesis of the theorem S is non-empty. Clearly S is open. If we show that S is also closed, then, from the connectedness of $\mathfrak{D}$, it will follow that S is all of $\mathfrak{D}$, implying the conclusion of the theorem.

Now suppose that given any y of $\mathfrak{D}$ we can construct a non-negative function $W^y(x)$ in $C^2(\mathfrak{D} - (y))$ such that

- (3.1) $W^y(x)$ has compact support in $\mathfrak{D}$,
- (3.2) $L^*W^y(x) \geq 0$ for all $x \neq y$,
- (3.3) $L^*W^y(x) > 0$ and $W^y(x) > 0$ in a spherical neighborhood of y whose radius $r(y)$ is bounded away from zero whenever y is restricted to a compact subdomain of $\mathfrak{D}$.

Let y belong to S . Then, in a certain spherical neighborhood N of y , $u = M$ almost everywhere. Change $W^y(x)$ in N in any way so as to make it twice continuously differentiable in all of N (i.e., also at y) and call the resulting function $w^y(x)$. This function will, because of the weak L -subharmonicity of u , satisfy*

$$\int u L^* w^y dx \geq 0.$$

Now

$$\int L^* w^y dx = \int w^y L1 dx = 0;$$

hence

$$\int_{x \notin N} L^* w^y dx = - \int_{x \in N} L^* w^y dx.$$

Also

$$\int_{x \notin N} u L^* w^y dx \geq - \int_N u L^* w^y dx = - \int_N M L^* w^y dx = \int_{x \notin N} M L^* w^y dx,$$

or

$$\int_{x \notin N} (u - M) L^* w^y(x) dx \geq 0.$$

Since, outside N , $w^y = W^y$ and, inside N , $u = M$ almost everywhere, it follows that $\int_{\mathfrak{D}} (u - M) L^* W^y dx \geq 0$ and, since $L^* W^y \geq 0$, $u = M$ almost everywhere in the support of $L^* W^y$. Now consider a sequence of points y_i in S approaching a point Y in $\mathfrak{D}$. We wish to show that Y belongs to S . (The points y_i will, of course, belong to a compact subdomain of $\mathfrak{D}$.) From the properties of W^y it

*All our integrations will be carried out over $\mathfrak{D}$, unless otherwise specified. Since the integrands will have compact support in $\mathfrak{D}$, we can also think of the integration as taken over R^n .

follows that $L^*W^{u_i} > 0$ for $|x - y| < r$, r independent of i . Thus each point y_i has an r -neighborhood in which $u = M$ almost everywhere. Since $y_i \rightarrow Y$ this implies that Y has a neighborhood in which $u = M$ almost everywhere. Thus Y belongs to S , proving that S is closed.

It remains, of course, to construct the function $W^y(x)$. We take

$$W^y(x) = \int_r^\infty \left(\frac{1}{r^k} - \frac{1}{R^k} \right) \phi(R) dR$$

where $r = |x - y|$ and $\phi(R)$ is a C^∞ function such that if R_0 is any positive number less than the distance of y from the boundary of $\mathfrak{D}$ (say half the distance) then

$$\begin{aligned} \phi(R) &= 0 & \text{for } R \leq 0, \\ \phi(R) &> 0 & \text{for } 0 < R < R_0, \\ \phi(R) &= 0 & \text{for } R \geq R_0. \end{aligned}$$

A straightforward calculation will show that for k sufficiently large this function satisfies the requirements 3.1–3.3 (see §A3).

4. Outline of the proof of Theorem 3. We may, without being less general, take y to be the origin. Our method of proof will consist in constructing a function $G(x)$ in C^2 for $x \neq 0$, having compact support in $\mathfrak{D}$, and satisfying, in addition, the following properties:

(4.1) $L^*G \geq 0$ in $\mathfrak{D} - (0)$ and $L^*G > 0$ in some deleted neighborhood N of the origin,

$$(4.2) \quad \int L^*G(x) dx = 1,$$

$$(4.3) \quad f(0) \leq \int f L^*G dx$$

for all weakly L -subharmonic functions f continuous at the origin.

