*Developments at an Analytic Corner of Solutions of Elliptic Partial Differential Equations**

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1. Introduction. This paper is concerned with the asymptotic behavior at a corner of a solution of the Dirichlet problem for a linear second order elliptic partial differential equation in two independent variables. The corner point is assumed to be a common end point of two analytic boundary arcs, on each of which the given boundary values are analytic. Thus the problem treated here is one with piecewise analytic boundary conditions. This work continues a line of investigation begun by H. LEWY [6] concerned with finding asymptotic developments for solutions of boundary value problems of this kind. In this paper we shall prove that it is possible to obtain an infinite asymptotic development for the solution of the Dirichlet problem. The case of Laplace's equation has already been studied by W. WASOW [11], and his results make it possible to reduce the boundary value problem considered here to one with zero boundary values.

Let D be an open domain of the (x, y) plane, the boundary of which in the neighborhood of the origin consists of two analytic arcs Γ_1 and Γ_2 . Suppose Γ_1 and Γ_2 meet at the origin to form a corner with interior angle $\alpha\pi > 0$, and suppose the origin is a regular point of both curves. To fix the notation, suppose that the tangent to Γ_1 at the origin goes into the tangent to Γ_2 at the origin by a positive rotation through the angle $\alpha\pi$. We shall consider the Dirichlet problem for the equation

$$(1.1) \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + A \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y} + Cu = f$$

where A , B , C , and f are analytic functions of the two variables x and y in a neighborhood of the origin. We shall study functions $u(x, y)$ which are solutions

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of (1.1) and in the class C^2 inside the domain D , are continuous on $D \cup \Gamma_1 \cup \Gamma_2$, and satisfy the boundary conditions

$$(1.2) \quad \begin{aligned} \mathfrak{u}(x, y) &= \varphi_1(x, y) \quad \text{on } \Gamma_1, \\ \mathfrak{u}(x, y) &= \varphi_2(x, y) \quad \text{on } \Gamma_2, \end{aligned}$$

where φ_1 and φ_2 are analytic in a neighborhood of the origin and take the same value at the origin. We shall also need a rather weak assumption concerning the behavior of the derivatives at the origin. We suppose that for $(x, y) \rightarrow (0, 0)$ in $D \cup \Gamma_1 \cup \Gamma_2$

$$(1.3) \quad \begin{aligned} \frac{\partial \mathfrak{u}}{\partial x} &= o(r^{-2+\delta}), \quad \frac{\partial \mathfrak{u}}{\partial y} = o(r^{-2+\delta}), \quad \text{if } \alpha \leq 1; \\ \frac{\partial \mathfrak{u}}{\partial x} &= o(r^{-1/\alpha-1+\delta}), \quad \frac{\partial \mathfrak{u}}{\partial y} = o(r^{-1/\alpha-1+\delta}), \quad \text{if } \alpha > 1, \end{aligned}$$

where $r = \sqrt{x^2 + y^2}$ and δ is a small positive constant.[†]

The problem treated here is entirely a local one. Consequently we make no assumptions concerning the domain, its boundary, or the boundary conditions outside a neighborhood of the origin. Thus D may be multiply connected or $\mathfrak{u}$ may satisfy entirely different boundary conditions on other portions of the boundary.

Results proved by the author in [5] make it possible to transform the problem for D into one for the upper half plane. A representation of the solution in terms of the ordinary Green's function for the half plane is then constructed. By an appropriate decomposition of the Green's function one obtains two-dimensional integrals which are analogous to certain one-dimensional integrals studied in [6] and [4]. These integrals can be extended as analytic functions of two complex variables. A study of them makes it possible to construct an iterative procedure for obtaining asymptotic developments.

In the extension to the complex domain we make essential use of a method of H. LEWY [7] which can be regarded as a generalization of the reflection principle to equation (1.1). In the case of analytic linear boundary conditions a significant statement concerning the extent of the domain of extension can be made; in the real plane the extension covers the same domain as reflection of a harmonic function with zero boundary values.

The developments obtained for the solution are valid not only in the real plane but also in a complex domain containing the real plane. An advantage of asymptotic developments of this type, which are valid in the complex domain

[†]Professors A. FRIEDMAN and W. LITTMAN have kindly pointed out to me that the estimates (1.3) are weak consequences of the other assumptions as can be seen by using the Schauder interior estimates. An exact argument could use the interior estimates together with LEWY's reflection principle discussed in §5 to obtain estimates for the derivatives in terms of distance from the corner.

as well as the real domain, is that they can be differentiated formally to obtain asymptotic developments for the derivatives of the solution. Thus it is possible to obtain information about the behavior of the derivatives at the corner. This information can be used to study the approximation of solutions of boundary value problems by finite difference methods (see LAASONEN [3], FORSYTHE [1], WASOW [12]).

The principal new results are stated in §3. Some additional results concerned with the case where the boundary values are discontinuous at the corner are stated in §10.

2. Notation. If the functions A , B , C , and f are real analytic functions of x and y in a neighborhood of the origin, they can be continued as analytic functions to complex values of x and y near the origin. It is convenient to introduce two complex variables z and z^* by the formulas

$$(2.1) \quad z = x + iy, \quad z^* = x - iy.$$

The functions A , B , C , and f can then be considered as analytic functions of z and z^* in a neighborhood of the origin. The real plane where x and y take real values is made up of the points where $z^* = \bar{z}$. We have the relations

$$(2.2) \quad \frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial z^*} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$$

which follow from (2.1) if x and y are considered as complex variables. On the other hand, these relations permit us to define differentiation with respect to z and z^* for points in the real plane, where $z^* = \bar{z}$, without knowing the function elsewhere. We set

$$(2.3) \quad U(z, z^*) = \mathfrak{U}(x, y).$$

The equation (1.1) then becomes

$$(2.4) \quad 4U_{zz^*} + (A + iB)U_z + (A - iB)U_{z^*} + CU = f.$$

The function $U(z, z^*)$ is thus defined originally for z in D and $z^* = \bar{z}$. We shall be interested in an analytic continuation of U to points (z, z^*) where z^* is not necessarily equal to $\bar{z}$.

Often we shall consider z and z^* as variables each ranging over a logarithmic Riemann surface with branch point at the origin. A point $z = re^{i\theta}$ on this surface is uniquely determined by specifying a positive real number r and a real number θ . As usual we set $|z| = r$, $\arg z = \theta$, and by $\log z$ we mean the branch of the logarithm which is real for $\arg z = 0$. By $\bar{z}$ we mean the point $re^{-i\theta}$.

Let S_1 be a sector $\theta_1 \leq \arg z \leq \theta_2$ on the logarithmic Riemann surface and S_2 a sector $\theta_3 \leq \arg z^* \leq \theta_4$. We shall be concerned with limit relations of a special type. Let (z_n, z_n^*) be a sequence of points of the Cartesian product $S_1 \times S_2$. Then by $(z_n, z_n^*) \xrightarrow{s} 0$ in $S_1 \times S_2$ we shall mean that $\lim_{n \rightarrow \infty} z_n = 0$, $\lim_{n \rightarrow \infty} z_n^* = 0$, and there is a constant $K > 1$ such that

$$(2.5) \quad \frac{1}{K} < \left| \frac{z_n}{z_n^*} \right| < K$$

for all n . By stating that some proposition holds for $(z, z^*) \xrightarrow{s} 0$ in $S_1 \times S_2$ we shall mean that the proposition holds for any sequence of points of $S_1 \times S_2$ which approaches the origin in this special sense.

For partial differential equations we are primarily interested in the behavior in the real plane. A limit statement in the complex domain implies a corresponding one in the real domain; for if z_n is any sequence of points which approaches the origin in a sector S_1 , then $(z_n, \bar{z}_n) \xrightarrow{s} 0$ in $S_1 \times \bar{S}_1$.

Let us consider power products of $z, z^*, \log z$, and $\log z^*$ with nonnegative exponents. We shall say that the product

$$z^\beta (\log z)^m z^{*\gamma} (\log z^*)^n$$

has degree $(\beta + \gamma, m + n)$. There is a natural ordering of these products analogous to the ordering of powers of z according to increasing degree. We say $(\mu, k) < (\mu', k')$ provided either $\mu < \mu'$ or $\mu = \mu'$ and $k > k'$. Then if $(\beta + \gamma, m + n) < (\beta' + \gamma', m' + n')$ we have

$$\frac{|z^{\beta'} (\log z)^{m'} z^{*\gamma'} (\log z^*)^{n'}|}{|z^\beta (\log z)^m z^{*\gamma} (\log z^*)^n|} \rightarrow 0$$

as $(z, z^*) \xrightarrow{s} 0$ in $S_1 \times S_2$ where S_1 and S_2 are any finite sectors. We emphasize here the use that we make of the restriction (2.5).

Now we can explain what type of asymptotic expansion we shall be considering. Let q be a positive integer and let

$$\mathfrak{Q}(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*)$$

be a formal power series in its six arguments. For each positive real number γ let

$$P_\gamma(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*)$$

be the "polynomial" obtained from $\mathfrak{Q}$ by omitting all terms of degree higher than $(\gamma, 0)$. Then we say

$$g(z, z^*) \sim \mathfrak{Q}(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*)$$

for $(z, z^*) \xrightarrow{s} 0$ in $S_1 \times S_2$ if for every γ

$$(2.6) \quad g(z, z^*) = P_\gamma(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*) + o(z^\gamma)$$

for $(z, z^*) \xrightarrow{s} 0$ in $S_1 \times S_2$.

We shall also consider finite asymptotic developments. If there is a polynomial P_γ with all terms of degree $\leq (\gamma, 0)$ such that (2.6) holds, we say $g(z, z^*)$ has a finite development with remainder term $o(z^\gamma)$.

We shall distinguish two different kinds of asymptotic expansions by the use of the letters " $\mathfrak{Q}$ " and " $\mathfrak{N}$ ". By writing $g(z, z^*) \sim \mathfrak{N}$ we shall mean that the constant term in the power series $\mathfrak{N}$ is not zero; by writing $g(z, z^*) \sim \mathfrak{Q}$ we admit the possibility that the constant term is zero.

To make the notation somewhat more flexible we shall sometimes write expressions such as $g(z, z^*) \sim z^{1/\alpha} \mathfrak{A}$ to mean the same as $z^{-1/\alpha} g(z, z^*) \sim \mathfrak{A}$, etc.

We have just explained our notation for power series and polynomials in six particular arguments, but of course similar definitions can be given for power series and polynomials in a finite number of arguments each of which is a power product of z , z^* , $\log z$, and $\log z^*$ containing a nonnegative power of z or z^* . When none of the arguments contains z^* or $\log z^*$, the expansions reduce to those considered in [6] and [5].

3. Principal results. We now state our main theorems. Theorems 3.1 and 3.2 require knowledge of the notation described in §2. Theorems 3.3 and 3.4 are special cases which do not require familiarity with our notation.

Let S be an arbitrary finite sector on the logarithmic Riemann surface, $\theta_1 \leq \arg z \leq \theta_2$ with $\theta_1 < \theta_2$. By a neighborhood of the origin in $S \times S$ we mean a set of points (z, z^*) such that

$$\begin{aligned} 0 < |z| < \epsilon, \quad \theta_1 \leq \arg z \leq \theta_2, \\ 0 < |z^*| < \epsilon, \quad \theta_1 \leq \arg z^* \leq \theta_2. \end{aligned}$$

Theorem 3.1. Let $U(z, \bar{z}) = \mathfrak{U}(x, y)$ be a solution of the boundary value problem stated in the introduction. Then U has an analytic continuation $U(z, z^*)$ defined on a neighborhood of the origin in $S \times S$. If α is irrational, then

$$U(z, z^*) \sim \mathfrak{A}(z, z^{1/\alpha}, z^*, z^{*1/\alpha})$$

for $(z, z^*) \rightarrow 0$ in $S \times S$, where $\mathfrak{A}$ is a formal power series in $z, z^{1/\alpha}, z^*,$ and $z^{*1/\alpha}$. If $\alpha = p/q$, a reduced fraction, then

$$U(z, z^*) \sim \mathfrak{A}(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*)$$

for $(z, z^*) \rightarrow 0$ in $S \times S$, where $\mathfrak{A}$ is a formal power series in its six arguments. Moreover, expansions for derivatives of arbitrary order are obtained by formally differentiating.

The expansion for $\mathfrak{U}$ in the real plane in terms of x and y can be obtained by using (2.1).

In the case of the homogeneous equation with zero boundary values, that is for $f \equiv 0$, $\varphi_1(x, y) \equiv 0$, $\varphi_2(x, y) \equiv 0$, we can prove a stronger result.

Theorem 3.2. Let $U(z, \bar{z}) = \mathfrak{U}(x, y)$ be a solution of the Dirichlet problem with $f \equiv 0$ and $\mathfrak{U} = 0$ on Γ_1 and Γ_2 . Then if α is irrational,

$$U(z, z^*) \sim z^{1/\alpha} \mathfrak{A}(z, z^{1/\alpha}, z^*, z^{*1/\alpha}) + z^{*1/\alpha} \mathfrak{A}'(z, z^{1/\alpha}, z^*, z^{*1/\alpha})$$

for $(z, z^*) \rightarrow 0$ in $S \times S$, where $\mathfrak{A}$ and $\mathfrak{A}'$ are formal power series in their arguments. If $\alpha = p/q$, a reduced fraction, then

$$\begin{aligned} U(z, z^*) \sim z^{1/\alpha} \mathfrak{A}(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*) \\ + z^{*1/\alpha} \mathfrak{A}'(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*) \end{aligned}$$

for $(z, z^*) \rightarrow 0$ in $S \times S$. Moreover, expansions for derivatives of arbitrary order are obtained by formally differentiating.

If one is interested in the leading terms of these expansions, the following corollaries are worthy of note.

