Functional Equations, Wave Propagation and Invariant Imbedding

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SUMMARY

It is shown that wave propagation in inhomogeneous media may be studied through the use of invariance principles similar to those used in radiative transfer

and neutron transport theory. For this purpose a wave localization principle is stated and proved. The analytical techniques derived from it furnish a new approach to some significant problems in mathematical physics, and, on the other hand, the physical concepts provide new insight into the analytic structure of the solutions of the one-dimensional wave equation.

1. Introduction. In recent years, functional equation techniques have been applied in a variety of ways to a number of mathematical models of fundamental processes. As examples, let us mention the work of AMBARZUMIAN, [1], CHANDRASEKHAR, [16], PREISENDORFER, [26], and BELLMAN & KALABA, [6, 7], in radiative transfer; the theory of biological growth processes and cascade processes via the theory of branching processes as developed by BELLMAN & HARRIS, [5], HARRIS, [18], and JANOSSY, [20]; the study of neutron fission by way of invariant imbedding carried out by BELLMAN, KALABA & WING, [12, 13, 14]; the theory of dynamic programming, [4]; and finally the papers of REDHEFFER, [27], and PASZKOWSKI & MYCIELSKI, [24], related to the results of BELLMAN & KALABA, [8], concerned with random walk processes.

Apart from the work of REDHEFFER, previous research has been devoted to the study of "particle processes," using a terminology which will be made precise below. In a brief note, [9], and in the paper [10] we sketched the application of functional equation techniques to the study of "wave processes." We wish to begin the systematic application of these methods here. Ultimately, we wish to describe quite general wave motion in these terms and, even further, to obtain a unified description of both wave and particle motion.

At the moment, our aims are more modest. We wish to study the propagation of plane waves, normally incident, through plane parallel media. The analytic techniques furnish a new approach to some significant problems in mathematical physics, and, reciprocally, the physical concepts will be used to provide a new insight into the analytic structure of the solutions of the one-dimensional wave equation.

Parallel with this, we wish to fashion tools which can be used to obtain numerical answers to numerical problems, using modern computational methods. This is of particular importance in connection with questions of propagation through randomly inhomogeneous media.

As so often happens in the field of mathematics, the most efficient way to provide these numerical answers is to furnish new analytic tools. It may well be that invariance principles and functional equations are the appropriate tools.

In order to apply functional equation techniques, we transform complex physical processes into the iteration of simple physical processes. This is, of course, the guiding idea behind the Lie theory of transformation groups, the Poincaré theory of differential equations in the large, the Birkhoff theory of dynamical systems via iteration, the Hadamard development of Huygen's principle, and finally the engulfing theory of semi-groups due to HILLE, [19], where many references will be found.

To perform this metamorphosis for wave processes, we introduce a new principle which we call the "*principle of localization*." We shall establish it rigorously for the case of plane waves, using for this purpose an interesting series due to BREMMER, [15], and, more generally, the theory of invariant imbedding.

No previous knowledge of mathematical physics, in general, or of wave motion, in particular, will be assumed. Of the reader, we shall expect only that he be familiar with the properties of the solutions of linear differential equations. Although from time to time we use such physical expressions as "wave," "reflection," and "transmission," these are used in quite precisely defined mathematical senses. Nonetheless, the terms are defined and retained to guide the intuition.

In subsequent papers we will apply the same techniques and concepts to the study of various types of partial differential equations and to the study of abstract operator equations. We propose to discuss cylindrical and spherical waves, plane waves that are obliquely incident on a stratified medium and the wave-particle duality mentioned §17. In addition, we propose to apply the functional equation approach to expand KELLER's new treatment of diffraction by means of rays, [21].

2. Particle Processes. The study of particle motion by way of functional equation techniques depends upon a very simple but important property of particles, of such simplicity that its basic nature is perhaps overlooked. We use the expression "functional equation techniques" to include many different types of formulation, since, as pointed out in [12], the transport equation is itself derived from invariance principles.

Consider, to illustrate this point, a particle traveling along the line segment $[a, b]$, suffering collisions which may cause it to reverse direction and change speed. In deriving equations describing its position as a function of time, we use the fact that its behavior in any segment of $[a, b]$ depends only upon the particular physical properties of this segment. This is a *localization principle*—which we take for granted in studying particle processes.

Another way of stating this principle is to postulate that the particle constantly behaves as if the entire interval ahead of it were homogeneous, possessing the properties of the infinitesimal interval directly in front of it.

We want to describe wave motion in a similar fashion which will permit the application of functional equation techniques.

3. A Stratified Medium. In setting up our mathematical models of physical processes, we shall use the familiar device of replacing a general continuous function by a step-function. Consider the semi-infinite region to the right of $x = 0$, and suppose that each x -line is specified, as to a physical property we shall discuss subsequently, by the function $k(x)$.

Then, in place of this continuous dependence, we think of the semi-infinite

region as being divided into a set of strata, with $k(x)$ having constant value in each *stratum*, $x_n \leq x < x_{n+1}$. The lines $x = x_i$ are called interfaces.

FIGURE 1. A Stratified Medium.

This, of course, is a familiar device, used, for example, by BERNOULLI to trace the path of a ray of light through an inhomogeneous medium by means of Snell's law, and used by STOKES, [29], by LUNEBERG, [23], and by many others. In LUNEBERG's paper, it is shown how the problem of wave propagation through inhomogeneous media may thus be reduced to the problem of the iteration of matrices or bilinear transformations. For the case of propagation through randomly inhomogeneous media, this permits the use of the methods given in BELLMAN, [3], and KERNER, [22], to obtain average transmission and reflection.