Then we shall have, using (4.2) and (4.3),

$$\int (M - u) L^*G dx \leq 0,$$

implying that $M = u$ almost everywhere in N , since $M - u \geq 0$ almost everywhere in $\mathfrak{D}$.

It remains, of course, to construct the function $G(x)$. To do this we need the existence of a local fundamental solution† — g of the operator L^* and the existence near the origin of a positive function†† p in C^2 such that $L^*p < 0$.

†By this we mean (symbolically) $L^*g = -\delta$; to be more precise, $-\int g L v dx = v(0)$ for any C^2 function v with support in a neighborhood of the origin. For an existence proof for such a g see [3]. g has the property that it becomes infinite at the origin.

††It is easily seen that the function $\beta - r^2$ will be such a function, where $r^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and where β is a positive constant.

Moreover, since $g \rightarrow \infty$ as $x \rightarrow 0$ (this is a property of the fundamental solution; see [3]) we shall accordingly consider only neighborhoods of the origin in which g is positive. Throughout we shall assume $g(x)$ and $p(x)$ to be defined in such a neighborhood $|x| \leq 2\rho$ of the origin. We define

$$(4.4) \quad G(x) = \begin{cases} F(g, p) = \int_{-\infty}^{g/p} \phi(\alpha)(g - \alpha p) d\alpha & \text{for } |x| \leq 2\rho \\ 0 & \text{for } |x| > 2\rho. \end{cases}$$

Here $\phi(\alpha)$ is chosen so that it is in C^2 and

$$(4.5) \quad \phi(\alpha) = 0 \quad \text{for } \alpha \leq \max_{\rho \leq |x| \leq 2\rho} \frac{g(x)}{p(x)} \equiv k,$$

$$(4.6) \quad \phi(\alpha) > 0 \quad \text{for } \alpha > k,$$

$$(4.7) \quad \phi(\alpha) \rightarrow 0 \quad \text{exponentially as } \alpha \rightarrow \infty,$$

$$(4.8) \quad \int_{-\infty}^{\infty} \phi(\alpha) d\alpha = 1.$$

It is easily seen that we have $G(x) = 0$ also for $\rho \leq |x| \leq 2\rho$. Namely, let $\rho \leq |x| \leq 2\rho$. Then $\phi(\alpha) = 0$ for $\alpha \leq g/p$, and the integral in (4.4) vanishes.

5. Proof of (4.1): $L^*G \geq 0$. We start with

$$(5.1) \quad G(x) = F(g, p) = \int_{-\infty}^{g/p} \phi(\alpha)(g - \alpha p) d\alpha.$$

From the chain rule,

$$D_i G = F_g D_i g + F_p D_i p,$$

$$D_i D_j G = D_i D_j g \cdot F_g + D_j F_g \cdot D_i g + D_i D_j p \cdot F_p + D_j F_p \cdot D_i p.$$

Inserting this into

$$L^*G = (b_{ij} D_{ij} + b_i D_i + b)G$$

and using the identity

$$F(g, p) = gF_g + pF_p$$

(a consequence of Euler's theorem on homogeneous functions), we get

$$(5.2) \quad L^*G = L^*g \cdot F_g + L^*p \cdot F_p + b_{ij} D_i F_g \cdot D_j g + b_{ij} D_i F_p \cdot D_j p.$$

Next we note that

$$F_g = \int_{-\infty}^{g/p} \phi(\alpha) d\alpha \geq 0, \quad F_p = -\int_{-\infty}^{g/p} \phi(\alpha) \alpha d\alpha \leq 0,$$

and that these become *strict* inequalities when $g = g(x)$ is sufficiently large,

i.e., for x in a neighborhood of the origin. Thus the first term in (5.2) vanishes, while the second term is non-negative, and actually positive for x sufficiently close to the origin.

Focusing our attention on the last two terms in (5.2), denoting their sum by I , and using the identities

$$D_i F_g = F_{gg} D_i g + F_{gv} D_i p, \quad D_i F_p = F_{vg} D_i g + F_{vp} D_i p,$$

we see that

$$I = b_{ii} \left[F_{gg} \cdot D_i g \cdot D_i g + F_{gv} D_i g \cdot D_i p \right. \\ \left. + F_{vg} D_i g \cdot D_i p + F_{vp} D_i p \cdot D_i p \right],$$

where, as usual, we use the summation convention.