Theorem 3.3. *Let $\mathfrak{u}(x, y)$ be a solution of the boundary value problem of the introduction. Then*

$$\mathfrak{u}(x, y) = \mathfrak{u}_1(x, y) + \mathfrak{u}_2(x, y)$$

where $\mathfrak{u}_1(x, y)$ and its partial derivatives of all orders remain bounded when (x, y) approaches the origin, and

$$\mathfrak{u}_2(x, y) = O(r^{1/\alpha})$$

for $\alpha \neq 1/m$ (m an integer) while

$$\mathfrak{u}_2(x, y) = O(r^{1/\alpha} \log r)$$

when $\alpha = 1/m$ ($m = 1, 2, 3, \dots$). These relations may be indefinitely formally differentiated.

We call attention to the fact that this theorem is exactly the same as WASOW's Theorem 5 in [11] for Laplace's equation.

Again we obtain a stronger result in the case of a homogeneous equation with zero boundary values. Let r and θ be polar coordinates with $\theta = 0$ the line tangent to Γ_1 and $\theta = \pi\alpha$ tangent to Γ_2 .

Theorem 3.4. *Let $\mathfrak{u}(x, y)$ be a real solution of the boundary value problem of the introduction with $f = 0$ and $\mathfrak{u} = 0$ on Γ_1 and Γ_2 . Then there is a constant γ such that, for $(x, y) \rightarrow (0, 0)$ in $D \cup \Gamma_1 \cup \Gamma_2$,*

$$\mathfrak{u}(x, y) = \gamma r^{1/\alpha} \sin(\theta/\alpha) + o(r^{1/\alpha}).$$

If $\alpha \neq 1/m$ ($m = 1, 2, \dots$), the relations obtained for partial derivatives of arbitrary order by differentiating formally are valid. If $\alpha = 1/m$, the relations obtained by formally differentiating are valid for derivatives of order $< m$. The derivatives of order $\leq 2m - 1$ are bounded, the derivatives of order $2m$ are $O(\log r)$, and those of order $n > 2m$ are $O(r^{2m-n})$.

Thus for higher derivatives we obtain better results in the exceptional cases $\alpha = 1/m$ than we do for nearby values of α .

A reduction of Theorem 3.4 to Theorem 3.2 can be carried out by showing that the sum of the constant term in $\mathfrak{a}$ times $z^{1/\alpha}$ and the constant term in $\mathfrak{a}'$ times $\bar{z}^{1/\alpha}$ must be $\gamma r^{1/\alpha} \sin(\theta/\alpha)$. This can be seen by using the fact that $\mathfrak{u}$ is 0 on Γ_1 and is real on a ray with argument $\frac{1}{2}\alpha\pi$. The remainder of the theorem follows easily by consideration of the asymptotic expansion.

For a particular equation or a particular boundary value problem it is, of course, possible to say much more concerning the structure of the corresponding

asymptotic expansions. Such additional information is probably best gained by taking into account the fact that the asymptotic expansion must formally satisfy the differential equation and formally satisfy the boundary conditions. For example, if one considers the equation $\Delta u + \lambda u = 0$ with boundary values given on Γ_1 and Γ_2 meeting at an angle $\alpha\pi$ with α irrational, one finds for a solution of such a problem

$$U(z, z^*) \sim \sum_{m,n} \left[A_{mn} \left(\sqrt{\frac{z}{z^*}} \right)^{m+n/\alpha} + B_{mn} \left(\sqrt{\frac{z^*}{z}} \right)^{m+n/\alpha} \right] J_{m+n/\alpha}(\sqrt{\lambda z z^*}),$$

where the sum is over nonnegative integers m and n and the function $J_{m+n/\alpha}$ is a Bessel function of order $m + n/\alpha$. In the real domain this reduces for $z = re^{i\theta}$ to an expansion

$$u(x, y) \sim \sum_{m,n} (a_{mn} \cos(m + n/\alpha)\theta + b_{mn} \sin(m + n/\alpha)\theta) J_{m+n/\alpha}(\sqrt{\lambda} r).$$

Now let us further restrict the functions considered by choosing particular boundary conditions. For example, suppose we require $u = 0$ on $\theta = 0$ and $\theta = \alpha\pi$. Then one finds

$$u(x, y) \sim \sum_{n=1}^{\infty} c_n \sin(n\theta/\alpha) J_{n/\alpha}(\sqrt{\lambda} r).$$

The way we prove the existence of asymptotic expansions in this paper, although the proof is constructive, is, nevertheless, far too complicated for finding asymptotic expansions for applications. Hence Theorems 3.1 and 3.2 should be regarded as existence theorems, and methods of deriving these expansions in particular cases, such as described above, are important. I hope to discuss questions connected with such applications in a future paper.

4. On asymptotic expansions. If two functions have asymptotic expansions of the type considered in §2 which are both formal power series in the same arguments, then a linear combination with constant coefficients also has an asymptotic expansion, the formal power series obtained in the obvious way. This follows immediately from the definition given. Also the formal power series obtained by multiplying the expansions for the two functions is an asymptotic expansion for the product of the two functions.

With regard to uniqueness of asymptotic expansions we must be more careful if $\alpha = p/q$, for then

$$(4.1) \quad z^{p/\alpha} = z^q, \quad z^{*p/\alpha} = z^{*q}.$$

Consequently we regard two power series $\mathcal{Q}$ and $\mathcal{Q}'$ in $z, z^{1/\alpha}, z^*, z^{*1/\alpha}, z^q \log z$, and $z^{*q} \log z^*$ as the same if use of the rules (4.1) plus collection of corresponding terms produces the same power series. In other words, with regard to uniqueness, we consider the canonical power series

$$\sum A_{klmn\mu\nu} z^{k+l/\alpha} z^{*m+n/\alpha} (z^q \log z)^\mu (z^{*q} \log z^*)^\nu$$

where k , m , μ , and ν run independently over all nonnegative integers while l and n run only over the integers $0, 1, 2, \dots, p-1$.

Similarly we consider two finite developments P_γ and P'_γ as the same if use of the rules (4.1) plus collection of corresponding terms produces the same polynomial.

Lemma 4.1. *Consider an arbitrary function $g(z, z^*)$ only for $z^* = \bar{z}$. Still $g(z, \bar{z})$ has at most one finite development with remainder term $o(z^\gamma)$ where $z^* = \bar{z}$ and $z \rightarrow 0$ in a sector S , $\theta_1 \leq \arg z \leq \theta_2$, $\theta_1 < \theta_2$. Moreover, $g(z, \bar{z})$ has at most one asymptotic expansion for $z \rightarrow 0$ in S .*

Proof. First of all we observe that if g has at most one finite development for each γ then it cannot have more than one asymptotic expansion. Hence it will suffice to prove the statement with respect to finite developments. This we prove by contradiction. Suppose there were a real number γ such that $g(z, \bar{z})$ had two finite developments with remainder term $o(z^\gamma)$. Then the difference of these developments would be a development for 0 with remainder term $o(z^\gamma)$ for $z \rightarrow 0$ in S which did not have all coefficients in its canonical series equal to 0. Suppose a term of degree (β, N) with $\beta \leq \gamma$, $N \geq 0$ is a term of lowest degree which does not vanish. Then we consider all terms of degree (β, k) where k is any nonnegative integer $\leq N$. If we set $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, we obtain for $r \rightarrow 0$ with $\theta_1 \leq \theta \leq \theta_2$

$$\begin{aligned} 0 &= \sum_{i=1}^J z^{\beta_i} \bar{z}^{\beta-\beta_i} Q_i(\log z, \log \bar{z}) + o(z^\beta) \\ &= \sum_{i=1}^J r^\beta (e^{i\beta_i\theta - i(\beta-\beta_i)\theta}) Q_i(\log r + i\theta, \log r - i\theta) + o(r^\beta), \end{aligned}$$

where the β_i are all distinct and the Q_i are polynomials in $(\log r + i\theta)$ and $(\log r - i\theta)$. It follows that

$$\sum_{i=1}^J e^{2i\beta_i\theta} Q_i(\log r + i\theta, \log r - i\theta) = 0$$

for $\theta_1 \leq \theta \leq \theta_2$, but by analytic continuation this identity must then hold for all complex numbers θ , and it is then easy to see that we must have

$$Q_j(\log r + i\theta, \log r - i\theta) = 0 \quad (j = 1, 2, \dots, J)$$

for all r and θ . It follows that all coefficients in the polynomials Q_j must vanish. This yields a contradiction and hence the lemma must hold.

A great advantage of asymptotic expansions of analytic functions is that under appropriate conditions they can be differentiated. We shall use the following lemma for functions of two complex variables.

Lemma 4.2. *Let S_1 be a sector on the logarithmic Riemann surface, $\theta_1 \leq \arg z \leq \theta_2$ with $\theta_1 < \theta_2$, and let S_2 be a sector $\theta_3 \leq \arg z^* \leq \theta_4$ with $\theta_3 < \theta_4$. Let $g(z, z^*)$ be analytic for $0 < |z| < \rho$, $0 < |z^*| < \rho$, (z, z^*) in $S_1 \times S_2$; and let β*

be any real number. Suppose, furthermore, that

$$g(z, z^*) = o(z^\beta)$$

for $(z, z^*) \xrightarrow{*} 0$ in $S_1 \times S_2$. Then if S'_1 is any closed sector in the interior of S_1 and S'_2 is any closed sector in the interior of S_2 ,

$$\frac{\partial^{m+n}}{\partial z^m \partial z^{*n}} g(z, z^*) = o(z^{\beta-m-n})$$

for $(z, z^*) \xrightarrow{*} 0$ in $S'_1 \times S'_2$.

Proof. By the hypothesis of the lemma, given $\epsilon > 0$ and a constant $K_0 > 1$ it is possible to find an $R < \rho$ such that (see definition of $\xrightarrow{*} 0$ in §2)

$$(4.2) \quad |g(\zeta, \zeta^*)| < \epsilon |\zeta|^\beta$$

provided (ζ, ζ^*) is in $S_1 \times S_2$, $0 < |\zeta| < R$, $0 < |\zeta^*| < R$, and $K_0^{-1} < |\zeta/\zeta^*| < K_0$. Since S'_1 and S'_2 are in the interiors of S_1 and S_2 , there is a constant $\delta > 0$ such that if (z, z^*) is any point in $S'_1 \times S'_2$, a circle of radius $\delta |z|$ about z will lie inside S_1 and a circle of radius $\delta |z^*|$ about z^* will lie inside S_2 .

Now consider any sequence of points $(z_k, z_k^*) \xrightarrow{*} 0$ in $S'_1 \times S'_2$, i.e., such that $z_k \rightarrow 0$, $z_k^* \rightarrow 0$, and $K_1^{-1} < |z_k/z_k^*| < K_1$ for some constant K_1 . Let C_k be the circle about z_k with radius $\delta |z_k|$, and let C_k^* be the circle about z_k^* with radius $\delta |z_k^*|$. Now we choose $K_0 = K_1(1 + \delta)/(1 - \delta)$. If k is large enough so that $|z_k| < \frac{1}{2}R$ and $|z_k^*| < \frac{1}{2}R$ then the product of the disks enclosed by C_k and C_k^* will lie inside the domain where g is analytic. Also if ζ is any point on C_k and ζ^* any point on C_k^* then (4.2) will be satisfied. By Cauchy's formula we have

$$\frac{\partial^{m+n}}{\partial z^m \partial z^{*n}} g(z_k, z_k^*) = \frac{-m!n!}{4\pi^2} \int_{C_k} \int_{C_k^*} \frac{g(\zeta, \zeta^*)}{(\zeta - z)^{m+1}(\zeta^* - z^*)^{n+1}} d\zeta d\zeta^*.$$

Hence for k sufficiently large

$$\begin{aligned} \left| \frac{\partial^{m+n}}{\partial z^m \partial z^{*n}} g(z_k, z_k^*) \right| &\leq \epsilon m!n! |z_k|^{\beta-m} |z_k^*|^{-n} \delta^{-m-n} \cdot \max((1 + \delta)^\beta, (1 - \delta)^\beta) \\ &\leq \epsilon |z_k|^{\beta-m-n} K_1^n \delta^{-m-n} m!n! \cdot \max((1 + \delta)^\beta, (1 - \delta)^\beta). \end{aligned}$$

The conclusion of the lemma follows by letting $\epsilon \rightarrow 0$.

Remark. We observe that a slight modification of this proof establishes that if in the lemma "o" is replaced by "O" the resulting statement is still true.

5. Extension by reflection. The solution $U(z, z^*)$ of (2.4) is originally defined only for z in $D \cup \Gamma_1 \cup \Gamma_2$, $z^* = \bar{z}$. In obtaining an asymptotic expansion of U we shall need to consider an analytic continuation of $U(z, z^*)$ where z and z^* are independent complex variables each ranging over a logarithmic Riemann surface. Let $\mathfrak{D}(\Theta, \rho)$ be the domain on the logarithmic Riemann surface defined by $0 < |z| < \rho$, $|\arg z| \leq \Theta$. In this section we shall prove that for every $\Theta > 0$

there is a number $\rho > 0$ such that $U(z, z^*)$ can be continued analytically to $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. We shall also obtain estimates for U and its derivatives as $(z, z^*) \rightarrow 0$ in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$.

To continue U analytically we use a reflection method due to H. LEWY [7] which allows reflection of a solution of (2.4) across an analytic curve where an analytic boundary condition is prescribed. In the case of a linear boundary condition this method takes a particularly simple form and it is then possible to determine exactly the domain covered by reflection. In case one stays within the domain of analyticity of the coefficients A, B, C , the free term f , and the boundary condition φ , reflection of a solution of (2.4) covers exactly the same domain as reflection of a harmonic function with zero boundary values.