4. Waves and Wave Motion. The equation

$$(1) \quad k^2(x)u_{tt} = u_{xx}$$

is called the *wave equation*. It governs the motion, for example, of an inhomogeneous vibrating string, [17, p. 291]. We are interested in solutions of the particular form

$$(2) \quad u(x, t) = e^{-i\omega t}u(x).$$

The function $u(x)$ then satisfies the ordinary differential equation

$$(3) \quad u''(x) + \omega^2 k^2(x)u(x) = 0,$$

which we shall refer to as the *reduced wave equation*. The function $k(x)$ is called the *wave number*. If it is constant, (3) has particular solutions of the form

$$(4) \quad u(x) = e^{\pm i\omega kx}.$$

For the case where $k(x)$ is constant, a solution of the form

$$(5) \quad u = e^{-i\omega t}e^{\pm i\omega kx} = e^{-i\omega(t \mp kx)}$$

is called a *wave*.

We shall drop the function $e^{-i\omega t}$, which is common to all solutions, and talk about the functions $e^{+i\omega kx}$ as *waves*. These are called *plane waves*. By proper choice of units, we can take ω to be one.

5. Two Semi-infinite Homogeneous Media. Given the two semi-infinite regions determined by $x > 0$ and $x < 0$, with

$$(1) \quad \begin{aligned} k(x) &= k_1, & x < 0, \\ k(x) &= k_2, & x > 0, \end{aligned}$$

we wish to determine the “transmitted” wave in $x > 0$, and the “reflected” wave in $x < 0$, due to a wave e^{ik_1x} starting at $x = -\infty$.

The meaning of this is the following. In $x < 0$, the function

$$(2) \quad u_1 = e^{ik_1x} + c_1e^{-ik_1x}$$

is a solution of

$$(3) \quad u'' + k_1^2u = 0,$$

while, in $x > 0$,

$$(4) \quad u_2 = c_2e^{ik_2x}$$

is a solution of

$$(5) \quad u'' + k_2^2u = 0.$$

The part of the solution $c_1e^{-ik_1x}$ is called the *reflected wave*, and the solution $c_2e^{ik_2x}$ is called the *transmitted* or *refracted wave*.

The justification for these terms will be given in subsequent sections in which we show that these expressions can be obtained as if certain simple reflection and transmission processes are taking place.

To determine the coefficients c_1 and c_2 , which measure the intensity of the reflected and transmitted waves, we postulate continuity of u and u' as we cross the interface $x = 0$. As we shall see subsequently, when dealing with transmission lines in §20, this merely asserts continuity of voltage and current across the interface.

These continuity conditions yield the equations

$$(6) \quad \begin{aligned} u_1(0) &= 1 + c_1 = c_2 = u_2(0), \\ u_1'(0) &= ik_1(1 - c_1) = ik_2c_2 = u_2'(0). \end{aligned}$$

The quantities $u_i(0)$ and $u_i'(0)$ are to be interpreted as limits of $u_i(x)$ and $u_i'(x)$ as $x \rightarrow 0$ from the appropriate direction.

The foregoing equations yield the well known relations

$$(7) \quad c_1 = \frac{k_1 - k_2}{k_2 + k_1}, \quad c_2 = \frac{2k_1}{k_2 + k_1}.$$

We have gone into these elementary results in great detail, since the extensions will employ the same basic ideas, applied to more general equations.

6. The WKB Approximation. The WKB approximation to the solution of

$$(1) \quad u'' + k^2(x)u = 0$$

(more properly the Liouville approximation; see [2]) is

$$(2) \quad u = \frac{\exp \left\{ \pm i \int_0^x k(y) dy \right\}}{k(x)^{\frac{1}{2}}}.$$

As is well known, it can be obtained by taking account only of the transmitted wave and ignoring all reflections. To see this, consider the stratified medium of Fig. 1 with an incoming wave e^{ik_0x} in $x < 0$. In the interval $0 \leq x < x_1$, we have a transmitted wave

$$(3) \quad \frac{2k_0}{k_0 + k_1} e^{ik_1x},$$

upon referring to the relations of (5.7).

This, in turn, produces in the stratum $x_1 \leq x < x_2$ the transmitted wave ($x_2 = x_1 + \Delta x_1$)

$$(4) \quad \left(\frac{2k_0}{k_0 + k_1} \right) e^{ik_1\Delta x_1} \cdot \frac{2k_1}{k_1 + k_2} e^{ik_2(x-x_1)}.$$

Generally, letting $x_{k+1} = x_k + \Delta x_k$, we see that the amplitude of the direct transmitted wave just to the left of $x = x_n$ is given by the expression

$$(5) \quad a_n = \left(\frac{2k_0}{k_0 + k_1} \right) \left(\frac{2k_1}{k_1 + k_2} \right) \cdots \left(\frac{2k_{n-1}}{k_{n-1} + k_n} \right) \exp \left\{ i \sum_{j=1}^n k_j \Delta x_j \right\}.$$

Letting $\Delta x_i = \Delta$, and $n\Delta \rightarrow x$ as $\Delta \rightarrow 0$, we obtain the approximation of (2),

$$(6) \quad u_0(x) = \left(\frac{k_0}{k(x)} \right)^{\frac{1}{2}} \exp \left\{ i \int_0^x k(s) ds \right\},$$

which plays such an essential role in quantum mechanics.

