The Time-Dependent Solution for a Storage Model with Poisson Input

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Summary. This paper is concerned with the time-dependent distribution function $F(z, t)$ of the content of an infinite dam, fed by a Poisson input and with a steady continuous release ceasing only when it becomes empty.

The forward Kolmogorov relation for $F(z, t)$ is a known difference-differential equation. By obtaining the probabilities of first emptiness of the dam, it is possible to find the probability $F(0, t)$ that the dam is empty after a period of time t , not necessarily for the first time.

This is finally used to solve the forward equation for $F(z, t)$ directly; the known stationary distribution of the dam content can then be derived as the limit of $F(z, t)$ when $t \rightarrow \infty$.

1. Introduction. In a recent note [1] and paper [2], the authors have discussed an infinite storage model, usually thought of as a dam, whose content $Z(t) \geq 0$ is defined in continuous time t ($0 \leq t < \infty$) by the equation

$$(1) \quad Z(t + \delta t) = Z(t) + \delta X(t) - \text{Min} \{Z(t), \delta t\}.$$

In this, $X(t)$ represents the input into the dam during time t , a Poisson process with parameter λ , such that in a small interval of time δt the quantity $\delta X(t) = 0$ or h may be added to the dam content. The distribution function (d.f.) of the input $X(t)$ is given by

$$K(x, t) = \sum_{r=0}^{\lfloor x/h \rfloor} e^{-\lambda t} \frac{(\lambda t)^r}{r!},$$

and it is clear from its Laplace-Stieltjes transform,

$$\int_{x=0}^{\infty} e^{-\theta x} dK(x, t) = \exp \{-\lambda t(1 - e^{-\theta h})\} = \exp \{-t\xi(\theta)\},$$

that inputs are additive, being independently distributed in non-overlapping

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intervals of time. The last term $\text{Min} \{Z(t), \delta t\}$ of equation (1) represents a continuous release which occurs at constant unit rate except if $Z(t) = 0$, when the release is also zero.

For this process, which is Markovian and time-homogeneous, it has been pointed out that the forward Kolmogorov equation can be obtained for the transition d.f.

$$F(z, t) = \text{Pr} \{Z(\tau + t) \leq z \mid Z(\tau) = z_0\}$$

of the dam content $Z(t)$, where $F(z, t)$ is independent of τ , the initial time, but not of the initial content z_0 . The function $F(z, t)$, satisfying the condition that $0 < F(z, t) \leq 1$ for all $z \geq 0$ and $0 \leq t < \infty$, is a point function everywhere non-decreasing and continuous to the right in z , and is everywhere differentiable with respect to t (see [3], pp. 269–70).

The forward equation is found to be

$$(2) \quad \frac{\partial}{\partial t} F(z, t) - \frac{\partial}{\partial z} F(z, t) = -\lambda \{F(z, t) - F(z - h, t)\} \quad (0 \leq z < \infty),$$

where it is understood that $F(z, t) = 0$ for all $z < 0$, $\text{Lim}_{z \rightarrow \infty} F(z, t) = 1$ for all t , and the derivatives $\partial F(z, t)/\partial z$ are appropriately interpreted at $z = 0$ and $z = h$. This result also arises as a special case of TAKÁCS' [4] integro-differential equation for the d.f. of the waiting time in a single server queue, where customers arrive in a Poisson process of constant parameter λ and have a fixed service time h . It is already known that $F(z, t)$ is everywhere continuous in $0 < z < \infty$, but has a discontinuity at $z = 0$, this being $F(0, t)$, the probability of emptiness of the dam at any time t . Further, $\partial F(z, t)/\partial z$ exists at all points z of the range, except $z = 0$ where there is only a right derivative and $z = h$ where the right and left derivatives are distinct and satisfy the equation

$$(3) \quad \frac{\partial}{\partial z} F(h + 0, t) - \frac{\partial}{\partial z} F(h - 0, t) = -\lambda F(0, t).$$