Theorem 5.1. (LEWY) *Let D be a simply connected open domain whose boundary contains an open segment σ of the real axis. Let A, B, C , and f be analytic for z and z^* in $D \cup \sigma \cup \bar{D}$ and let $\varphi(z)$ be analytic for z in $D \cup \sigma \cup \bar{D}$. Suppose $U(z, \bar{z})$ is in C^1 for z in $D \cup \sigma$, is a solution of the equation (2.4) for z in D , $z^* = \bar{z}$, and satisfies the boundary condition $U(z, \bar{z}) = \varphi(z)$ for z on σ . Then U can be continued as an analytic function of z and z^* to the Cartesian product $(D \cup \sigma \cup \bar{D}) \times (D \cup \sigma \cup \bar{D})$.*

In LEWY's proof it is assumed that D is a schlicht domain, but the result can be immediately extended to domains D on a logarithmic Riemann surface such as we consider.

The above theorem assumes that U is in the class C^1 for z in $D \cup \sigma$. However, it follows from a result of L. LICHTENSTEIN (see [8], especially pp. 561–568) that if U is continuous in $D \cup \sigma$ and the function φ , considered as a function of arc length, is in C^2 , then a solution of the Dirichlet problem must be in C^1 for z in $D \cup \sigma$. LICHTENSTEIN's proof actually refers to boundaries which are curves with continuous curvature. In order to apply his result in our case we need only consider a subsegment of σ and join the ends with a curve inside D to form a boundary curve with continuous curvature. Also LICHTENSTEIN requires that the boundary values be in C^2 for the entire boundary, but to obtain the local result requires only a local assumption. Hence without loss of generality we can assume that U is in C^1 for z in $D \cup \sigma$.

By means of conformal mapping we can interpret LEWY's theorem as a statement concerning reflection across any analytic curve. The domain covered in the real plane will be the same as would be obtained by reflection of a harmonic function with zero boundary values. Let us, for brevity, describe the region between Γ_1 and Γ_2 in a neighborhood of the origin by the inequality $0 \leq \arg z \leq \pi\alpha$ although actually, even after a rotation, the lines $\arg z = 0$ and $\arg z = \pi\alpha$ are only the tangents to Γ_1 and Γ_2 at the origin. Then reflection across Γ_1 extends $U(z, \bar{z})$ to $-\pi\alpha \leq \arg z \leq \pi\alpha$. Following this by reflection across Γ_2 extends $U(z, \bar{z})$ to $-\pi\alpha \leq \arg z \leq 3\pi\alpha$. Then reflection across Γ_1 extends U to $-3\pi\alpha \leq \arg z \leq 3\pi\alpha$, etc. Alternating in this fashion, we can extend U so that it is defined in a neighborhood of the origin in any finite sector

however large, but this repeated reflection may shrink the circle about the origin in which U is analytic. Thus we find that, for every $\Theta > 0$, $U(z, \bar{z})$ is defined in a domain $\mathfrak{D}(\Theta, \rho)$. It follows from LEWY's theorem that $U(z, z^*)$ can be continued as an analytic function to the product domain $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$.

We shall need an estimate for $U(z, z^*)$ as $(z, z^*) \rightarrow_s 0$ in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. To obtain this estimate we repeat a portion of LEWY's proof using the explicit formulas for reflection. (See [7], equations (3.12)–(3.16)). We shall consider first the special case in which one of the arcs Γ_1 or Γ_2 is a segment σ of the positive real axis with one end point at the origin. The general case can then be reduced to this one by a short additional argument involving conformal mapping.

The Riemann function $R(z, z^*, t, t^*)$ is an analytic function of four complex variables which, because of the assumptions concerning the coefficients, is analytic in a neighborhood of $(0, 0, 0, 0)$. Let (p) be a rectifiable path between $\bar{z}^*$ and z which stays within a domain for which the integrands below are defined. Also let

$$a(z, z^*) = \frac{1}{4}[A(z, z^*) + iB(z, z^*)], \quad b(z, z^*) = \frac{1}{4}[A(z, z^*) - iB(z, z^*)].$$

Then

$$\begin{aligned} U(z, z^*) &= U(\bar{z}^*, z^*)R(z, z^*, \bar{z}^*, z^*) + \int_{\bar{z}^*}^z \{ [U(t, \bar{l})R(z, z^*, t, \bar{l})]_t \\ (5.1) \quad &- U(t, \bar{l})[R_t(z, z^*, t, \bar{l}) - b(t, \bar{l})R(z, z^*, t, \bar{l})] \\ &+ [U(t, \bar{l})(R_t - aR)] d\bar{l} \} - \frac{1}{4} \int_{\bar{z}^*}^z \left\{ \int_{z^*}^{\bar{l}} R(z, z^*, t, t^*)f(t, t^*) dt^* \right\} dt, \end{aligned}$$

where in both integrals the variable t ranges over the path (p) joining $\bar{z}^*$ to z , and in the last integral t^* ranges over the segment of the conjugate path $(\bar{p})$ between z^* and $\bar{l}$. It can be shown that the sum of the two integrals is independent of the path (p) joining $\bar{z}^*$ to z .

Let ω be an arbitrary point of the segment σ of Theorem 5.1. We have ([7], (3.14); there ω is the origin)

$$\begin{aligned} U(z, \bar{z}) &= -U(\omega, \omega)R(z, \bar{z}, \omega, \omega) + U(z, \omega)R(z, \bar{z}, z, \omega) \\ &\quad + U(\omega, \bar{z})R(z, \bar{z}, \omega, \bar{z}) \\ (5.2) \quad &- \int_{\omega}^z U(t, \omega)[R_t(z, \bar{z}, t, \omega) - b(t, \omega)R(z, \bar{z}, t, \omega)] dt \\ &- \int_{\omega}^{\bar{z}} U(\omega, \bar{l})[(R_t(z, \bar{z}, \omega, \bar{l}) - a(\omega, \bar{l})R(z, \bar{z}, \omega, \bar{l})] d\bar{l} \\ &+ \frac{1}{4} \int_{\omega}^z \left\{ \int_{\omega}^{\bar{z}} R(z, \bar{z}, t, t^*)f(t, t^*) dt^* \right\} dt. \end{aligned}$$

A special case of this formula is obtained when z is on σ where $z = \bar{z}$ and $U(z, \bar{z}) = \varphi(z)$. We have

$$\begin{aligned}
(5.3) \quad \varphi(z) = & -\varphi(\omega)R(z, z, \omega, \omega) + U(z, \omega)R(z, z, z, \omega) + U(\omega, z)R(z, z, \omega, z) \\
& - \int_{\omega}^z U(t, \omega)(R_i(z, z, t, \omega) - b(t, \omega)R) dt \\
& - \int_{\omega}^z U(\omega, t)(R_i(z, z, \omega, t) - a(\omega, t)R) dt \\
& + \frac{1}{4} \int_{\omega}^z \int_{\omega}^z R(z, z, t, t^*)f(t, t^*) dt^* dt.
\end{aligned}$$

Using these formulas let us see how one continues $U(z, z^*)$, which is defined originally only for z in $D \cup \sigma$, $z^* = \bar{z}$, to the larger domain $(D \cup \sigma \cup \bar{D}) \times (D \cup \sigma \cup \bar{D})$. First of all, knowing $U(t, \bar{t})$ for all t in $D \cup \sigma$, we can use (5.1) to find $U(z, \omega)$ for all z in $D \cup \sigma$. Similarly, by (5.1) we can find $U(\omega, z^*)$ for all z^* in $\bar{D} \cup \sigma$. Then (5.3) is a Volterra integral equation for $U(\omega, z)$ for z in $D \cup \sigma$ since both $\varphi(z)$ and $U(z, \omega)$ are known there. Consequently, solution of this integral equation provides an analytic continuation of $U(\omega, z)$ to $D \cup \sigma \cup \bar{D}$. Similarly, (5.3) is also a Volterra integral equation for $U(z, \omega)$ with z in $\bar{D} \cup \sigma$. Solution of it provides a continuation of $U(z, \omega)$ to $D \cup \sigma \cup \bar{D}$. Next (5.2) can be used to obtain $U(z, \bar{z})$ for z in $D \cup \sigma \cup \bar{D}$, and finally by using (5.1) we obtain $U(z, z^*)$ for the entire product domain $(D \cup \sigma \cup \bar{D}) \times (D \cup \sigma \cup \bar{D})$.

Now let us see how estimates of the form

$$(5.4) \quad U_z(z, \bar{z}) = o(z^{\lambda-1}), \quad U_{\bar{z}}(z, \bar{z}) = o(\bar{z}^{\lambda-1})$$

with $\lambda < 0$, known originally for $z \rightarrow 0$ in $D \cup \sigma$, can be used to find estimates for U , U_z , and $U_{\bar{z}}$ as $(z, z^*) \rightarrow 0$ in $(D \cup \sigma \cup \bar{D}) \times (D \cup \sigma \cup \bar{D})$. We shall not use the assumption that $U(z, \bar{z})$ is continuous at the origin, but instead we derive a weaker estimate by integration of the derivatives.

We have for $z \rightarrow 0$ in $D \cup \sigma$

$$\begin{aligned}
U(z, \bar{z}) &= \int_{\omega}^z \{U_i(t, \bar{t}) dt + U_{\bar{i}}(t, \bar{t}) d\bar{t}\} + U(\omega, \omega) \\
&= \int_{\omega}^z \{o(t^{\lambda-1}) dt + o(\bar{t}^{\lambda-1}) d\bar{t}\} + O(1) = o(z^{\lambda})
\end{aligned}$$

since $\lambda < 0$; we assume that the path of integration has the following two properties: (i) For every point t on the path $|z| \leq |t| \leq |\omega|$. (ii) For every $r > 0$ the length of the part of the path for which $|t| \leq r$ is less than some constant times r . For domains of the type we are considering such a path can always be found. Similar assumptions are made for all paths of integration considered in the following discussion.

Now we repeat the extension of U , this time estimating the functions obtained. By (5.1) we have, for $z \rightarrow 0$ in $D \cup \sigma$,

$$U(z, \omega) = O(1) + \int_{\omega}^z \{o(\bar{t}^{\lambda-1}) d\bar{t} + o(t^{\lambda}) dt\} + O(1) = o(z^{\lambda}).$$

Similarly, by (5.1) we find $U(\omega, z^*) = o(z^{*\lambda})$ for $z^* \rightarrow 0$ in $\bar{D} \cup \sigma$. Next, the customary iterative solution of the Volterra integral equation (5.3) yields the same estimates for $z \rightarrow 0$ in the larger domain $D \cup \sigma \cup \bar{D}$ and $z^* \rightarrow 0$ in $D \cup \sigma \cup \bar{D}$. Then by (5.2) we have

$$U(z, \bar{z}) = O(1) + o(z^\lambda) + o(z^\lambda) - \int_\omega^z o(t^\lambda) dt - \int_\omega^{\bar{z}} o(\bar{t}^\lambda) d\bar{t} + O(1) = o(z^\lambda)$$

for $z \rightarrow 0$ in $D \cup \sigma \cup \bar{D}$. It is also necessary to have an estimate for $U_z(z, \bar{z})$ as $z \rightarrow 0$ in $D \cup \sigma \cup \bar{D}$. Such an estimate can be obtained by differentiating (5.2). It is then easy to see that the dominant term for $U_z(z, \bar{z})$ is $U_z(z, \omega) R(z, \bar{z}, z, \omega)$. This term can then be estimated for $z \rightarrow 0$ in a sector which, near the origin, is inside $D \cup \sigma \cup \bar{D}$ by Lemma 4.2. It follows that $U_z(z, \bar{z}) = o(z^{\lambda-1})$ for $z \rightarrow 0$ in a sector in the interior of $D \cup \sigma \cup \bar{D}$. Let S be such a sector. Then we can use these estimates with (5.1) to obtain an estimate for $U(z, z^*)$. We find

$$U(z, z^*) = o(z^\lambda)$$

for $(z, z^*) \xrightarrow{s} 0$ in $S \times S$. It follows from Lemma 4.2 that

$$(5.5) \quad U_z(z, z^*) = o(z^{\lambda-1}), \quad U_{z^*}(z, z^*) = o(z^{\lambda-1})$$

as $(z, z^*) \xrightarrow{s} 0$ in $S \times S$.

Thus the estimates (5.4) are preserved under reflection, or, more precisely, they hold in any sector which near the origin is inside the domain obtained by reflection.

To reflect, when the analytic arc Γ_1 reflected across is not a segment of the real axis, we use a conformal mapping. Let $\psi(z)$ be analytic at the origin with $\psi(0) = 0$, $\psi'(0) \neq 0$, and suppose ψ maps the analytic arc Γ_1 onto the segment σ . We set†

$$U_1(z, z^*) = U(\psi(z), \bar{\psi}(z^*)).$$

Then U_1 satisfies an equation of the same type as (2.3) in the domain D_1 which is the image of a part of D near the origin. It is also easy to verify that the derivatives of U_1 satisfy estimates of the form (5.4) for $z \rightarrow 0$ in D_1 . The extension of U_1 then satisfies the estimates (5.5) in the larger domain. Then, since

$$U(z, z^*) = U_1(\psi^{-1}(z), \bar{\psi}^{-1}(z^*)),$$

we obtain the estimates for U in a larger domain. In the same way we can see that the estimates (5.4) are preserved for reflection across Γ_2 . By alternately reflecting across Γ_1 and Γ_2 we find after a finite number of reflections

†Throughout this paper, if ψ is an analytic function of one complex variable, we denote by $\bar{\psi}$ the analytic function for which $\bar{\psi}(\bar{z}) = \overline{\psi(z)}$. If ψ has a power series expansion about the origin, the coefficients of the series for $\bar{\psi}$ are the conjugates of those for ψ . Also we use ψ^{-1} to denote the function which is the inverse of ψ and $\bar{\psi}^{-1}$ to denote the function which is the inverse of $\bar{\psi}$.

$$(5.6) \quad U(z, z^*) = o(z^\lambda)$$

for $(z, z^*) \xrightarrow{s} 0$ in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. By (1.3) we can take $\lambda = -1 + \delta$ for $\alpha \leq 1$ and $\lambda = -1/\alpha + \delta$ for $\alpha > 1$.