7. The Bremmer Series. The importance of the foregoing interpretation lies in the fact that the physical model of reflections and transmissions can be used to obtain further approximations, and ultimately a solution, in a very simple fashion.

Following BREMMER, [15], we obtain these approximations by adding the effects of successive reflections. The contribution at x due to those waves which are the result of *one* reflection is easily seen to be

$$(1) \quad u_1(x) = -\frac{1}{2k(x)^{\frac{1}{2}}} \int_x^\infty \frac{k'(s)}{k(s)^{\frac{1}{2}}} u_0(s) \exp \left\{ i \int_x^s k(t) dt \right\} ds.$$

Similarly, if we define $u_n(x)$ as the contribution due to those waves which have been reflected n times, we see that

$$(2) \quad \begin{aligned} u_{2N}(x) &= \frac{1}{2(k(x))^{\frac{1}{2}}} \int_0^x \frac{k'(s)}{k(s)^{\frac{1}{2}}} u_{2N-1}(s) \exp \left\{ i \int_s^x k(t) dt \right\} ds, \\ u_{2N+1}(x) &= -\frac{1}{2(k(x))^{\frac{1}{2}}} \int_0^\infty \frac{k'(s)}{k(s)^{\frac{1}{2}}} u_{2N}(s) \exp \left\{ i \int_x^s k(t) dt \right\} ds. \end{aligned}$$

The details will be found in BREMMER, [15].

It follows that a solution of the wave equation of (6.1) should be obtained by adding up the contribution at x due to all reflections. Hence we form the series

$$(3) \quad u(x) = \sum_{i=0}^{\infty} u_i(x),$$

which we shall call the *Bremmer series*.

8. Discussion. As will be shown below, the convergence of the Bremmer series can be established under the conditions that

$$(1) \quad \begin{aligned} (a) \quad &|k(s)| \geq a > 0, \quad s \geq 0, \\ (b) \quad &\int_0^\infty |k'(t)| dt = b < 2a. \end{aligned}$$

One would suspect, however, that condition (1b) could be replaced by the weaker condition

$$(2) \quad \int_0^\infty |k'(s)| ds < \infty.$$

Although it is possible to improve (1b) slightly by more sophisticated analysis, which we do not present, we are unable to determine whether the series in (7.3) converges under the condition in (2) or not. Our guess is that it does not.

Fortunately, for our ultimate purposes, we can circumvent this difficulty by means of the techniques of invariant imbedding, as will be shown further below.

9. The Statement of the Result. The result we wish to demonstrate is

Theorem 1. *Let the function $k(x)$ satisfy the conditions*

$$(1) \quad \begin{aligned} (a) \quad &|k(x)| \geq a > 0, \quad x \geq 0, \\ (b) \quad &\int_0^\infty |k'(x)| dx = b < 2a. \end{aligned}$$

Then the Bremmer series, defined by (7.3), converges to a solution of

$$(2) \quad u'' + k^2(x)u = 0.$$

10. The Proof of Convergence. To establish the preceding result, we argue as follows. Using the expression given for $u_0(x)$ in (6.6), we see that

$$(1) \quad |u_0(x)| \leq \left(\frac{|k(0)|}{a} \right)^{\frac{1}{2}} = q_1.$$

Referring to (7.1), we have

$$(2) \quad |u_1(x)| \leq \frac{q_1}{2a^{\frac{1}{2}}} \int_x^\infty \frac{|k'(s)|}{a^{\frac{1}{2}}} ds \leq \frac{q_1}{2a} \int_0^\infty |k'(s)| ds = q_1 \frac{b}{2a}.$$

Let us then show inductively that for some constant q_2 we have

$$(3) \quad |u_k(x)| \leq q_2 \left(\frac{b}{2a} \right)^k.$$

The result is clearly valid for $k = 0$ and 1.

Referring to the recurrence relations of (7.2), we have

$$(4) \quad \begin{aligned} |u_{2N}(x)| &\leq \frac{1}{2a^{\frac{1}{2}}} \int_0^x \frac{|k'(s)|}{a^{\frac{1}{2}}} \left(q_2 \left(\frac{b}{2a} \right)^{2N-1} \right) ds \\ &\leq \frac{1}{2a} \left(\int_0^\infty |k'(s)| ds \right) \left(q_2 \left(\frac{b}{2a} \right)^{2N-1} \right) = q_2 \left(\frac{b}{2a} \right)^{2N}, \\ |u_{2N+1}(x)| &\leq \frac{1}{2a^{\frac{1}{2}}} \int_x^\infty \frac{|k'(s)|}{a^{\frac{1}{2}}} \left(q_2 \left(\frac{b}{2a} \right)^{2N} \right) ds \\ &\leq \frac{1}{2a} \left(\int_0^\infty |k'(s)| ds \right) \left(q_2 \left(\frac{b}{2a} \right)^{2N} \right) = q_2 \left(\frac{b}{2a} \right)^{2N+1}. \end{aligned}$$

It follows that the Bremmer series converges uniformly for $x \geq 0$, and that $u(x)$ is uniformly bounded for $x \geq 0$. As noted above, the foregoing result can be improved in various ways. We cannot, however, establish the desired result that the series converges under the condition of (8.2) rather than that of (9.1b).

Considering the fact that the series is generally used for perturbation purposes, it is perhaps amazing that it converges at all.