When $\lambda h < 1$, a stationary solution $\Phi(z) = \text{Lim}_{t \rightarrow \infty} F(z, t)$ exists which is independent of z_0 , the initial dam content. This was first obtained explicitly by TAKÁCS [4] in the form

$$(4) \quad \Phi(z) = (1 - \lambda h) \sum_{r=0}^{\lfloor z/h \rfloor} e^{-\lambda(rh-z)} \frac{\{\lambda(rh-z)\}^r}{r!},$$

where the Laplace-Stieltjes transform of the distribution is

$$(5) \quad \psi(\theta) = \int_0^\infty e^{-\theta z} d\Phi(z) = \frac{(1 - \lambda h)\theta}{\theta - \lambda(1 - e^{-\theta h})} \quad (\theta > 0).$$

Although the Laplace-Stieltjes transform with respect to z of the d.f. $F(z, t)$ is known from [2], [4] and [5] to be

$$\begin{aligned}
 \psi(\theta, t) &= \int_{z=0}^{\infty} e^{-\theta z} dF(z, t) \\
 (6) \quad &= e^{\{\theta - \lambda(1 - e^{-\theta h})\}t} \left\{ e^{-\theta z_0} - \theta \int_0^t F(0, u) e^{-(\theta + \lambda(1 - e^{-\theta h}))u} du \right\} \quad (\theta > 0),
 \end{aligned}$$

a function depending on the probability of emptiness $F(0, t)$ of the dam, the explicit solution $F(z, t)$ of the difference-differential equation (2) has not so far been obtained. We proceed to find it in this paper; beginning with the probabilities of first emptiness of the dam, we derive the probability $F(0, t)$ that the dam be empty at any time t , and finally use this to construct the general time-dependent d.f. $F(z, t)$ of the dam content $Z(t)$, given that the initial content is z_0 .

2. Probabilities of first emptiness of the dam. Consider the instants $\tau + T$ at which our dam, whose initial content is $Z(\tau) = z_0$, may become empty for the first time. Since the inputs, each of magnitude h , must occur in such a way that the dam is never empty before $\tau + T$, first emptiness can occur only after the time-intervals $T = z_0 + rh$ ($r = 0, 1, 2, \dots$) have elapsed, depending on the number r of inputs which prevent the dam from emptying during this period. The probability distribution of first emptiness times is thus discrete, and will be expressed at the times $\tau + T = \tau + z_0 + rh$ ($r = 0, 1, 2, \dots$) by

$$g(z_0, T) =$$

$$\Pr \{Z(\tau + T) = 0 \mid Z(\tau) = z_0; Z(\tau + \beta T) > 0 \text{ for all } 0 < \beta < 1\}.$$

Since we are dealing with a time-homogeneous process, $g(z_0, T)$ will be independent of τ .

Following some recent work of KENDALL [6] on dams with additive inputs having continuous distributions, it may be shown quite generally that, when the input is an additive process with the discrete probability $k(x, t)$ that $x = rh$ ($r = 0, 1, 2, \dots$) units flow into the dam during time t , then the probability $g(z_0, T)$ that the dam, whose initial content is z_0 , will be empty for the first time after a discrete time-interval $T = z_0 + rh$ ($r = 0, 1, 2, \dots$) is

$$(7) \quad g(z_0, T) = \frac{z_0}{T} k(T - z_0, T).$$

The Laplace-Stieltjes transform of this T -distribution is given by $G(\theta) = e^{-z_0 \eta(\theta)}$, $\eta(\theta)$ being a solution of the equation

$$(8) \quad \eta(\theta) = \theta + \xi(\eta(\theta)) \quad (\theta > 0),$$

where $\exp \{-t \xi(\theta)\}$ is the transform for the distribution $k(x, t)$ of the input.

By applying these results when the input is of Poisson type, we may obtain the distribution of first emptiness times as

$$\begin{aligned}
 (9) \quad g(z_0, z_0 + rh) &= \frac{z_0}{z_0 + rh} e^{-\lambda(z_0 + rh)} \frac{\{\lambda(z_0 + rh)\}^r}{r!} \\
 &= e^{-\lambda(z_0 + rh)} \frac{\lambda^r}{r!} z_0 (z_0 + rh)^{r-1} \quad (r = 0, 1, 2, \dots),
 \end{aligned}$$

with a transform $G(\theta) = e^{-z_0 \eta(\theta)}$, where $\eta(\theta)$ is a solution of the equation

$$(10) \quad \eta(\theta) = \theta + \lambda(1 - e^{-\eta(\theta)h}) \quad (\theta > 0)$$