6. A preliminary transformation. We now make a preliminary transformation which simplifies the domain at the expense of complicating the differential equation. For this purpose we consider the mapping function $F(z)$ which conformally maps the upper half plane onto a neighborhood of the origin in D in such a way that the origin is mapped onto the origin, the arc Γ_1 is mapped onto a segment of the positive real axis, and Γ_2 is mapped onto a segment of the negative real axis. We shall use results from [5] concerning the asymptotic behavior of $F(z)$ for $z \rightarrow 0$.

We define a new function of z and z^* by the formula (see the footnote on p. 739)

$$(6.1) \quad u(z, z^*) = U(F(z), \bar{F}(z^*)).$$

If $U(z, z^*)$ is known for z in D , $z^* = \bar{z}$, this formula determines $u(z, z^*)$ for z in the upper half plane $0 \leq \arg z \leq \pi$, $z^* = \bar{z}$. A simple calculation shows that $u(z, z^*)$ satisfies the equation

$$(6.2) \quad 4 \frac{\partial^2 u(z, z^*)}{\partial z \partial z^*} + a_1(z, z^*) \frac{\partial u}{\partial z} + a_2(z, z^*) \frac{\partial u}{\partial z^*} + a_3(z, z^*) u = a_4(z, z^*)$$

where

$$(6.3) \quad \begin{aligned} a_1(z, z^*) &= [A(F(z), \bar{F}(z^*)) + iB(F(z), \bar{F}(z^*))]\bar{F}'(z^*), \\ a_2(z, z^*) &= [A(F(z), \bar{F}(z^*)) - iB(F(z), \bar{F}(z^*))]F'(z), \\ a_3(z, z^*) &= C(F(z), \bar{F}(z^*))F'(z)\bar{F}'(z^*), \\ a_4(z, z^*) &= f(F(z), \bar{F}(z^*))F'(z)\bar{F}'(z^*). \end{aligned}$$

We shall use the following lemma proved in [5].†

Lemma 6.1. *The mapping function $F(z)$ can be continued analytically for $|z|$ sufficiently small to any finite sector S . If α is irrational*

$$F(z) \sim z^\alpha \mathfrak{N}_1(z, z^\alpha)$$

for $z \rightarrow 0$ in S , where $\mathfrak{N}_1(z, z^\alpha)$ is a formal power series with nonvanishing constant term. If $\alpha = p/q$, a reduced fraction,

†I would like to take this opportunity to correct an error in [5]. L. LICHTENSTEIN [9] proved that $dF^{-1}(z)/dz \sim cz^{1/\alpha-1}$ for $z \rightarrow 0$ with $c \neq 0$. I stated incorrectly that he had proved this result only for irrational α . In fact, LICHTENSTEIN proved this result for all α . He also stated the result obtained by formal differentiation for higher derivatives but proved it only for irrational α . However, this lack of proof for the result concerning higher derivatives can perhaps be attributed to the fact that it was not well known at that time that one can differentiate such asymptotic relations, a fact which seems to have been first pointed out by RRRR [10].

$$F(z) \sim z^\alpha \mathfrak{N}_1(z, z^\alpha, z^p \log z)$$

for $z \rightarrow 0$ in S , where $\mathfrak{N}_1$ is a formal power series in z, z^α , and $z^p \log z$ with non-vanishing constant term. Also if α is irrational, the inverse function

$$F^{-1}(z) \sim z^{1/\alpha} \mathfrak{N}_2(z, z^{1/\alpha});$$

if $\alpha = p/q$, then

$$F^{-1}(z) \sim z^{1/\alpha} \mathfrak{N}_2(z, z^{1/\alpha}, z^q \log z).$$

The functions $a_i(z, z^*)$ are, in general, not analytic at the origin. They are however analytic functions on a product domain $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. More precisely, for every $\Theta > 0$ there is a number $\rho > 0$ such that $a_i(z, z^*)$ is analytic for (z, z^*) in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$ where $\mathfrak{D}(\Theta, \rho)$ is the domain $0 < |z| < \rho$, $|\arg z| \leq \Theta$. We shall need expansions for the functions $a_i(z, z^*)$ as $(z, z^*) \xrightarrow{*} 0$ in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. Using the fact that A, B, C , and f are analytic at the origin together with Lemmas 4.2 and 6.1, we find for $\alpha = p/q$, a reduced fraction,

$$\begin{aligned} (6.4) \quad a_1(z, z^*) &\sim z^{*\alpha-1} \mathfrak{A}_1(z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*), \\ a_2(z, z^*) &\sim z^{\alpha-1} \mathfrak{A}_2(z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*), \\ a_3(z, z^*) &\sim z^{\alpha-1} z^{*(\alpha-1)} \mathfrak{A}_3(z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*), \\ a_4(z, z^*) &\sim z^{\alpha-1} z^{*(\alpha-1)} \mathfrak{A}_4(z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*). \end{aligned}$$

If α is irrational we obtain similar expansions but with no powers of $z^p \log z$ or $z^{*p} \log z^*$. These expansions can be obtained in a straightforward manner by using the finite developments given by the power series for A, B, C , and f about the origin and the finite developments obtained by using the initial sections of the asymptotic expansions for $F(z), \bar{F}(z^*), F'(z)$, and $\bar{F}'(z^*)$.

In the same way we obtain estimates for the coefficients of (6.2) as $(z, z^*) \xrightarrow{*} 0$ in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. We have

$$\begin{aligned} (6.5) \quad a_1(z, z^*) &= O(z^{\alpha-1}), & a_2(z, z^*) &= O(z^{\alpha-1}), \\ a_3(z, z^*) &= O(z^{2\alpha-2}), & a_4(z, z^*) &= O(z^{2\alpha-2}). \end{aligned}$$

Similarly, by (6.1) and (5.6) we have

$$(6.6) \quad u(z, z^*) = o(z^{\alpha\lambda}) = \begin{cases} o(z^{-\alpha+\alpha\delta}) & \text{if } \alpha \leq 1, \\ o(z^{-1+\alpha\delta}) & \text{if } \alpha > 1 \end{cases}$$

for $(z, z^*) \xrightarrow{*} 0$ in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. Hence also

$$(6.7) \quad \frac{\partial u}{\partial z}(z, z^*) = o(z^{\alpha\lambda-1}), \quad \frac{\partial u}{\partial z^*}(z, z^*) = o(z^{\alpha\lambda-1}).$$

We shall also use the following theorem due to Wasow [11].

Theorem 6.1. *Let $W(z, \bar{z})$ be a harmonic function in D which is continuous at the origin and satisfies the boundary conditions $W(z, \bar{z}) = \varphi_1(z)$ on Γ_1 , $W(z, \bar{z}) =$*

$\varphi_2(z)$ on Γ_2 . Then $W(z, \bar{z}) = \operatorname{Re} G(z)$ where $G(z)$ has an asymptotic expansion in powers of z and $z^{1/\alpha}$ when α is irrational and in powers of $z, z^{1/\alpha}$, and $z^\alpha \log z$ when $\alpha = p/q$.

For our study we let $W(z, \bar{z})$ be the harmonic function which takes the same boundary values as $U(z, \bar{z})$. By the above theorem

$$W(z, \bar{z}) = \frac{1}{2}[G(z) + \overline{G(z)}] = \frac{1}{2}[G(z) + \bar{G}(\bar{z})].$$

Hence W can be extended as an analytic function of z and z^* with

$$W(z, z^*) = \frac{1}{2}[G(z) + \bar{G}(z^*)].$$

In essence, Wasow's work will permit us to reduce the problem of studying the asymptotic behavior to the special one for zero boundary values.

We let $w(z, z^*)$ be the *harmonic function* which takes the same boundary values as $u(z, z^*)$ so that

$$w(z, z^*) = W(F(z), \bar{F}(z^*)).$$

Combining Wasow's expansion for $G(z)$ with Lemma 6.1, we find

$$(6.8) \quad w(z, z^*) \sim \alpha_5(z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*)$$

for $(z, z^*) \rightarrow_s 0$ in $\mathfrak{D}(\Theta, \rho) \times \mathfrak{D}(\Theta, \rho)$. This can be readily verified by considering initial segments of the asymptotic expansions for $G(z)$, $\bar{G}(z^*)$, $F(z)$, and $\bar{F}(z^*)$. If α is irrational we obtain an expansion of this type without powers of $z^p \log z$ and $z^{*p} \log z^*$.

7. A representation in terms of Green's function and its continuation. In this section we shall obtain a representation of $u(z, z^*)$ using the ordinary Green's function for the upper half plane. As in §6 we let $w(z, z^*)$ be the harmonic function which has the same boundary values as u . Let Θ be a number $> 2\pi$. Let $S(\Theta, R)$ be the Cartesian product domain consisting of those points (z, z^*) for which

$$\begin{aligned} 0 < |z| &\leq R, & |\arg z| &\leq \Theta, \\ 0 < |z^*| &\leq R, & |\arg z^*| &\leq \Theta. \end{aligned}$$

The results of the previous sections may be summarized by saying that for every Θ there is a positive number R such that $u(z, z^*)$, $w(z, z^*)$, and $a_i(z, z^*)$ ($i = 1, 2, 3, 4$) are analytic in the domain $S(\Theta, R)$. Generally speaking, if Θ is large we must expect to have to choose a small R .

Let z be a point such that $0 < |z| < R$, $0 < \arg z < \pi$. Let H_ϵ be the domain in the $\zeta = \xi + i\eta$ plane consisting of the half circle $|\zeta| \leq R$, $\eta \geq 0$, with the exception of a small circle of radius ϵ about the point z and a half circle of radius ϵ about the origin. Applying Green's formula to this domain, we obtain

$$\iint_{H_\epsilon} [(u(\zeta, \bar{\zeta}) - w(\zeta, \bar{\zeta}))\Delta v - v\Delta(u - w)] d\xi d\eta + \int_{C_\epsilon} \left[(u - w) \frac{\partial v}{\partial n} - v \frac{\partial(u - w)}{\partial n} \right] ds = 0$$

where C_ϵ is the boundary of H_ϵ and differentiation is with respect to the inner normal. For v we use the Green's function for the upper half plane

$$v = G(z, \zeta) = -\frac{1}{2\pi} \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right|$$

which takes the value 0 for $\eta = 0$ and satisfies the equation $\Delta v = 0$ in the interior of H_ϵ .

Let γ be the circumference of the circle of radius ϵ about z , and let γ' be the arc of the small half circle about the origin. Letting $\epsilon \rightarrow 0$ we observe that

$$\int_\gamma \left[(u - w) \frac{\partial G}{\partial n} - G \frac{\partial(u - w)}{\partial n} \right] ds \rightarrow w(z, \bar{z}) - u(z, \bar{z})$$

while

$$\begin{aligned} \int_{\gamma'} \left[(u - w) \frac{\partial G}{\partial n} - G \frac{\partial(u - w)}{\partial n} \right] ds &= [o(\epsilon^{-1+\alpha\delta})O(1) - O(\epsilon)o(\epsilon^{-2+\alpha\delta})]O(\epsilon) \\ &= o(\epsilon^{\alpha\delta}) = o(1), \end{aligned}$$

as can be seen by using (6.6) and the fact that $G(z, \zeta)$ is the real part of a function of ζ which is analytic at the origin.

Thus we obtain for $0 < |z| < R$, $0 < \arg z < \pi$

$$\begin{aligned} (7.1) \quad u(z, \bar{z}) &= \iint_H G(z, \zeta) \left[a_1(\zeta, \bar{\zeta}) \frac{\partial u}{\partial \zeta}(\zeta, \bar{\zeta}) + a_2(\zeta, \bar{\zeta}) \frac{\partial u}{\partial \bar{\zeta}}(\zeta, \bar{\zeta}) \right. \\ &\quad \left. + a_3(\zeta, \bar{\zeta})u - a_4(\zeta, \bar{\zeta}) \right] d\xi d\eta \\ &\quad + w(z, \bar{z}) + \int_\Gamma \left[(u - w) \frac{\partial G}{\partial n} - G \frac{\partial(u - w)}{\partial n} \right] ds \end{aligned}$$

where H is the half circle $|\zeta| \leq R$, $\eta \geq 0$ and Γ is the portion of the circumference in the upper half plane. The ξ axis from $-R$ to R yields no contribution to the boundary integral because both $u - w$ and G vanish there. The integral over H is absolutely convergent, for at $\zeta = z$ the integrand has at most a logarithmic singularity and at $\zeta = 0$ the integrand is $o(\zeta^{\alpha\delta-2})$ by (6.5)–(6.7).

The formula (7.1) was derived under the assumption that $0 < |z| < R$, $0 < \arg z < \pi$, $z^* = \bar{z}$. We wish to obtain a continuation of this formula to a larger range of values of z and z^* . In particular, we want to drop the restriction that $z^* = \bar{z}$. To begin with we observe that an analytic continuation of the integral over Γ can be obtained by replacing $G(z, \zeta)$ by

$$-\frac{1}{4\pi} \log \frac{(\zeta - z)(\bar{\zeta} - z^*)}{(\zeta - z^*)(\bar{\zeta} - z)}$$

which for $z^* = \bar{z}$ agrees with $G(z, \zeta)$. The integral over Γ where $|\zeta| = R$ then becomes an analytic function of z and z^* for $|z| < R$, $|z^*| < R$. The branch of the logarithm is only considered for $|\zeta| = R$ and is the one which takes real values for $z^* = \bar{z}$, $|z| < R$ and is continued through values of z and z^* for which $|z| < R$, $|z^*| < R$. Similarly, we replace $w(z, \bar{z})$ by $w(z, z^*)$ which is analytic on $S(\Theta, R)$.