11. The Verification of the Solution. It remains to show that $u(x)$ is actually a solution of the wave equation. Let

$$(1) \quad \begin{aligned} v(x) &= u_0(x) + u_2(x) + u_4(x) + \cdots, \\ w(x) &= u_1(x) + u_3(x) + u_5(x) + \cdots. \end{aligned}$$

Then the recurrence relations of (7.2) yield the integral equations

$$(2) \quad \begin{aligned} v(x) &= u_0(x) + \frac{1}{2k(x)^{\frac{1}{2}}} \int_0^x \frac{k'(s)}{k(s)^{\frac{1}{2}}} w(s) \exp \left\{ i \int_s^x k(t) dt \right\} ds, \\ w(x) &= -\frac{1}{2k(x)^{\frac{1}{2}}} \int_x^\infty \frac{k'(s)}{k(s)^{\frac{1}{2}}} v(s) \exp \left\{ i \int_x^s k(t) dt \right\} ds. \end{aligned}$$

Thus we have the relations

$$(3) \quad \frac{d}{dx} \left(v(x)k(x)^{\frac{1}{2}} \exp \left\{ -i \int_0^x k(t) dt \right\} \right) = \frac{1}{2} \left(\frac{k'(x)}{k(x)^{\frac{1}{2}}} \right) w(x) \exp \left\{ -i \int_0^x k(t) dt \right\}$$

(since

$$\frac{d}{dx} \left(u_0(x)k(x)^{\frac{1}{2}} \exp \left\{ -i \int_0^x k(t) dt \right\} \right) = 0 \quad)$$

and

$$(4) \quad \frac{d}{dx} \left(w(x)k(x)^{\frac{1}{2}} \exp \left\{ i \int_0^x k(t) dt \right\} \right) = \frac{1}{2} \left(\frac{k'(x)}{k(x)^{\frac{1}{2}}} \right) v(x) \exp \left\{ i \int_0^x k(t) dt \right\}.$$

From these equations we obtain

$$(5) \quad \begin{aligned} v'(x)k(x)^{\frac{1}{2}} + \frac{1}{2} \frac{k'(x)}{k(x)^{\frac{1}{2}}} v(x) - ik(x)^{\frac{3}{2}} v(x) &= \frac{1}{2} \left(\frac{k'(x)}{k(x)^{\frac{1}{2}}} \right) w(x), \\ w'(x)k(x)^{\frac{1}{2}} + \frac{1}{2} \left(\frac{k'(x)}{k(x)^{\frac{1}{2}}} \right) w(x) + ik(x)^{\frac{3}{2}} w(x) &= \frac{1}{2} \left(\frac{k'(x)}{k(x)^{\frac{1}{2}}} \right) v(x). \end{aligned}$$

Adding, we have

$$(6) \quad [v'(x) + w'(x)] + ik(x)[w(x) - v(x)] = 0.$$

Hence

$$(7) \quad [v''(x) + w''(x)] + ik'(x)[w(x) - v(x)] + ik(x)[w'(x) - v'(x)] = 0.$$

Subtracting the equations in (5), we see that

$$(8) \quad k(x)[w'(x) - v'(x)] + k'(x)[w(x) - v(x)] + ik^2(x)[w(x) + v(x)] = 0.$$

Combining (7) and (8), we have the desired equation for $u(x) = v(x) + w(x)$,

$$(9) \quad u''(x) + k^2(x)u(x) = 0.$$

12. The Principle of Localization. The results established above permit us to state the following

Principle of Localization. *A plane wave traveling through an inhomogeneous medium proceeds as if there were an instantaneous reflection and transmission at each interface of a stratum $[y, y + \Delta y]$. These reflections and transmissions occur as if the stratum $[y, y + \Delta y]$ were actually a semi-infinite homogeneous medium with wave number $k(y)$ for $x \geq y$.*

Note the complete analogy with the localization principle for particles, stated in §2.

13. Discussion. The convergence of the Bremmer series, under the conditions stated in Theorem 1, establishes the localization principle, provided that $k(x)$

satisfies the stated conditions. In order to establish the result in general, we follow a different approach. Prior to this, let us digress and indicate an interesting application of the Bremmer series.

14. The Asymptotic Behavior of the Solution. Let us now demonstrate that the Bremmer series enables us to establish in an elegant way a known result concerning the asymptotic behavior of the solutions of the equation

$$(1) \quad u'' + (1 + \phi(x))u = 0,$$

cf. [2], p. 135. We wish to show that the equation has solutions of the form

$$(2) \quad \begin{aligned} u_1 &= \exp \left\{ i \int_0^x (1 + \phi(s))^{\frac{1}{2}} ds \right\} (1 + o(1)), \\ u_2 &= \exp \left\{ -i \int_0^x (1 + \phi(s))^{\frac{1}{2}} ds \right\} (1 + o(1)), \end{aligned}$$

as $x \rightarrow \infty$, provided that

$$(3) \quad \begin{aligned} (a) \quad &\phi(x) \rightarrow 0, \\ (b) \quad &\int^\infty |\phi'(t)| dt < \infty. \end{aligned}$$

To see this, choose x_0 large enough so that $\int_{x_0}^\infty |\phi'(s)| ds$ is sufficiently small so as to satisfy the conditions of Theorem 1. From the explicit representations for the $u_i(x)$, we have, for $x > x_0$,

$$(4) \quad \begin{aligned} u_0(x) &= \exp \left\{ i \int_{x_0}^x (1 + \phi(s))^{\frac{1}{2}} ds \right\} (1 + o(1)), \\ u_{2N}(x) &= \left(\frac{1}{4} \int_{x_0}^\infty \left[\frac{\phi'(s)}{(1 + \phi(s))} u_{2N-1}(s) \exp \left\{ -i \int_{x_0}^s (1 + \phi(r))^{\frac{1}{2}} dr \right\} \right] ds \right) \\ &\quad \cdot \exp i \left\{ \int_{x_0}^x (1 + \phi(s))^{\frac{1}{2}} ds \right\} (1 + o(1)), \\ u_{2N+1}(x) &\rightarrow 0 \quad \text{as } x \rightarrow \infty. \end{aligned}$$

Using the previously obtained bounds, we see that the series $\sum_{k=0}^\infty u_k(x)$ has the stated asymptotic form.