We now denote the linear transformation corresponding to the matrix b_{ii} by B , and the vectors $\text{grad } g$ and $\text{grad } p$ by M and N respectively. Then, since B is symmetric and $F_{vg} = F_{gv}$,

$$I = F_{gg}(BM, M) + 2F_{gv}(BM, N) + F_{vp}(BN, N).$$

Since B is positive definite, it has a positive definite square root and, writing $\tilde{M} = \sqrt{B}M$, $\tilde{N} = \sqrt{B}N$,

$$I = F_{gg}(\tilde{M}, \tilde{M}) + 2F_{gv}(\tilde{M}, \tilde{N}) + F_{vp}(\tilde{N}, \tilde{N})$$

or, in coordinates,

$$I = \sum_{i=1}^n (F_{gg} \tilde{M}_i^2 + 2F_{gv} \tilde{M}_i \tilde{N}_i + F_{vp} \tilde{N}_i^2).$$

We shall now show that the form $F_{gg}\xi^2 + 2F_{gv}\xi\eta + F_{vp}\eta^2$ is non-negative, from which will follow that $I \geq 0$, and hence that L^*G is non-negative, and actually positive for x close to the origin. Using the easily derived identities

$$F_{gg} = \frac{1}{p} \phi\left(\frac{g}{p}\right), \quad F_{gv} = \frac{g^2}{p^3} \phi\left(\frac{g}{p}\right), \quad F_{vp} = \frac{g}{p^2} \phi\left(\frac{g}{p}\right),$$

we see that

$$F_{gg}F_{vp} - F_{gv}^2 = \frac{g^2}{p^4} \phi^2\left(\frac{g}{p}\right) - \frac{g^2}{p^4} \phi^2\left(\frac{g}{p}\right) = 0,$$

showing that the above form is semidefinite; hence it must be non-negative or non-positive. Since F_{gg} and F_{vp} are each non-negative the form is non-negative.

6. Proof of (4.2): $\int L^* G dx = 1$. We have

$$(6.1) \quad G(x) = \int_{-\infty}^{g/p} \phi(\alpha)(g - \alpha p) d\alpha \\ = g(x) + \int_{g/p}^{\infty} \phi(\alpha)(\alpha p - g) d\alpha - \int_{-\infty}^{\infty} \alpha p \phi(\alpha) d\alpha.$$

It is easily verified that $G(x) - g(x)$ belongs to C^2 for x near the origin. (Here we use the fact that $g(x) \rightarrow \infty$ like a negative power of $|x|$ as $x \rightarrow 0$, and that $\varphi(\alpha) \rightarrow 0$ at a faster rate as $\alpha \rightarrow \infty$.)

For ϵ sufficiently small let $g_\epsilon(x)$ agree with $g(x)$ for $|x| \geq \epsilon$, and be modified for $|x| < \epsilon$ in such a way as to be in C^2 , to satisfy $g_\epsilon \rightarrow g$ in the L_1 norm as $\epsilon \rightarrow 0$, and to satisfy

$$(6.2) \quad G_\epsilon(x) \equiv G(x) - g(x) + g_\epsilon(x) \geq 0.$$

Then, by definition,

$$\int L^*G \, dx = \lim_{\epsilon \rightarrow 0} \int_{|x| > \epsilon} L^*G_\epsilon \, dx.$$