The continuation of the integral over H presents more difficulty. To obtain such a continuation we introduce a linear operator T , acting on analytic functions, which corresponds to multiplication by $G(z, \zeta)$ and integration over H . The introduction of this operator is the major new idea of this paper, and a study of this operator will enable us to set up an iteration process to obtain an asymptotic expansion for $u(z, z^*)$.

We shall now state three lemmas which will be proved in this section and §8. These lemmas are concerned with functions $g(z, z^*)$ which are analytic in $S(\Theta, R)$ and satisfy the estimate

$$(7.2) \quad g(z, z^*) = o(z^{-2+\epsilon})$$

for $(z, z^*) \rightarrow 0$ in $S(\Theta, R)$ where ϵ is some positive constant. (Observe the restriction on the domain where g is assumed to be analytic.)

Lemma 7.1. *There is a linear operator T given explicitly by (7.4)–(7.6) which transforms a function $g(z, z^*)$ which is analytic in $S(\Theta, R)$ and satisfies the estimate (7.2) into a function $Tg(z, z^*)$ analytic in the smaller domain $S(\Theta' - 2\pi, R')$ where Θ' is any number less than Θ and R' is any number less than R . For $0 < \arg z < \pi$, $z^* = \bar{z}$ the operator reduces to one given by the formula*

$$Tg(z, \bar{z}) = \iint_H G(z, \zeta) g(\zeta, \bar{\zeta}) d\xi d\eta$$

where H is the domain $|\zeta| \leq R$, $0 \leq \arg \zeta \leq \pi$.

Lemma 7.2. Let

$$g(z, z^*) = z^\beta z^{*\gamma} (\log z)^m (\log z^*)^n$$

where β and γ are real numbers > -1 and m and n are nonnegative integers. Then

$$Tg(z, z^*) = z^{\beta+\gamma+2} P_1(\log z) + z^{*\beta+\gamma+2} P_2(\log z^*) \\ + z^{\beta+1} z^{*\gamma+1} P_3(\log z, \log z^*) + p_1(z) + p_2(z^*)$$

where P_1 and P_2 are polynomials in their arguments of degrees $\leq m+n$ if $\beta+\gamma$ is not an integer and of degrees $\leq m+n+1$ if $\beta+\gamma$ is an integer; P_3 is a polynomial in $\log z$ and $\log z^*$ which is of degree $\leq m$ in $\log z$ and of degree $\leq n$ in $\log z^*$; $p_1(z)$ is a power series in z convergent for $|z| < R$ with $p_1(0) = 0$; and $p_2(z^*)$ is a power series in z^* convergent for $|z^*| < R$ with $p_2(0) = 0$.

Lemma 7.3. *Let γ be a real number > -2 which is not an integer. Suppose $g(z, z^*) = o(z^\gamma)$ for $(z, z^*) \rightarrow_s 0$ in $S(\Theta, R)$. Then*

$$Tg(z, z^*) = Q_1(z) + Q_2(z^*) + o(z^{\gamma+2})$$

for $(z, z^) \rightarrow_s 0$ in $S(\Theta - 2\pi, R)$, where Q_1 and Q_2 are polynomials of degrees $< \gamma + 2$. Also $Q_1(0) = Q_2(0) = 0$, and if $-2 < \gamma < -1$ both polynomials vanish identically.*

To motivate the definition of the operator T and at the same time obtain a proof of Lemma 7.1 we begin with the integral

$$\iint_H G(z, \zeta) g(\zeta, \bar{\zeta}) d\zeta d\eta.$$

For $z^* = \bar{z}$ we have

$$\begin{aligned} G(z, \zeta) &= -\frac{1}{2\pi} \log \left| \frac{\zeta - z}{\zeta - \bar{z}} \right| \\ &= -\frac{1}{4\pi} \left\{ \log \left(1 - \frac{z}{\zeta} \right) - \log \left(1 - \frac{z}{\bar{\zeta}} \right) + \log \left(1 - \frac{z^*}{\bar{\zeta}} \right) - \log \left(1 - \frac{z^*}{\zeta} \right) \right\}, \end{aligned}$$

provided the branches of the logarithms are chosen appropriately. If we break the above integral into four corresponding parts, we obtain two-dimensional integrals which are analogous to one-dimensional integrals studied in [4] and [6]. We can, therefore, hope that an appropriate treatment of these integrals will lead to an iterative procedure for obtaining asymptotic expansions.

We introduce polar co-ordinates or, more precisely, we introduce two new variables ρ and t with ρ ranging over real numbers between 0 and R and t a complex variable ranging over the unit circle. We set $\zeta = \rho t$. With $z^* = \bar{z}$ we find

$$\begin{aligned} (7.3) \quad G(z, \zeta) &= -\frac{1}{4\pi} \left\{ \log \left(1 - \frac{z}{\rho t} \right) - \log \left(1 - \frac{t\bar{z}}{\rho} \right) \right. \\ &\quad \left. + \log \left(1 - \frac{tz^*}{\rho} \right) - \log \left(1 - \frac{z^*}{\rho t} \right) \right\}. \end{aligned}$$

In making this decomposition we must specify what branches of the logarithms are meant. Since later we shall regard ρ as a complex variable, let us fix z and t and consider $\log(1 - z/\rho t)$ as a function of ρ . We make a cut in the ρ plane along the ray from 0 to z/t and consider in the remainder of the plane the branch of $\log(1 - z/\rho t)$ whose imaginary part is $> -\pi$ and $< \pi$. The logarithm is thus defined uniquely for all complex ρ except those on the cut. There, for the moment, we leave it undefined. We observe that for $\arg \rho = \arg z/t$, $|\rho| > |z/t|$ the logarithm is real.

The branches of the other three logarithms in (7.3) are specified similarly. For $\log(1 - tz^*/\rho)$ the cut is along the ray from 0 to tz^* , for $\log(1 - z^*/\rho t)$ it

runs from 0 to tz^* , and for $\log(1 - z^*/\rho t)$ it is from 0 to z^*/t . Each branch is required to have imaginary part between $-\pi$ and π .

With these definitions equation (7.3) is then valid for z in H , $z^* = \bar{z}$, $|t| = 1$, and $\rho > 0$, provided $\rho > |z|$ if $\arg t = \arg z$. To verify this, we observe that the following congruences hold modulo $2\pi i$:

$$\begin{aligned}\log\left(1 - \frac{z}{\rho t}\right) + \log\left(1 - \frac{t\bar{z}}{\rho}\right) &\equiv \log\left|1 - \frac{z}{\rho t}\right|^2, \\ \log\left(1 - \frac{tz}{\rho}\right) + \log\left(1 - \frac{\bar{z}}{\rho t}\right) &\equiv \log\left|1 - \frac{\bar{z}}{\rho t}\right|^2.\end{aligned}$$

However, each of the logarithms on the left has imaginary part $> -\pi$ and $< \pi$. Hence the sum of two of them must have imaginary part $> -2\pi$ and $< 2\pi$, and consequently each of the two sums is real and we can replace congruence by equality. Equation (7.3) then follows with $\zeta = \rho t$, $\bar{\zeta} = \rho/t$.

Observe that for fixed z the set of ζ for which (7.3) does not hold because the right side is undefined is just the portion of the ray $\arg \zeta = \arg z$ for which $|\zeta| \leq |z|$. This set has plane measure zero, and hence can be ignored in the following integrals; in fact, it is convenient to omit the entire ray $\arg \zeta = \arg z$. For z in H , $z^* = \bar{z}$ we have

$$-4\pi i \iint_H G(z, \zeta) g(\zeta, \bar{\zeta}) d\xi d\eta = T_1 g(z, z^*) - T_2 g(z) + T_3 g(z, z^*) - T_4 g(z^*)$$

where

$$\begin{aligned}T_1 g(z, z^*) &= \int_1^{e^{i\pi}} \int_0^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{z}{\rho t}\right) \rho d\rho \frac{dt}{t}, \\ T_2 g(z) &= \int_1^{e^{i\pi}} \int_0^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{tz}{\rho}\right) \rho d\rho \frac{dt}{t}, \\ T_3 g(z, z^*) &= \int_1^{e^{i\pi}} \int_0^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{tz^*}{\rho}\right) \rho d\rho \frac{dt}{t}, \\ T_4 g(z^*) &= \int_1^{e^{i\pi}} \int_0^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{z^*}{\rho t}\right) \rho d\rho \frac{dt}{t}.\end{aligned}$$

Our aim is to extend the functions $T_i g$ analytically to points where $z^* \neq \bar{z}$.

Let us consider $T_1 g(z, z^*)$. We fix z in H , fix $z^* = \bar{z}$, and consider the inner integral over ρ for fixed t . Taking advantage of the analyticity of g , we treat the integral over ρ as a Cauchy integral. We change the path of integration by a continuous deformation into a path which runs alongside the cut from 0 to z/t and then goes from z/t to R in such a way that for all points of this part of the path, except the two end points, $|z/t| < |\rho| < R$ and $\arg \rho$ is between 0 and $\arg(z/t)$. Observe that we obtain two essentially different deformations depending upon whether $\arg t < \arg z$ or $\arg t > \arg z$ (see Figs. 1 and 2). If $\arg t < \arg z$

FIG. 1. The path of integration in the ρ -plane for $\arg t < \arg z$.

the ray from 0 to R is to the right of the ray from 0 to z/t , but if $\arg t > \arg z$ the ray from 0 to R is to the left of the ray from 0 to z/t . As we have mentioned previously, we need not consider the points where $\arg t = \arg z$.

Now let us determine the imaginary part of $\log(1 - z/\rho t)$ on the part of the path which is along the cut. First assume $\arg t < \arg z$. Then on a nearby path $z/\rho t$ would be in the upper half plane and have absolute value greater than 1; therefore along the cut $\text{Im} \log(1 - z/\rho t) = -\pi$. On the other hand, if $\arg t > \arg z$, then on a nearby path $z/\rho t$ would be in the lower half plane and have absolute value greater than 1. Hence, in this case, along the cut $\text{Im} \log(1 - z/\rho t) = \pi$.

FIG. 2. The path of integration in the ρ -plane for $\arg t > \arg z$.

Setting $\tau = \rho t/z$, we find for $\arg t < \arg z$

$$\int_0^{z/t} g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{z}{\rho t}\right) \rho d\rho = \int_0^1 g\left(\tau z, \frac{\tau z}{t^2}\right) \left[\log\left(\frac{1}{\tau} - 1\right) - \pi i \right] \left(\frac{z}{t}\right)^2 \tau d\tau,$$

while for $\arg t > \arg z$

$$\int_0^{z/t} g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{z}{\rho t}\right) \rho d\rho = \int_0^1 g\left(\tau z, \frac{\tau z}{t^2}\right) \left[\log\left(\frac{1}{\tau} - 1\right) + \pi i \right] \left(\frac{z}{t}\right)^2 \tau d\tau.$$

If $z^* = \bar{z}$, then $\arg z = \arg \sqrt{z/z^*}$.[†] Moreover, $\sqrt{z/z^*}$ is a point on the unit circle, and hence it is the point where the changeover from $\arg t < \arg z$ to $\arg t > \arg z$ occurs. Hence for z in H , $z^* = \bar{z}$ we have

$$\begin{aligned} T_1 g(z, z^*) &= \int_1^{e^{i\pi}} \left\{ \int_0^1 g\left(\tau z, \frac{\tau z}{t^2}\right) \left[\log\left(\frac{1}{\tau} - 1\right) - \pi i \right] \left(\frac{z}{t}\right)^2 \tau d\tau \right\} \frac{dt}{t} \\ &\quad + \int_{\sqrt{z/z^*}}^{e^{i\pi}} \left\{ \int_0^1 g\left(\tau z, \frac{\tau z}{t^2}\right) 2\pi i \left(\frac{z}{t}\right)^2 \tau d\tau \right\} \frac{dt}{t} \\ &\quad + \int_1^{e^{i\pi}} \left\{ \int_{z/t}^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{z}{\rho t}\right) \rho d\rho \right\} \frac{dt}{t} \\ &\equiv I_1(z) + I_2(z, z^*) + I_3(z). \end{aligned} \tag{7.4}$$

It is this formula which we will use to extend T_1 in $S(\Theta - 2\pi, R)$ to values of z and z^* where $z^* \neq \bar{z}$. The only additional stipulation necessary is the path of integration from $\sqrt{z/z^*}$ to $e^{i\pi}$. We require for every point t of the path that

$$\min(1, |\sqrt{z/z^*}|) \leq |t| \leq \max(1, |\sqrt{z/z^*}|)$$

and also that $\arg t$ be between $\arg \sqrt{z/z^*}$ and π .