15. The Classical Solution. The usual technique for determining the reflected and transmitted waves is the following. In the interval $[-\infty, z]$, we have the equation

$$(1) \quad u'' + k_1^2 u = 0$$

with a solution of the form

$$(2) \quad u = e^{ik_1(x-z)} + c_2 e^{-ik_1(x-z)}.$$

The term $e^{ik_1(x-z)}$ represents a normalized incoming wave from $-\infty$, while $c_1 e^{-ik_1(x-z)}$ represents the reflected wave. The coefficient $c_1 = c_1(z)$ we call the *reflection coefficient*.

In $[z, b]$ we have the equation

$$(3) \quad u'' + k^2(x)u = 0$$

with a solution of the form

$$(4) \quad u = c_2 u_1(x) + c_3 u_2(x),$$

where $u_1(x)$ and $u_2(x)$ are particular linearly independent solutions determined for the sake of convenience by the conditions

$$(5) \quad \begin{aligned} u_1(b) &= 1, & u_1'(b) &= 0, \\ u_2(b) &= 0, & u_2'(b) &= 1. \end{aligned}$$

In $[b, \infty]$, we have the equation

$$(6) \quad u'' + k_2^2 u = 0$$

with the solution

$$(7) \quad u = c_4 e^{ik_2(x-b)}.$$

To determine the four unknowns c_1 , c_2 , c_3 and c_4 , we postulate continuity of the solution and its derivative at the interfaces $x = z$, $x = b$. We obtain thereby the conditions

$$(8) \quad c_2 = c_4, \quad c_3 = ik_2 c_4,$$

from $x = b$, and

$$(9) \quad \begin{aligned} 1 + c_1 &= c_2 u_1(z) + c_3 u_2(z), \\ ik_1(1 - c_1) &= c_2 u_1'(z) + c_3 u_2'(z), \end{aligned}$$

from $x = z$.

Using the values for c_2 and c_3 obtained from (8) in (9), we have the two equations

$$(10) \quad \begin{aligned} 1 + c_1 &= c_4(u_1(z) + ik_2 u_2(z)), \\ ik_1(1 - c_1) &= c_4(u_1'(z) + ik_2 u_2'(z)). \end{aligned}$$

Solving for c_1 , we have

$$(11) \quad c_1 = - \frac{\begin{vmatrix} 1 & u_1(z) + ik_2 u_2(z) \\ ik_1 & u_1'(z) + ik_2 u_2'(z) \end{vmatrix}}{\begin{vmatrix} 1 & u_1(z) + ik_2 u_2(z) \\ -ik_1 & u_1'(z) + ik_2 u_2'(z) \end{vmatrix}} = - \frac{\frac{\phi'(z)}{\phi(z)} - ik_1}{\frac{\phi'(z)}{\phi(z)} + ik_1},$$

where $\phi(z) = u_1(z) + ik_2 u_2(z)$.

Since $\phi(x)$ satisfies the wave equation of (3), the function $w = \phi'(z)/\phi(z)$ satisfies the Riccati equation

$$(12) \quad w' + w^2 + k^2(z) = 0.$$

Next we make the assumption that $k(x)$ is differentiable and $k(z) = k_1$. Since

$$(13) \quad w = ik_1 \frac{1 - c_1}{1 + c_1} = ik(z) \frac{1 - c_1(z)}{1 + c_1(z)},$$

we see that $c_1(z)$ satisfies the equation

$$(14) \quad c_1'(z) = -2ik(z)c_1(z) + \frac{k'(z)}{2k(z)}(1 - c_1^2(z)).$$

As we shall see, this is precisely the result we obtain using invariant imbedding and the principle of localization.

16. Invariant Imbedding. The basic idea of invariant imbedding, [6], is to consider a particular physical process as a member of a family of similar processes. Using the intrinsic structure of the process, one then obtains relations connecting different members of the family. In this way we obtain an insight into the process which is quite different from that obtained by consideration of the isolated process. Not only do we derive new analytic results in this way, but frequently we forge new computational approaches as well.

The problem we have been considering is that of determining the reflected and transmitted waves associated with a plane parallel slab, homogeneous in the y - and z -directions, but inhomogeneous in the x -direction. Here we are treating only the case of normally incident plane waves.

To perform the imbedding, we keep the right interface fixed at $x = b$ and allow the left interface to be variable, $x = z$, where $-\infty < z < b$ (see Figure 2). The intensities of the transmitted and reflected waves will thus be functions of z .

FIGURE 2.

The medium to the left of $x = z$ will be taken to be homogeneous with wave number $k(z)$, or, more precisely, $k(z - 0)$, while the medium to the right of $x = b$ will be taken to be homogeneous with wave number $k(b)$, again $k(b + 0)$. Let us write $k_1 = k(b)$.