Since G_ϵ is in C^2 with compact support, we have (cf. (6.2))

$$\lim_{\epsilon \rightarrow 0} \int_{|x| > \epsilon} L^*G_\epsilon \, dx = -\lim_{\epsilon \rightarrow 0} \int_{|x| < \epsilon} L^*G_\epsilon \, dx$$

and, since $G_\epsilon - g_\epsilon$ is in C^2 and independent of ϵ , the above quantity equals

$$-\lim_{\epsilon \rightarrow 0} \int_{|x| < \epsilon} L^*g_\epsilon \, dx = -\lim_{\epsilon \rightarrow 0} \int_{\mathfrak{D}} L^*g_\epsilon \, dx,$$

the last equality following from the fact that $g_\epsilon = g$ for $|x| \geq \epsilon$ and g , being a fundamental solution to L^* , satisfies $L^*g = 0$ for $x \neq 0$. We wish to show that the last integral equals -1 . Since g is a fundamental solution of L , we have

$$-v(0) = \int gLv \, dx$$

for all smooth v with compact support in $\mathfrak{D}$. Since $g_\epsilon \rightarrow g$ in L_1 ,

$$\lim_{\epsilon \rightarrow 0} \int_{\mathfrak{D}} L^*g_\epsilon v \, dx = \lim_{\epsilon \rightarrow 0} \int_{\mathfrak{D}} g_\epsilon Lv \, dx = \int_{\mathfrak{D}} gLv \, dx = -v(0).$$

If we now choose for v a function with compact support in $\mathfrak{D}$ but equal to one for $|x| < \epsilon'$ (ϵ' sufficiently small) then, since eventually $L^*g_\epsilon = L^*g = 0$ for $|x| > \epsilon'$, we have the desired result:

$$(6.3) \quad \lim_{\epsilon \rightarrow 0} \int_{\mathfrak{D}} L^*g_\epsilon \, dx = -1.$$

7. Proof of (4.3): $f(0) \leq \int_{\mathfrak{D}} fL^*G \, dx$. From the definition of weak L -subharmonicity we have

$$\int_{\mathfrak{D}} fL^*G_\epsilon \, dx \geq 0$$

and hence

$$(7.1) \quad \int_{|x| > \epsilon} fL^*G \, dx = \int_{|x| > \epsilon} fL^*G_\epsilon \, dx \geq -\int_{|x| < \epsilon} fL^*G_\epsilon \, dx.$$

Since $g_\epsilon - G_\epsilon$ is in C^2 (cf. (6.1), (6.2)) and is independent of ϵ , we have

$$\lim_{\epsilon \rightarrow 0} \int_{|x| < \epsilon} u L^* G_\epsilon \, dx = \lim_{\epsilon \rightarrow 0} \int_{|x| < \epsilon} u L^* g_\epsilon \, dx.$$

Now let g_ϵ be chosen in such a way that, in addition to the previous requirements on g_ϵ , we have

$$L^* g_\epsilon \leq 0.$$

(Such a g_ϵ can easily be created by a construction similar to the one used to construct G from g and p . See the appendix.) Since u is continuous at the origin, given $\eta > 0$ we can find an $\epsilon > 0$ such that $\epsilon \rightarrow 0$ as $\eta \rightarrow 0$ and

$$|f(0) - f(x)| \leq \eta \quad \text{in } |x| < \epsilon.$$

Hence

$$\left| \int_{|x| < \epsilon} (f(x) - f(0)) L^* g_\epsilon \, dx \right| \leq -\eta \int_{|x| < \epsilon} L^* g_\epsilon \, dx,$$

and in the limit, as $\eta \rightarrow 0$, $\epsilon \rightarrow 0$ and the integral on the right approaches -1 by (6.3), implying that the integral on the left tends to zero and that

$$\begin{aligned} (7.2) \quad \lim_{\epsilon \rightarrow 0} \int_{|x| < \epsilon} f(x) L^* G_\epsilon \, dx &= f(0) \lim_{\epsilon \rightarrow 0} \int_{|x| < \epsilon} L^* G_\epsilon \, dx \\ &= f(0) \lim_{\epsilon \rightarrow 0} \int_{|x| < \epsilon} L^* g_\epsilon \, dx = -f(0). \end{aligned}$$

Finally, letting $\epsilon \rightarrow 0$ in (7.1) and applying (7.2), we are left with the desired result.