Similar considerations lead to the following formulas:

$$\begin{aligned} T_2 g(z) &= \int_1^{e^{i\pi}} \left\{ \int_0^1 g(t^2 \tau z, \tau z) \left[\log\left(\frac{1}{\tau} - 1\right) - \pi i \right] (tz)^2 \tau d\tau \right\} \frac{dt}{t} \\ &\quad + \int_1^{e^{i\pi}} \left\{ \int_{tz}^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{tz}{\rho}\right) \rho d\rho \right\} \frac{dt}{t}, \\ T_3 g(z, z^*) &= \int_1^{e^{i\pi}} \left\{ \int_0^1 g(t^2 \tau z^*, \tau z^*) \left[\log\left(\frac{1}{\tau} - 1\right) + \pi i \right] (tz^*)^2 \tau d\tau \right\} \frac{dt}{t} \\ &\quad - \int_{\sqrt{z/z^*}}^{e^{i\pi}} \left\{ \int_0^1 g(t^2 \tau z^*, \tau z^*) 2\pi i (tz^*)^2 \tau d\tau \right\} \frac{dt}{t} \\ &\quad + \int_1^{e^{i\pi}} \left\{ \int_{tz^*}^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{tz^*}{\rho}\right) \rho d\rho \right\} \frac{dt}{t}, \end{aligned} \tag{7.5}$$

[†]By $\sqrt{z/z^*}$ we mean the point on the logarithmic Riemann surface with absolute value $\sqrt{|z|/|z^*|}$ and argument $\frac{1}{2}(\arg z - \arg z^*)$.

$$T_4 g(z^*) = \int_1^{e^{i\pi}} \left\{ \int_0^1 g\left(\tau z^*, \frac{\tau z^*}{t^2}\right) \left[\log\left(\frac{1}{\tau} - 1\right) + \pi i \right] \left(\frac{z^*}{t}\right)^2 \tau d\tau \right\} \frac{dt}{t}, \\ + \int_1^{e^{i\pi}} \left\{ \int_{z^*/t}^R g\left(\rho t, \frac{\rho}{t}\right) \log\left(1 - \frac{z^*}{\rho t}\right) \rho d\rho \right\} \frac{dt}{t}.$$

We then define the linear operator T by the equation

$$(7.6) \quad Tg(z, z^*) = \frac{i}{4\pi} \{T_1 g(z, z^*) - T_2 g(z) + T_3 g(z, z^*) - T_4 g(z^*)\}.$$

Observe that for T_2 and T_4 there is no change in the imaginary part of the logarithm for different values of t . This is the case because when z is in H the ray from 0 to R is always to the right of the ray from 0 to tz and the ray from 0 to R is always to the left of the one from 0 to $\bar{z}/t$.

To complete the proof of Lemma 7.1 we must prove that if $g(z, z^*)$ is analytic in $S(\Theta, R)$ then $Tg(z, z^*)$ is defined and analytic in $S(\Theta' - 2\pi, R)$ where $\Theta' < \Theta$. We have

$$T_1 g(z, z^*) = \int_0^{e^{i\pi}} \int_0^1 + \int_{\sqrt{z/z^*}}^{e^{i\pi}} \int_0^1 + \int_1^{e^{i\pi}} \int_{z/t}^R = I_1(z) + I_2(z, z^*) + I_3(z).$$

Let us restrict z to a closed domain $\mathfrak{D}'$ in the interior of $0 < |z| < R$, $|\arg z| \leq \Theta - 2\pi$. Then if $0 < \tau < 1$, $|t| = 1$, $0 \leq \arg t \leq \pi$, the integrand of $I_1(z)$ for fixed τ and t is an analytic function of z ; for $0 < |\tau z| < R$, $0 < |\tau z/t^2| < R$, $|\arg \tau z| \leq \Theta$, $|\arg(\tau z/t^2)| \leq \Theta - 2\pi$ and hence $(\tau z, \tau z/t^2)$ is within the domain of analyticity of g . Moreover, the integrand is by (7.2) less in absolute value than an absolutely integrable function of τ and t . Consequently $I_1(z)$ is an analytic function of z for z in $\mathfrak{D}'$.

Next we consider $I_2(z, z^*)$. We restrict z to the closed domain $\mathfrak{D}'$ and also restrict z^* to $\mathfrak{D}'$. Then if t is restricted to the path from $\sqrt{z/z^*}$ to $e^{i\pi}$, the integrand of the inner integral is for fixed τ an analytic function of z and t . To see this, recall that we have required $|t|$ to be between 1 and $|\sqrt{z/z^*}|$, and $\arg t$ to be between $\arg \sqrt{z/z^*}$ and π . Hence when $0 < \tau < 1$, we have $0 < |\tau z| < R$ and $0 < |\tau z/t^2| < R$. Also $\arg \tau z = \arg z$, and $\arg(\tau z/t^2)$ is between $\arg z^*$ and $\arg z - 2\pi$. Consequently $(\tau z, \tau z/t^2)$ is in the domain of analyticity of g . By (7.2) the integrand is less in absolute value than an integrable function of τ . Hence the inner integral is an analytic function of t and z . Because the limit of integration is an analytic function of z and z^* and the integrand is less in absolute value than an integrable function of t , we conclude that $I_2(z, z^*)$ is an analytic function for (z, z^*) in $\mathfrak{D}' \times \mathfrak{D}'$.

For z in $\mathfrak{D}'$ we have

$$I_3(z) = - \int_1^{e^{i\pi}} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \int_{z/t}^R \left(\frac{z}{\rho t}\right)^k g\left(\rho t, \frac{\rho}{t}\right) \rho d\rho \right\} \frac{dt}{t}.$$

The interchange of summation and integration is justified by the convergence of

$$\sum_{k=1}^{\infty} \frac{1}{k} \int_{z/t}^R \left| \frac{z}{\rho t} \right|^k \left| g\left(\rho t, \frac{\rho}{t}\right) \right| |\rho| |d\rho|$$

for $|z| < R$. The series converges absolutely and uniformly for z in $\mathfrak{D}'$, t on the path from 1 to $e^{i\pi}$. Hence $I_3(z)$ is analytic for z in $\mathfrak{D}'$.

Thus we have proved that $T_1g(z, z^*)$ is analytic in $S(\Theta' - 2\pi, R')$ where Θ' is any number $< \Theta$ and R' any number $< R$. The proofs for $T_2g(z)$, $T_3g(z, z^*)$, and $T_4g(z^*)$ are similar and contain no essentially different features. Consequently $Tg(z, z^*)$ is analytic in $S(\Theta' - 2\pi, R')$ and the proof of Lemma 7.1 is complete.

8. Proofs of lemmas. In this section we shall prove Lemmas 7.2 and 7.3. These lemmas are quite similar to lemmas proved by LEWY in [6] and by the author in [4]. The reader may find it somewhat easier to follow the computations here if he is familiar with either [4] or [6].

Proof of Lemma 7.2: Suppose $g(z, z^*) = z^\beta z^{*\gamma} (\log z)^m (\log z^*)^n$ where β and γ are real numbers > -1 and m and n are nonnegative integers. To study $T_1g(z, z^*)$ given by (7.4) we write as before, $T_1g(z, z^*) = I_1(z) + I_2(z, z^*) + I_3(z)$ and consider these three integrals separately.

We have

$$I_1(z) = \int_1^{e^{i\pi}} \left\{ \int_0^1 \tau^{\beta+\gamma+1} t^{-2\gamma} z^{\beta+\gamma} (\log z + \log \tau)^m (\log z + \log \tau - 2 \log t)^n \right. \\ \left. \cdot \left[\log \left(\frac{1}{\tau} - 1 \right) - \pi i \right] \left(\frac{z}{t} \right)^2 d\tau \right\} \frac{dt}{t} = z^{\beta+\gamma+2} P_4(\log z)$$

where $P_4(\log z)$ is a polynomial in $\log z$ of degree $m+n$.

To evaluate the integral

$$I_2(z, z^*) = \int_{\sqrt{z/z^*}}^{e^{i\pi}} \left\{ \int_0^1 \tau^{\beta+\gamma+1} t^{-2\gamma-2} z^{\beta+\gamma+2} (\log \tau z)^m (\log \tau z/t^2)^n \cdot 2\pi i d\tau \right\} \frac{dt}{t},$$

we make a change of variable by setting $\sigma = z/t^2$. We find

$$I_2(z, z^*) = - \int_{z^*}^{ze^{-2\pi i}} \left\{ \int_0^1 \tau^{\beta+\gamma+1} \sigma^{\gamma+1} z^{\beta+1} \right. \\ \left. \cdot (\log \tau + \log z)^m (\log \tau + \log \sigma)^n \cdot \pi i d\tau \right\} \frac{d\sigma}{\sigma}.$$

The inner integral is equal to

$$\sigma^{\gamma+1} z^{\beta+1} P_5(\log z, \log \sigma)$$

where P_5 is a polynomial in $\log z$ and $\log \sigma$ which is of degree $\leq m$ in $\log z$ and of degree $\leq n$ in $\log \sigma$. In view of the formula

$$(8.1) \quad \int \sigma^\lambda (\log \sigma)^\nu d\sigma = \sigma^{\lambda+1} \left[\frac{(\log \sigma)^\nu}{(\lambda+1)} - \nu \frac{(\log \sigma)^{\nu-1}}{(\lambda+1)^2} \right. \\ \left. + \nu(\nu-1) \frac{(\log \sigma)^{\nu-2}}{(\lambda+1)^3} - \cdots + (-1)^\nu \frac{\nu(\nu-1) \cdots 2 \cdot 1}{(\lambda+1)^{\nu+1}} \right],$$

which is valid for any real number $\lambda \neq -1$, we have

$$\begin{aligned} I_2(z, z^*) &= -z^{\beta+1} \int_{z^*}^{ze^{-2\pi i}} \sigma^\gamma P_5(\log z, \log \sigma) d\sigma \\ &= z^{\beta+\gamma+2} P_6(\log z) + z^{\beta+1} z^{*\gamma+1} P_7(\log z, \log z^*) \end{aligned}$$

where P_6 is a polynomial in $\log z$ of degree $m+n$ and P_7 is a polynomial in $\log z$ and $\log z^*$ which is of degree $\leq m$ in $\log z$ and of degree $\leq n$ in $\log z^*$.

Finally, to study $I_3(z)$ we proceed as in the proof of analyticity and write

$$\begin{aligned} I_3(z) &= -\int_1^{e^{i\pi}} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{t} \right)^k \int_{z/t}^R \rho^{\beta+\gamma+1-k} \right. \\ &\quad \left. \cdot (\log \rho + \log t)^m (\log \rho - \log t)^n d\rho \right\} t^{\beta-\gamma-1} dt. \end{aligned}$$

Using (8.1) we find, if $\beta + \gamma$ is not an integer,

$$(8.2) \quad \left\{ \sum_{k=1}^{\infty} \right\} = \sum_{k=1}^{\infty} \left(\frac{z}{R} \right)^k \psi_k(t) + z^{\beta+\gamma+2} P_8(\log z, \log t)$$

where P_8 is a polynomial in $\log z$ and $\log t$ of total degree $m+n$, and there is a constant M such that for t on the path of integration $|\psi_k(t)| \leq M$ ($k = 1, 2, 3, \dots$). It follows that

$$(8.3) \quad I_3(z) = z^{\beta+\gamma+2} P_9(\log z) + p_3(z)$$

where P_9 is a polynomial in $\log z$ of degree $m+n$ and $p_3(z)$ is a power series in z convergent for $|z| < R$ and such that $p(0) = 0$.

If $\beta + \gamma$ is an integer μ , we obtain an exceptional term in the series for $k = \mu + 2$ where (8.1) is not applicable. Instead we use the formula

$$\int (\log \rho)^\nu \frac{d\rho}{\rho} = \frac{(\log \rho)^{\nu+1}}{\nu+1}.$$

We find, in this case, that we again obtain a result of the type (8.2), but now $P_8(\log z, \log t)$ is a polynomial of total degree $m+n+1$, and consequently also $P_9(\log z)$ in (8.3) is a polynomial of degree $m+n+1$.

This completes our study of $T_1g(z, z^*)$. The other terms $T_2g(z)$, $T_3g(z, z^*)$, and $T_4g(z^*)$ can be handled in a manner which is completely analogous. One finds that $T_2g(z)$ is equal to the sum of a power series in z and a polynomial in $\log z$ times $z^{\beta+\gamma+2}$. The term $T_3g(z, z^*)$ is equal to the sum of a power series in z^* , a polynomial in $\log z^*$ times $z^{*(\beta+\gamma+2)}$, and a polynomial in $\log z$ and $\log z^*$ times $z^{\beta+1} z^{*(\gamma+1)}$; and $T_4g(z^*)$ is equal to the sum of a power series in z^* and a polynomial in $\log z^*$ times $z^{*(\beta+\gamma+2)}$. Combining these results we obtain Lemma 7.2.

Proof of Lemma 7.3: Before beginning the proof I would like to remark that the proof cannot be obtained easily by computation, in as much as the lemma is not correct for integral γ . Let $g(\zeta, \zeta^*)$ be analytic for (ζ, ζ^*) in $S(\Theta, R)$ and suppose that $g(\zeta, \zeta^*) = o(\zeta^\gamma)$ for $(\zeta, \zeta^*) \rightarrow_s 0$ in $S(\Theta, R)$ where γ is a real number

> -2 which is not an integer. We wish to prove that for $(z, z^*) \rightarrow 0$ in $S(\Theta - 2\pi, R)$

$$Tg(z, z^*) = Q_1(z) + Q_2(z^*) + o(z^{\gamma+2})$$

where Q_1 and Q_2 are polynomials of degrees $< \gamma + 2$. By definition this means that given any sequence (z_n, z_n^*) of points for which

$$(z_n, z_n^*) \in S(\Theta - 2\pi, R), \quad \frac{1}{K} < \left| \frac{z_n}{z_n^*} \right| < K$$

and $z_n \rightarrow 0, z_n^* \rightarrow 0$, and given any $\epsilon > 0$ there is a number n_0 such that for $n \geq n_0$

$$|Tg(z_n, z_n^*) - Q_1(z_n) - Q_2(z_n^*)| < \epsilon |z_n|^{\gamma+2}.$$

In the following discussion we regard K as fixed throughout. By assumption, for every positive ϵ there is a positive number R_1 such that

$$|g(\zeta, \zeta^*)| < \epsilon |\zeta|^\gamma$$

if (ζ, ζ^*) is in $S(\Theta, R_1)$ and $K^{-1} < |\zeta/\zeta^*| < K$. Also there is a constant M such that

$$|g(\zeta, \zeta^*)| < M |\zeta|^\gamma$$

for all (ζ, ζ^*) in $S(\Theta, R)$ for which $K^{-1} < |\zeta/\zeta^*| < K$.

In the course of the proof we shall use various constants C_1, C_2 , etc. For the proof it is important to verify that these constants are independent of z, z^* , and ϵ . However they will, in general, depend on the other variables considered. It will be found that they can all be chosen so as to depend only on Θ, R, γ, K , and M .