We now wish to determine the wave reflected in the half-space $x \leq z$, as a result of the wave $e^{ik(z)(x-z)}$ incident at $x = z$. We use this expression rather

than the more usual $e^{ik(z)x}$ in order to normalize the incoming intensity, and simultaneously to simplify the algebra.

The reflected wave will have the form

$$(1) \quad u(z)e^{-ik(z)(x-z)},$$

where the function $u(z)$ (corresponding to $c_1(z)$ of §15) is the function we wish to determine. When $z = 0$, we have the original problem, while $z = b$ is readily resolved. From §5, we see that

$$(2) \quad u(b) = \frac{k(b-0) - k_1}{k(b-0) + k_1}.$$

We now wish to employ the functional equation technique of invariant imbedding, [6, 7], and the *localization principle* stated above, to obtain a differential equation for $u(z)$.

17. Functional Equations. Consider Figure 3, a one-dimensional version of Figure 2.

FIGURE 3.

In the infinitesimal interval $[z, z + \Delta]$, we take the wave number $k(x)$ to have the constant value $k(z + \Delta)$.

Taking account of zeroth and first order terms in Δ , the total reflected wave from $[z, b]$ will be a sum of three parts:

- (a) The immediate reflected wave from $[z, z + \Delta]$.
- (b) The wave that arises from transmission through $[z, z + \Delta]$, reflection from the interface $x = z + \Delta$, and transmission through the interface
- (1) $x = z$.
- (c) The wave that results from transmission through $x = z$, reflection at $x = z + \Delta$, reflection at $x = z$, reflection again at $x = z + \Delta$, and finally transmission through $x = z$.

Schematically, the cases are shown in Figure 4.

FIGURE 4.

The intensities of the reflected and transmitted waves are obtained in terms of the function $u(z)$ we have defined above and the principle of localization stated in §12.

The analytic transliteration is the equation

$$\begin{aligned}
 u(z) &= \frac{k(z-0) - k(z+\Delta-0)}{k(z-0) + k(z+\Delta-0)} \\
 &+ \frac{2k(z-0)}{k(z-0) + k(z+\Delta-0)} u(z+\Delta) e^{2ik(z+\Delta-0)\Delta} \frac{2k(z+\Delta-0)}{k(z-0) + k(z+\Delta-0)} \\
 (2) \quad &+ \frac{2k(z-0)}{k(z-0) + k(z+\Delta-0)} e^{ik(z+\Delta-0)\Delta} u(z+\Delta) e^{ik(z+\Delta-0)\Delta} \\
 &\cdot \frac{k(z+\Delta-0) - k(z-0)}{k(z-0) + k(z+\Delta-0)} e^{ik(z+\Delta-0)\Delta} u(z+\Delta) e^{ik(z+\Delta-0)\Delta} \\
 &\cdot \frac{2k(z+\Delta-0)}{k(z+\Delta-0) + k(z-0)} + o(\Delta).
 \end{aligned}$$

Passing to the limit as $\Delta \rightarrow 0$, we obtain the Riccati equation

$$(3) \quad u'(z) = \frac{k'}{2k} - 2iku - \frac{k'}{2k} u^2$$

($k' = k'(z)$), precisely the equation derived in §15, using the conventional approach.

Notice the similarity of this equation with that of equation (6.2) of [11], which arose in connection with reflection of neutrons in fissionable material. We propose to discuss this wave-particle duality in more detail in a subsequent paper.

18. Discussion. The preceding result furnishes a proof of the validity of the localization principle insofar as obtaining reflection coefficients is concerned. A similar discussion will show that it is equally safe to use it in deriving the transmission coefficient. We shall merely give the equation corresponding to (17.3) below.

19. The Transmitted Wave. Referring to §16 or §17, let the transmitted wave, $x > b$, have the form

$$(1) \quad u(x) = v(z) e^{ik_1(x-b)}.$$

To obtain a differential equation for $v(z)$, we use the fact that, to terms in Δ^2 , the total transmitted wave consists of one component which is transmitted through the interface $x = z$, is transmitted through $x = z + \Delta$, and emerges at $x = b$, and next a second component which arises from transmission through $x = z$, reflection at $x = z + \Delta$, reflection at $x = z$ and transmission at $x = z + \Delta$, and finally which emerges at $x = b$.

The resulting equation is

$$(2) \quad v(z) = \frac{2k(z-0)}{k(z-0) + k(z+\Delta-0)} e^{ik(z+\Delta-0)\Delta} \left\{ v(z+\Delta) + u(z+\Delta) e^{ik(z+\Delta-0)\Delta} \right. \\ \left. + \frac{k(z+\Delta-0) - k(z-0)}{k(z+\Delta-0) + k(z-0)} e^{ik(z+\Delta-0)\Delta} v(z+\Delta) \right\} + o(\Delta).$$

A passage to the limit as $\Delta \rightarrow 0$ yields the required differential equation

$$(3) \quad v' = \left(\frac{k'}{2k} - u \frac{k'}{2k} - ik \right) v,$$

where $k' = k'(z)$, for $z < b$, with

$$(4) \quad v(b) = \frac{2k(b-0)}{k(b-0) + k_1}.$$

20. The Non-uniform Transmission Line. Let us now apply the same ideas to the study of non-uniform transmission lines.

Let $z(x)$ be the series impedance per unit length and $y(x)$ be the shunt admittance per unit length of a non-uniform electrical transmission line, where x is measured from the load to the generator. In addition, let the voltage and current on the line be denoted by $v(x)$ and $i(x)$, respectively. These variables will, in general, be complex. The transmission line equations become, after factoring out the harmonic time factor,

$$(1) \quad \frac{di}{dx} = y(x)v, \quad \frac{dv}{dx} = z(x)i.$$

If the generator is at distance d from the load, as boundary conditions we have

$$(2) \quad v(d) = v_{\text{gen}}, \quad v(0) = z_L i(0),$$

where v_{gen} is the generator voltage and z_L is the load impedance (see Figure 5).