APPENDIX

A1. Calabi's definition. According to the definition of E. CALABI [1]

$$Lu \geq 0 \quad \text{weakly}$$

in a domain if u is upper semicontinuous there, and if given any ϵ and any point x_0 in the domain there exists a twice continuously differentiable function $v(x)$, defined in a neighborhood $N(\epsilon, x_0)$ of x_0 and depending on ϵ, x_0 , such that

$$u(x) - v(x)$$

assumes its minimum at x_0 and such that

$$Lv \geq -\epsilon$$

in N .

Now suppose u satisfies CALABI's definition. Then for x in $N(\epsilon, x_0)$ we have

$$u(x) - v(x) \geq u(x_0) - v(x_0)$$

and

$$u(x) - u(x_0) \geq v(x) - v(x_0).$$

Since v is in C^2 , hence continuous,

$$\liminf_{x \rightarrow x_0} (u(x) - u(x_0)) \geq \lim_{x \rightarrow x_0} v(x) - v(x_0) = 0,$$

from which it follows that

$$\liminf_{x \rightarrow x_0} u(x) \geq u(x_0),$$

that is, u is lower semicontinuous at x_0 . But since x_0 was arbitrary and $u(x)$ is assumed to be upper semicontinuous, we may conclude that $u(x)$ must be continuous in the whole domain.

A2. Construction of g_ϵ . From the function g we wish to construct g_ϵ agreeing with g for $|x| \geq \epsilon$, being twice continuously differentiable (even at the origin) and satisfying $L^*g_\epsilon \leq 0$.

If $p(x)$ again denotes the positive L^* -superharmonic function $p(x) = \beta - r^2$ of Section 4, then define

$$g_\epsilon = \int_0^\infty \min(g, \alpha p) \varphi_\epsilon(\alpha) d\alpha = p \int_0^{g/\nu} \alpha \varphi_\epsilon(\alpha) d\alpha + g \int_{g/\nu}^\infty \varphi_\epsilon(\alpha) d\alpha \equiv Q(g, p).$$

By calculations similar to those carried out in Section 5, it follows that if $\varphi_\epsilon(\alpha)$ is an appropriately chosen non-negative function with compact support, depending on ϵ , with integral equal to one, then the above function g_ϵ will have all the required properties.

A3. $L^*W^\nu \geq 0$. We have

$$W^\nu(x) = \int_r^\infty \left(\frac{1}{r^k} - \frac{1}{R^k} \right) \phi(R) dR,$$

where ϕ is as defined in section 3.

$$D_i W^\nu = -\frac{\xi_i}{r^{k+1}} k \int_r^\infty \phi(R) dR,$$

where $\xi_i = x_i/r$. Noting that

$$D_i \xi_j = \frac{\delta_{ij}}{r} - \frac{\xi_i \xi_j}{r}$$

there results (no summation)

$$D_i D_j W^\nu = \xi_i \xi_j \frac{(k+2)k}{r^{k+2}} \int_r^\infty \phi(R) dR + \xi_i \xi_j \frac{k}{r^{k+1}} \phi(r) - \frac{\delta_{ij}}{r^{k+2}} k \int_r^\infty \phi(R) dR.$$

Hence

$$\begin{aligned} L^*W^u &= b_{ij}\xi_i\xi_j\left[\frac{k(k+2)}{r^{k+2}}\int_r^\infty\phi(R)\,dR+\frac{k}{r^{k+1}}\phi(r)\right] \\ &\quad -\frac{k}{r^{k+2}}\operatorname{Trace}\,[b_{ij}]\int_r^\infty\phi(R)\,dR-b_i\xi_i\left[\frac{k}{r^{k+1}}\int_r^\infty\phi(R)\,dR\right] \\ &\quad +b\frac{1}{r^k}\int_r^\infty\phi(R)\,dR-b\int_r^\infty\frac{1}{R^k}\phi(R)\,dR. \end{aligned}$$

Taking $k \geq 1$, $R_0 \leq 1$, we see, using the ellipticity and the fact that $a_{ij} = b_{ij}$, that

$$L^*W^u \geq [mk(k+2) - kK]\frac{1}{r^{k+2}}\int_r^\infty\phi(R)\,dR.$$

Here m , K are positive constants depending only on the coefficients, not on k . Clearly, we can, by choosing k large, make the bracketed expression positive.

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