We shall carry out the proof in detail only for $T_1g(z, z^*)$. Again we consider separately $I_1(z), I_2(z, z^*)$, and $I_3(z)$. Let τ and t be points on the paths of integration of the integrals appearing in $I_1(z)$ so that $0 < \tau < 1, |t| = 1, 0 < \arg t < \pi$. Then if (z, z^*) is in $S(\Theta - 2\pi, R_1)$, the point $(\tau z, \tau z/t^2)$ is in $S(\Theta, R_1)$ and $|\tau z/(\tau z/t^2)| = 1$. Hence, referring to (7.4), we see that

$$(8.4) \quad |I_1(z)| \leq \epsilon |z|^{\gamma+2} \left| \int_1^{e^{i\pi}} \left\{ \int_0^1 \tau^\gamma \left| \log \left(\frac{1}{\tau} - 1 \right) - \pi i \right| \tau d\tau \right\} dt \right| \\ \leq C_1 \epsilon |z|^{\gamma+2}.$$

Next we consider $I_2(z, z^*)$, the second integral on the right side of (7.4). We assume that (z, z^*) is in $S(\Theta - 2\pi, R_1)$, $K^{-1} < |z/z^*| < K$, and that t and τ are points on the paths of integration. Then, as we have already seen in §7, the point $(\tau z, \tau z/t^2)$ is in $S(\Theta, R_1)$. Also $K^{-\frac{1}{2}} < |t| < K^{\frac{1}{2}}$, and hence $K^{-1} < |\tau z/(\tau z/t^2)| < K$. Consequently we obtain the estimate

$$(8.5) \quad |I_2(z, z^*)| \leq 2\pi \epsilon |z|^{\gamma+2} \int_{\sqrt{z/z^*}}^{e^{i\pi}} \left\{ \int_0^1 \tau^{\gamma+1} d\tau \right\} |t|^{-3} |dt| \leq C_2 \epsilon |z|^{\gamma+2}.$$

Next

$$I_3(z) = - \int_1^{e^{i\pi}} \left\{ \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{z}{t} \right)^k \int_{z/t}^R g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho \right\} \frac{dt}{t}.$$

Let (z, z^*) be a point in $S(\Theta - 2\pi, R_1)$ and let $\psi = \arg(z/t)$. Then

$$\begin{aligned} \left\{ \sum_{k=1}^{\infty} \right\} &= \sum_{k=1}^m \frac{1}{k} \left(\frac{z}{t} \right)^k \int_0^R g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho - \sum_{k=1}^m \frac{1}{k} \left(\frac{z}{t} \right)^k \int_0^{z/t} g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho \\ &\quad + \sum_{k=m+1}^{\infty} \frac{1}{k} \left(\frac{z}{t} \right)^k \int_{z/t}^{R_1 e^{i\psi}} g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho \\ &\quad + \sum_{k=m+1}^{\infty} \frac{1}{k} \left(\frac{z}{t} \right)^k \int_{R_1 e^{i\psi}}^R g\left(\rho t, \frac{\rho}{t}\right) \rho^{1/k} d\rho. \end{aligned}$$

Because of the analyticity of g this transformation is certainly valid if we let m be the integer for which $\gamma + 1 < m < \gamma + 2$. The integral from 0 to R is to be taken along the real axis and the integrals from 0 to z/t and from z/t to $R_1 e^{i\psi}$ are to be taken along the ray $\arg \rho = \psi$. The absolute convergence of the integrals and series is guaranteed by the assumption that γ is greater than -2 .

We set

$$\left\{ \sum_{k=1}^{\infty} \right\} = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4$$

and estimate the sums Σ_2 , Σ_3 , and Σ_4 . First we have for $k \leq m$

$$\left| \left(\frac{z}{t} \right)^k \int_0^{z/t} g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho \right| \leq \epsilon |z|^k \int_0^{|z|} \rho^{\gamma+1-k} d\rho = \frac{\epsilon |z|^{\gamma+2}}{\gamma+2-k}.$$

Hence

$$(8.6) \quad \int_1^{e^{i\pi}} |\Sigma_2| \left| \frac{dt}{t} \right| \leq C_3 \epsilon |z|^{\gamma+2}.$$

Next we consider the terms in Σ_3 . We have for $k \geq m+1$

$$\begin{aligned} \left| \left(\frac{z}{t} \right)^k \int_{z/t}^{R_1 e^{i\psi}} g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho \right| &\leq \epsilon |z|^k \int_{|z|}^{R_1} \rho^{\gamma+1-k} d\rho \\ &\leq \epsilon |z|^k \int_{|z|}^{\infty} \rho^{\gamma+1-k} d\rho = \frac{\epsilon |z|^{\gamma+2}}{k - \gamma - 2} \end{aligned}$$

since $m > \gamma + 1$. Consequently

$$(8.7) \quad \int_1^{e^{i\pi}} |\Sigma_3| \left| \frac{dt}{t} \right| \leq C_4 \epsilon |z|^{\gamma+2}.$$

Similarly we can estimate the terms in Σ_4 . We have for $k \geq m+1$

$$\begin{aligned} \left| \left(\frac{z}{t} \right)^k \int_{R_1 e^{i\psi}}^R g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho \right| &\leq M |z|^k \int_{R_1 e^{i\psi}}^R \rho^{\gamma+1-k} |d\rho| \\ &\leq C_5 R_1^{\gamma+1} \left(\frac{|z|}{R_1} \right)^k. \end{aligned}$$

Hence

$$|\Sigma_4| \leq \frac{C_5 R_1^{\gamma-m} |z|^{m+1}}{1 - |z|/R_1}.$$

We have $m+1 > \gamma+2$. Consequently, if $|z|$ is sufficiently small, $< R_2$ say, then

$$(8.8) \quad \int_1^{e^{i\pi}} |\Sigma_4| \cdot \left| \frac{dt}{t} \right| \leq \epsilon |z|^{\gamma+2}.$$

Combining the estimates (8.4)–(8.8), we find

$$|T_1 g(z, z^*) - Q(z)| \leq C_6 \epsilon |z|^{\gamma+2}$$

if (z, z^*) is in $S(\Theta - 2\pi, R_2)$, $K^{-1} < |z/z^*| < K$, and $Q(z)$ is the polynomial of degree $\leq m$ given explicitly by the formula

$$Q(z) = - \sum_{k=1}^m \frac{z^k}{k} \left[\int_1^{e^{i\pi}} \left\{ \int_0^R g\left(\rho t, \frac{\rho}{t}\right) \rho^{1-k} d\rho \right\} t^{-k-1} dt \right].$$

Hence by letting $\epsilon \rightarrow 0$ we find

$$T_1 g(z, z^*) = Q(z) + o(z^{\gamma+2})$$

for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta - 2\pi, R)$.

A similar argument proves that $T_2 g(z)$ is the sum of a polynomial in z and a function which is $o(z^{\gamma+2})$, while $T_3 g(z, z^*)$ and $T_4 g(z^*)$ are each equal to a polynomial in z^* plus a function which is $o(z^{\gamma+2})$. All of these polynomials have constant terms equal to zero and are of degrees $< \gamma+2$. If $-2 < \gamma < -1$ it follows that the polynomials vanish identically. This completes the proof of Lemma 7.3.

9. Obtaining the asymptotic expansions. We shall now apply Lemmas 7.2 and 7.3 to obtain an asymptotic expansion for $u(z, z^*)$. It is then possible to find an expansion for $U(z, z^*)$ by using Lemma 6.1 together with equation (6.1).

By (7.1) and Lemma 7.1 we have

$$(9.1) \quad u(z, z^*) = w(z, z^*) + V(z, z^*) + T[a_1(z, z^*)u_z(z, z^*) + a_2 u_{z^*} + a_3 u - a_4]$$

where $V(z, z^*)$ is the function of z and z^* contributed by the integral over Γ . The function $V(z, z^*)$ is analytic for $|z| < R$, $|z^*| < R$. The asymptotic expansion is obtained by an iterative process. To begin, we use the estimate $u(z, z^*) = o(z^{-\alpha+\alpha\delta})$ for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta, R)$ which is given by equation (6.6). Then if Θ' is any positive number $< \Theta$, we have by (6.5)

$$a_1 u_z + a_2 u_{z^*} + a_3 u - a_4 = o(z^{\alpha\delta-2})$$

for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta', R)$.

Let β be any positive number which is smaller than $\alpha\delta$ and 1 and is such that $\beta + k\alpha$ is not an integer for $k = 0, 1, 2, \dots$. Then, since $o(z^{\alpha\delta-2}) = o(z^{\beta-2})$, by using Lemma 7.3 we find

$$u(z, z^*) = w(z, z^*) + V(z, z^*) + o(z^\beta)$$

for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta' - 2\pi, R)$. We separate out the constant term of the power series for $V(z, z^*)$ and the constant term in the asymptotic expansion for $w(z, z^*)$; and we incorporate the rest of the power series and asymptotic expansion into the remainder term. Hence we find, since $\beta < 1$,

$$(9.2) \quad u(z, z^*) = c_0 + o(z^\beta)$$

for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta' - 2\pi, R)$ where c_0 is a constant. The equation (9.2) provides a finite development for $u(z, z^*)$ with remainder term $o(z^\beta)$.

We shall prove the following lemma by induction.

Lemma 9.1. *For each integer $N \geq 0$ the function $u(z, z^*)$ has a finite development with remainder term $o(z^{\beta+N\alpha})$ which is valid for $(z, z^*) \xrightarrow{s} 0$ in*

$$S(\Theta' - 2(N+1)\pi, R).$$

If α is irrational the finite development is a polynomial in z, z^α, z^ , and $z^{*\alpha}$. If $\alpha = p/q$, a reduced fraction, then the development is a polynomial in $z, z^\alpha, z^p \log z, z^*, z^{*\alpha}$, and $z^{*p} \log z^*$.*

Remark. For any given number Θ' the assertion is trivial for N sufficiently large because $S(\Theta' - 2(N+1)\pi, R)$ is then empty.

Proof. The lemma holds for $N = 0$ by (9.2). We assume the statement of the lemma for N and prove it for $N + 1$. Both $w(z, z^*)$ and $V(z, z^*)$ have finite developments of arbitrarily high order because V is analytic at the origin and w has the asymptotic expansion (6.8). Hence certainly

$$w(z, z^*) + V(z, z^*) = \mathfrak{F}_0 + o(z^{\beta+(N+1)\alpha})$$

where $\mathfrak{F}_0$ is a polynomial in the arguments z, z^α, z^* , and $z^{*\alpha}$ for irrational α and a polynomial in $z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*$ when $\alpha = p/q$, a reduced fraction. Similarly by (6.4) and the induction hypothesis there are polynomials $\mathfrak{F}_i$ of the type described such that

$$\begin{aligned} a_1(z, z^*) &= z^{*\alpha-1}(\mathfrak{F}_1 + o(z^{\beta+N\alpha})), \\ a_2(z, z^*) &= z^{\alpha-1}(\mathfrak{F}_2 + o(z^{\beta+N\alpha})), \\ a_3(z, z^*) &= z^{\alpha-1}z^{*\alpha-1}(\mathfrak{F}_3 + o(z^{\beta+N\alpha})), \\ a_4(z, z^*) &= z^{\alpha-1}z^{*\alpha-1}(\mathfrak{F}_4 + o(z^{\beta+N\alpha})), \\ u(z, z^*) &= \mathfrak{F}_5 + o(z^{\beta+N\alpha}), \\ u_z(z, z^*) &= z^{-1}(\mathfrak{F}_6 + o(z^{\beta+N\alpha})), \\ u_{z^*}(z, z^*) &= z^{*-1}(\mathfrak{F}_7 + o(z^{\beta+N\alpha})). \end{aligned}$$

All of these developments are valid for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta' - 2(N+1)\pi, R)$. It follows that there are polynomials $\mathfrak{F}_8$ and $\mathfrak{F}_9$ of the type described such that

$$(9.3) \quad a_1 u_z + a_2 u_{z^*} + a_3 u - a_4 = z^{\alpha-1} z^{*-1} \mathfrak{F}_8 + z^{*\alpha-1} z^{-1} \mathfrak{F}_9 + o(z^{\beta+(N+1)\alpha-2})$$

for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta' - 2(N+1)\pi, R)$.

Next we insert the right side of (9.3) into (9.1) and apply Lemmas 7.2 and 7.3 to obtain a higher order development for u . Observe that T operating on the remainder term $o(z^{\beta+(N+1)\alpha-2})$ yields, besides polynomials in z and z^* , a function which is $o(z^{\beta+(N+1)\alpha})$ for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta' - 2(N+2)\pi, R)$.

The lemma will now follow if it can be shown that T operating on terms in $z^{\alpha-1} z^{*-1} \mathfrak{F}_8$ or $z^{*\alpha-1} z^{-1} \mathfrak{F}_9$ yields, besides power series in z or z^* , only polynomials in $z, z^\alpha, z^*, z^{*\alpha}$ when α is irrational and polynomials in $z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*$ when $\alpha = p/q$. First let us assume α is irrational. Then a typical term of $z^{\alpha-1} z^{*-1} \mathfrak{F}_8$ or $z^{*\alpha-1} z^{-1} \mathfrak{F}_9$ has the form $cz^{k-1+l\alpha} z^{*m-1+n\alpha}$ where k, l, m, n are non-negative integers and either $l \geq 1$ or $n \geq 1$. By Lemma 7.2 we have

$$T(cz^{k-1+l\alpha} z^{*m-1+n\alpha}) = c_1 z^{k+m+(l+n)\alpha} + c_2 z^{k+l\alpha} z^{*m+n\alpha} + c_3 z^{*k+m+(l+n)\alpha} + p_1(z) + p_2(z^*)$$

where c_1, c_2 , and c_3 are constants and $p_1(z)$ and $p_2(z^*)$ are power series. No logarithmic terms are introduced because $k+m+(l+n)\alpha-2$ is not an integer since $l+n \geq 1$. Since all of the terms obtained are of the sort described, the proof of the lemma for irrational α is complete.