In the case of the uniform line, for which $y(x)$ and $z(x)$ are constant, this boundary value problem is readily solved. In addition, solutions are known for

FIGURE 5. The Transmission Line

exponential and certain other lines. In the case of the non-uniform line, it has been customary to attack the problem by considering the second order equations which are obtained by elimination from equation (1):

$$(3) \quad \begin{aligned} v'' &= \frac{z'}{z} v' + yzv, \\ i'' &= \frac{y'}{y} i' + yzi, \end{aligned}$$

along with the two-point boundary conditions in equation (2).

We shall follow an entirely different path by dealing directly with the *impedance*. Once again we are speaking in physical terms, but operating along purely mathematical lines. Let $Z(x)$ be the impedance of the line looking toward the load and at a distance x from it. Here we are imbedding the problem of determining the impedance of the line of length d within the class of problems involving lines of length x for $x \geq 0$. Several other authors have shown an awareness of the theoretical advantages in doing this, [25, 28, 30, 31]. As we shall see, $Z(x)$ satisfies a Riccati equation and the transformation

$$(4) \quad Z(x) = \frac{v(x)}{i(x)} = \frac{1}{y(x)} \frac{i'(x)}{i(x)},$$

interpretable on physical grounds, and well known in the mathematical literature, provides the link between the general second order linear homogeneous equation and the Riccati equation.

FIGURE 6. An Incremental Section of a Transmission Line

Let us first derive the differential equation satisfied by $Z(x)$ on physical grounds. We have, referring to Fig. 6 and using standard series-parallel circuit considerations,

$$(5) \quad \begin{aligned} Z(x+h) &= z(x)h + \frac{1}{\frac{1}{Z(x)} + y(x)h} + o(h), \\ &= z(x)h + Z(x) \frac{1}{1 + y(x)h} + o(h) \\ &= z(x)h + Z(x)[1 - y(x)h] + o(h), \end{aligned}$$

so that

$$Z'(x) = z(x) - y(x)Z^2(x),$$

which is a nonlinear first order equation of Riccattian type. As initial condition, we have

$$(6) \quad Z(0) = z_L.$$

For some choices of the functions $v(x)$ and $y(x)$, $Z(x)$ may be determined explicitly, for example if $y(x)$ and $z(x)$ are constant. In most instances, though, the equation will have to be integrated numerically (as a system, one equation for the real part and one for the imaginary part), which is perfectly straightforward using, for example, a Runge-Kutta method.

Once $Z(x)$ has been determined on the interval of interest, $0 \leq x \leq d$, where we assume the generator is at the distance d from the load, we may determine $v(x)$ by a quadrature by noting that

$$(7) \quad \begin{aligned} Z(x) &= \frac{v(x)}{i(x)} = \frac{v(x)z(x)}{v'(x)}, \\ \frac{v'(x)}{v(x)} &= \frac{z(x)}{Z(x)}, \quad v(d) = v_{\text{gen}}, \end{aligned}$$

so that

$$(8) \quad v(x) = v_{\text{gen}} \exp \left\{ \int_d^x z(s)Z^{-1}(s) ds \right\}.$$

In particular, $v(0)$ is determined.

Another quadrature yields

$$(9) \quad i(x) = \frac{v(0)}{z_L} + \int_0^x y(s)v(s) ds.$$

It is a simple matter to verify that the functions $v(x)$ and $i(x)$ defined by the relations

$$(10) \quad \begin{cases} v(x) = v_{\text{gen}} \exp \left\{ \int_d^x z(s)Z^{-1}(s) ds \right\}, \\ i(x) = \frac{v(0)}{z_L} + \int_0^x y(s)v(s) ds, \end{cases}$$

where $Z(x)$ is given by

$$(11) \quad Z'(x) = z(x) - y(x)Z^2(x), \quad Z(0) = z_L,$$

satisfy the relations

$$(12) \quad \begin{aligned} i'(x) &= y(x)v(x), & v(0) &= z_L i(0), \\ v'(x) &= z(x)i(x), & v(d) &= v_{\text{gen}}. \end{aligned}$$

This, in effect, reduces the two-point boundary value problem to an initial value problem well suited for numerical integration.

As a matter of convenience, one would determine the admittance $Y(x) = [Z(x)]^{-1}$ rather than the impedance, in view of equation (10). This function satisfies the differential equation

$$(13) \quad Y' = y(x) - z(x)Y^2(x), \quad Y(0) = [Z(0)]^{-1}.$$

21. Stochastic Wave Propagation. The foregoing methods can be applied to the problem of determining the reflected and transmitted waves associated with a randomly inhomogeneous slab. Since this has been discussed in detail in [10], we refer the interested reader to this source.

22. The N -dimensional Case. Let us now briefly indicate how the foregoing results may be extended to include linear differential equations of the N^{th} order. Consider the linear differential equation

$$(1) \quad u^{(N)} + p_1(x)u^{(N-1)} + \cdots + p_N(x)u = 0.$$

and suppose that

$$(2) \quad p_i(x) = p_i + q_i(x), \quad i = 1, 2, \cdots, N,$$

where the p_i are constant, the $q_i(x)$ are small, and $q_i(x) \rightarrow 0$ as $x \rightarrow \infty$. Further, suppose that the roots of the polynomial equation

$$(3) \quad r^N + p_1 r^{N-1} + \cdots + p_N = 0,$$

which we shall call $\lambda_1, \lambda_2, \cdots, \lambda_N$, are distinct. Then the roots of the equation

$$(4) \quad r^N + p_1(x)r^{N-1} + \cdots + p_N(x) = 0,$$

which we shall call $\lambda_1(x), \lambda_2(x), \cdots, \lambda_N(x)$ are distinct for $x \geq 0$ if the $|q_i(x)|$ are sufficiently small—which we shall assume.

Let the $q_i(x)$ be identically zero for $x \leq 0$. By a “wave” we shall now mean a solution of (1) for $x < 0$ having the form

$$(5) \quad u = e^{\lambda_i x}.$$

If the medium for $x > 0$ is homogeneous, which is to say if the $p_i(x)$ are constant for $x > 0$, we can ask for the analogues of the reflected and transmitted waves previously found.

Let λ'_i be the natural continuation of λ_i as x passes through zero. By this we mean that $\lambda'_i \rightarrow \lambda_i$ as the $q_i \rightarrow 0$. Then we look for a solution of (1) of the form

$$(6) \quad u = c_i e^{\lambda'_i x}$$

in $x > 0$, and of the form

$$(7) \quad u = e^{\lambda_i x} + \sum_{j \neq i} c_j e^{\lambda_j x}$$

for $x < 0$.

The term $c_i e^{\lambda_i' x}$ we can call a *transmitted wave*, and the sum $\sum_{i \neq i} c_i e^{\lambda_i x}$ a *reflected wave*. As above, the N coefficients $c_1, c_2, \dots, c_N$ are to be determined by continuity conditions, continuity of u and its first $N - 1$ derivatives as x crosses zero. The relations are

$$(8) \quad \begin{aligned} c_i &= 1 + \sum_{j \neq i} c_j, \\ \lambda_i' c_i &= \lambda_i + \sum_{j \neq i} \lambda_j c_j, \\ (\lambda_i')^{N-1} c_i &= \lambda_i^{N-1} + \sum_{j \neq i} \lambda_j^{N-1} c_j. \end{aligned}$$

Since by assumption the λ_i are distinct, these N equations have a non-vanishing determinant, the Vandermonde determinant, and so yield a unique solution for the c_i .

Let us now change our notation slightly and let $c_i = c_{ij}$, $j \neq i$, but leave c_i unchanged. Then, performing the same operation for each i , $i = 1, 2, \dots, N$, we see that an oncoming vector wave

$$(9) \quad z = \begin{bmatrix} e^{\lambda_1 x} \\ e^{\lambda_2 x} \\ \vdots \\ e^{\lambda_N x} \end{bmatrix}$$

results in a transmitted wave

$$(10) \quad y = Bz', \quad x > 0,$$

where

$$(11) \quad B = \begin{bmatrix} c_1 & 0 & & \\ & c_2 & \ddots & \\ & & \ddots & \\ 0 & & & c_N \end{bmatrix}, \quad z' = \begin{bmatrix} e^{\lambda_1' x} \\ e^{\lambda_2' x} \\ \vdots \\ e^{\lambda_N' x} \end{bmatrix},$$

and a reflected wave

$$(12) \quad w = Cz, \quad x < 0,$$

where

$$(13) \quad C = \begin{bmatrix} 0 & c_{12} & \cdots & c_{1N} \\ c_{21} & 0 & \cdots & c_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1} & \cdots & c_{N,N-1} & 0 \end{bmatrix}.$$

The matrices B and C may then be called the transmission and reflection matrices. If we proceed in the same fashion, but set the left interface at $x = z$ instead of at $x = 0$, we find that these are functions of z , $B(z)$ and $C(z)$.

It is now easy to obtain Riccati matrix equations for $B(z)$ and $C(z)$ using the principle of localization and invariance principles. Furthermore, we can obtain Bremmer series and use them, under various assumptions, to obtain asymptotic results of the type discussed in [2], due to CESARI and LEVINSON.

23. The Matrix Case. Similarly, we can treat the matrix equation

$$(1) \quad X'' + A^2(t)X = 0.$$

First we consider the case where

$$(2) \quad \begin{aligned} A(t) &= A_1, & t < 0, \\ &= A_2, & t > 0. \end{aligned}$$

By a *wave*, we now mean a matrix of the form

$$(3) \quad X = e^{iBt}.$$

For $t < 0$, we construct a solution of the form

$$(4) \quad X = e^{iA_1t} + e^{-iA_1t}C_1.$$

The part e^{iA_1t} we call an *oncoming wave*, the part $e^{-iA_1t}C_1$ the *reflected wave*.

In $t > 0$, we write

$$(5) \quad X = e^{iA_2t}C_2$$

and call this the *transmitted wave*.

The usual continuity conditions at the interface yield the relations

$$(6) \quad I + C_1 = C_2, \quad A_1 - A_1C_1 = A_2C_2,$$

leading to

$$(7) \quad C_1 = (A_1 + A_2)^{-1}(A_1 - A_2), \quad C_2 = 2(A_1 + A_2)^{-1}A_1.$$

Having obtained these results, we can now proceed to derive Riccati equations, Bremmer series and so on. As indicated in [9], this yields an analogue of the WKB approximation for matrix systems of the type shown above in (1).

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