If $\alpha = p/q$, a reduced fraction, then a typical term from $z^{\alpha-1} z^{*-1} \mathfrak{F}_8$ or $z^{*\alpha-1} z^{-1} \mathfrak{F}_9$ has the form

$$cz^{i-1+j\alpha} z^{*k-1+l\alpha} (z^p \log z)^m (z^{*p} \log z^*)^n$$

where i, j, k, l, m , and n are nonnegative integers and either $j \geq 1$ or $l \geq 1$. By Lemma 7.2 the operator T applied to such a term yields

$$z^{(i+k)+(j+l)\alpha} z^{p(m+n)} P_1(\log z) + z^{*(i+k)+(j+l)\alpha} z^{*p(m+n)} P_2(\log z^*) + z^{i+j\alpha} z^{*k+l\alpha} z^{pm} z^{*pn} P_3(\log z, \log z^*) + p_1(z) + p_2(z^*).$$

We must verify that besides the power series $p_1(z)$ and $p_2(z^*)$ this expression is a polynomial in the arguments $z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*$. The only difficulty in proving this is in seeing that $\log z$ always occurs with a factor of z^p and $\log z^*$ always occurs with a factor of z^{*p} . By Lemma 7.2 the polynomial P_3 is of degree $\leq m$ in $\log z$ and of degree $\leq n$ in $\log z^*$; therefore there is no difficulty for the term with P_3 . Also if $(j+l)\alpha$ is not an integer P_1 and P_2 have degrees $\leq m+n$, therefore again there is no trouble. On the other hand, if $(j+l)\alpha$ is an integer P_1 and P_2 can, according to Lemma 7.2, be of degree $m+n+1$ and we must then obtain an extra factor of z^p or z^{*p} . Indeed, if $(j+l)\alpha = (j+l)p/q$ is an integer, then q must divide $(j+l)$. Furthermore, since $j+l \geq 1$ the integer $(j+l)/q$ is at least 1. Hence we can separate out an extra factor of z^p or z^{*p} from $z^{(j+l)\alpha}$ or $z^{*(j+l)\alpha}$, respectively. The remainder of the argument is exactly the same as in the case of irrational α ; therefore Lemma 9.1 is proved.

Since we may take Θ arbitrarily large and also N arbitrarily large, Lemma 9.1

guarantees the existence of finite developments of arbitrarily high order which are valid in $S \times S$ for arbitrarily large finite sectors S . By Lemma 4.1 it is not possible for a function to have two different finite developments with remainder term $o(z^\gamma)$ which are valid in $S \times S$. Hence, if we obtain a finite development with remainder term $o(z^{\gamma'})$ with $\gamma' > \gamma$, its terms of degree $\leq (\gamma, 0)$ must be exactly the same as those in a development with remainder term $o(z^\gamma)$. It follows that u has exactly one asymptotic expansion for $(z, z^*) \xrightarrow{*} 0$ in $S \times S$, where S is any finite sector. If α is irrational

$$(9.4) \quad u(z, z^*) \sim \mathcal{Q}_0(z, z^\alpha, z^*, z^{*\alpha}),$$

while if $\alpha = p/q$, a reduced fraction,

$$(9.5) \quad u(z, z^*) \sim \mathcal{Q}_0(z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*).$$

Now, since by (6.1)

$$(9.6) \quad U(z, z^*) = u(F^{-1}(z), \bar{F}^{-1}(z^*)),$$

we can use Lemma 6.1 to obtain an expansion for U . To carry this out, one can replace the asymptotic expansion for $F^{-1}(z)$ by a finite development of arbitrarily high order and then compute developments for $(F^{-1}(z))^\alpha$, $\bar{F}^{-1}(z)$, and $(\bar{F}^{-1}(z^*))^\alpha$.

For the moment, let us assume α is irrational. Then by Lemma 6.1,

$$F^{-1}(z) \sim z^{1/\alpha} \mathcal{U}_2(z, z^{1/\alpha}).^\dagger$$

It is then easy to verify that

$$(F^{-1}(z))^\alpha \sim z \mathcal{U}_3(z, z^{1/\alpha})$$

and similarly that

$$\begin{aligned} \bar{F}^{-1}(z^*) &\sim z^{*1/\alpha} \mathcal{U}_4(z^*, z^{*1/\alpha}), \\ (\bar{F}^{-1}(z^*))^\alpha &\sim z^* \mathcal{U}_5(z^*, z^{*1/\alpha}). \end{aligned}$$

Using these developments together with (9.4) we then find

$$U(z, z^*) \sim \mathcal{Q}(z, z^{1/\alpha}, z^*, z^{*1/\alpha})$$

for $(z, z^*) \xrightarrow{*} 0$ in $S_1 \times S_1$ where S_1 is an arbitrarily large finite sector.

If $\alpha = p/q$, we have

$$F^{-1}(z) \sim z^{1/\alpha} \mathcal{U}_2(z, z^{1/\alpha}, z^q \log z).$$

A similar but more complicated calculation leads to the result

$$U(z, z^*) \sim \mathcal{Q}(z, z^{1/\alpha}, z^q \log z, z^*, z^{*1/\alpha}, z^{*q} \log z^*).$$

Since by Lemma 4.2 we can differentiate asymptotic expansions, the proof of Theorem 3.1 is now complete.

[†]Note that according to the convention of §2 expansions $\mathcal{U}_i$ all have nonvanishing constant terms.

Proof of Theorem 3.2. Now we assume $f = 0$ and $\mathfrak{U} = 0$ on Γ_1 and Γ_2 . Consequently for the harmonic function taking the same boundary values near the origin we can take $w(z, z^*) \equiv 0$. We repeat the above proof taking into account any improvements that can be obtained. First of all we observe that in (9.2) the constant c_0 must be 0. In place of Lemma 9.1 we prove the following lemma.

Lemma 9.2. *For every integer $N \geq 0$ there are polynomials $\mathfrak{F}_{11}$ and $\mathfrak{F}_{12}$ such that*

$$u(z, z^*) = z\mathfrak{F}_{11} + z^*\mathfrak{F}_{12} + o(z^{\beta+N\alpha})$$

for $(z, z^*) \xrightarrow{s} 0$ in $S(\Theta' - 2(N+1)\pi, R)$. If α is irrational $\mathfrak{F}_{11}$ and $\mathfrak{F}_{12}$ are polynomials in z, z^α, z^* , and $z^{*\alpha}$. If $\alpha = p/q$ they are polynomials in $z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*$.

Proof. We prove the lemma by induction. By (9.2) with $c_0 = 0$ the statement holds for $N = 0$. We assume the statement for N and prove it for $N + 1$. Observe that by (9.1) and Lemmas 7.2 and 7.3 the function $V(z, z^*)$ vanishes at the origin. Hence the development $\mathfrak{F}_0$ has the form $z\mathcal{P}_0 + z^*\mathcal{Q}_0$ where $\mathcal{P}_0$ and $\mathcal{Q}_0$ are polynomials. Similarly, it follows from the induction hypothesis that $\mathfrak{F}_5, \mathfrak{F}_6$ and $\mathfrak{F}_7$ have the property that each is of the form $z\mathcal{P}_i + z^*\mathcal{Q}_i$ where $\mathcal{P}_i$ and $\mathcal{Q}_i$ are polynomials in the above mentioned arguments. Because the differential equation is homogeneous $a_4(z, z^*) \equiv 0$. Consequently $\mathfrak{F}_8$ and $\mathfrak{F}_9$ also have this property, and an examination of the remainder of the proof shows that T preserves this property. The statement for $N + 1$ then follows.

We then obtain

$$u(z, z^*) \sim z\mathcal{Q}_0 + z^*\mathcal{Q}'_0$$

where $\mathcal{Q}_0$ and $\mathcal{Q}'_0$ are formal power series in z, z^α, z^* , and $z^{*\alpha}$ for irrational α and formal power series in $z, z^\alpha, z^p \log z, z^*, z^{*\alpha}$, and $z^{*p} \log z^*$ for $\alpha = p/q$. Use of (9.6) together with Lemma 6.1 then yields Theorem 3.2. Conversely, from the expansion for $U(z, z^*)$ given by Theorem 3.2 the expansion for $u(z, z^*)$ can be found.

10. The problem with discontinuous boundary values. Now let us investigate briefly the boundary value problem of the introduction when the requirement that $\varphi_1(0, 0) = \varphi_2(0, 0)$ is dropped and $\mathfrak{U}$ is allowed to be discontinuous at the origin. We keep the requirements (1.3), however, because they are essential to our method.

All of the considerations in §5 concerning extension by reflection still hold because we made no use there of the assumption of continuity at the origin. Also the preliminary transformation in §6 is valid without change. Although Wasow did not study the Dirichlet problem with discontinuous boundary values, the method used by him generalizes easily to such a problem. One finds that a harmonic function $W(z, \bar{z})$ which satisfies such a boundary value problem can be written

$$W(z, \bar{z}) = \operatorname{Re} \{c_0 \log z + G(z)\}$$

where $G(z)$ has an asymptotic expansion in powers of z and $z^{1/\alpha}$ for irrational α and in powers of z , $z^{1/\alpha}$, and $z^a \log z$ for $\alpha = p/q$. Paralleling the discussion before (6.8) we obtain for $\alpha = p/q$

$$w(z, z^*) - c \log z - \bar{c} \log z^* \sim \alpha_s(z, z^\alpha, z^p \log z, z^*, z^{*\alpha}, z^{*p} \log z^*).$$

where c is a constant. If α is irrational, we obtain an expansion of the same type but with no powers of $z^p \log z$ or $z^{*p} \log z^*$.

The argument of §9 can then be repeated with the changes which are made necessary by the extra logarithmic terms. One obtains the following theorem.

Theorem 10.1. *Let $\mathfrak{U}(x, y) = U(z, \bar{z})$ be a solution of the equation (1.1) which is in the class C^2 inside D , is continuous on $(D \cup \Gamma_1 \cup \Gamma_2) - 0$, and satisfies the boundary conditions $\mathfrak{U}(x, y) = \varphi_1(x, y)$ on Γ_1 , $\mathfrak{U}(x, y) = \varphi_2(x, y)$ on Γ_2 , where φ_1 and φ_2 are analytic functions near the origin and $\varphi_1(0, 0)$ need not equal $\varphi_2(0, 0)$. Suppose also that the conditions (1.3) are satisfied. Then if S is any finite sector, U has an analytic continuation $U(z, z^*)$ defined on a neighborhood of the origin in $S \times S$; and*

$$U(z, z^*) \sim \log z \, \mathfrak{A} + \log z^* \mathfrak{A}^* + \mathfrak{A}'$$

for $(z, z^*) \rightarrow 0$ in $S \times S$ where $\mathfrak{A}$, $\mathfrak{A}^*$, and $\mathfrak{A}'$ are formal power series in z , $z^{1/\alpha}$, z^* , and $z^{*1/\alpha}$ when α is irrational, and formal power series in z , $z^{1/\alpha}$, $z^a \log z$, z^* , $z^{*1/\alpha}$, $z^{*a} \log z^*$ for $\alpha = p/q$, a reduced fraction. Moreover, expansions for derivatives of arbitrary order are obtained by formally differentiating.

As in §3 one can derive a statement concerning the order of magnitude of the derivatives of $\mathfrak{U}$ by consideration of the leading terms alone. One finds that the n^{th} derivatives of $\mathfrak{U}$ are $O(z^{-n})$. For Laplace's equation this result was obtained by P. LAASONEN [2].

BIBLIOGRAPHY

- [1] GEORGE E. FORSYTHE, *Asymptotic lower bounds for the fundamental frequency of convex membranes*, Pacific Journal of Mathematics, **5** (1955), pp. 691-702.
- [2] PENTTI LAASONEN, *On the behavior of the solution of the Dirichlet problem at analytic corners*, Annales Academiæ Scientiarum Fennicæ Series A, I. Mathematica, No. 241 (1957).
- [3] ———, *On the degree of convergence of discrete approximations for the solutions of the Dirichlet problem*, Annales Academiæ Scientiarum Fennicæ Series A, I. Mathematica, No. 246 (1957).
- [4] R. SHERMAN LEHMAN, *Developments in the neighborhood of the beach of surface waves over an inclined bottom*, Communications on Pure and Applied Mathematics, **7** (1954), pp. 393-439.
- [5] ———, *Development of the mapping function at an analytic corner*, Pacific Journal of Mathematics, **7** (1957), pp. 1437-1449.
- [6] HANS LEWY, *Developments at the confluence of analytic boundary conditions*, University of California Publications in Mathematics, Vol. 1, No. 7 (1950), pp. 247-280.
- [7] ———, *On the reflection laws of second order differential equations in two independent variables*, to appear in the Bulletin of the American Mathematical Society.

- [8] LEON LICHTENSTEIN, *Zur Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung des elliptischen Typus*, Mathematische Annalen, **67** (1909), pp. 559–575.
- [9] ———, *Über die konforme Abbildung ebener analytischer Gebiete mit Ecken*, Journal für die Reine und Angewandte Mathematik, **140** (1911), pp. 100–119.
- [10] J. F. RITT, *On the differentiability of asymptotic series*, Bulletin of the American Mathematical Society, **24** (1918), pp. 225–227.
- [11] WOLFGANG WASOW, *Asymptotic development of the solution of Dirichlet's problem at analytic corners*, Duke Mathematical Journal, **24** (1957), pp. 47–56.
- [12] ———, *The accuracy of difference approximations to plane Dirichlet problems with piecewise analytic boundary values*, Quarterly of Applied Mathematics, **15** (1957), pp. 53–63.

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