in this case. That the probabilities (9) form a proper distribution may be readily verified from the limit of the transform $G(\theta)$ as $\theta \rightarrow 0+$. For, since $\lambda h < 1$, the only non-negative root of the equation (10) as $\theta \rightarrow 0+$ is $\eta(0+) = 0$, so that

$$G(0+) = \sum_{r=0}^{\infty} g(z_0, z_0 + rh) = 1.$$

It is equally simple, however, to derive this result by evaluating $G(\theta)$ directly; this is

$$\begin{aligned}
 (11) \quad G(\theta) &= \sum_{r=0}^{\infty} g(z_0, z_0 + rh) e^{-\theta(z_0 + rh)} \\
 &= e^{-(\theta + \lambda)z_0} \left\{ \sum_{r=0}^{\infty} \frac{\{\lambda(z_0 + rh)\}^r e^{-(\lambda + \theta)h}}{r!} \right. \\
 &\quad \left. - \lambda h e^{-(\lambda + \theta)h} \sum_{r=0}^{\infty} \frac{\{(\lambda z_0 + [r + 1]h)\}^r e^{-(\lambda + \theta)h}}{r!} \right\} \\
 &= G_1(\theta) - G_2(\theta) \quad (\theta > 0).
 \end{aligned}$$

These terms are clearly the constants in α of the function

$$\begin{aligned}
 G(\theta, \alpha) &= e^{-(\theta + \lambda)z_0} \left\{ \sum_{r=0}^{\infty} (\lambda e^{-(\lambda + \theta)h})^r e^{\alpha(z_0 + rh)} \alpha^{-r} \right. \\
 &\quad \left. - \lambda h e^{-(\lambda + \theta)h} \sum_{r=0}^{\infty} (\lambda e^{-(\lambda + \theta)h})^r e^{\alpha(z_0 + rh + h)} \alpha^{-r} \right\} \\
 &= e^{(\alpha - \lambda - \theta)z_0} \{1 - \lambda h e^{(\alpha - \lambda - \theta)h}\} \{1 - \lambda \alpha^{-1} e^{(\alpha - \lambda - \theta)h}\}^{-1}
 \end{aligned}$$

where α may be complex and $|\alpha^{-1} \lambda e^{(\alpha - \lambda - \theta)h}| < 1$ for convergence. This condition is satisfied if $\lambda < |\alpha| < \lambda_1$, where λ, λ_1 ($\lambda < h^{-1} < \lambda_1$) are the roots of the equation $\alpha e^{-\alpha h} = \lambda e^{-\lambda h}$.

To find $G(\theta)$, we require the residue of $G(\theta, \alpha)\alpha^{-1}$ at its pole $\alpha_0(\theta) \leq \lambda$, where this is the smaller root of the equation

$$(12) \quad \alpha e^{-\alpha h} = \lambda e^{-(\lambda + \theta)h}.$$

Thus we obtain

$$\begin{aligned}
 (13) \quad G(\theta) &= \lim_{\alpha \rightarrow \alpha_0(\theta)} e^{(\alpha - \lambda - \theta)z_0} \frac{1 - \lambda h e^{(\alpha - \lambda - \theta)h}}{\alpha - \lambda e^{(\alpha - \lambda - \theta)h}} \{\alpha - \alpha_0(\theta)\} \\
 &= e^{(\alpha_0(\theta) - \lambda - \theta)z_0} = e^{-z_0 \eta(\theta)},
 \end{aligned}$$

$$g_s = \Pr \left\{ \sum_{i=0}^r n_i \geq j+1 (j=0, \dots, s-2), \sum_{i=0}^{j-1} n_i = s, n_s = 0 \right\}.$$

Since the restriction (14) must hold, there will be $\sum_{i=s+1}^r n_i = r-s$ inputs in $(\tau + z_0 + sh, \tau + z_0 + rh)$, the probability of these being

$$\Pr \left\{ \sum_{i=s+1}^r n_i = r-s \right\} = e^{-\lambda(r-s)h} \frac{\{\lambda(r-s)h\}^{r-s}}{(r-s)!}.$$

Summing over all values of $s = 0, 1, \dots, r$, to include every possible way in which the r inputs flow into the dam during the interval $(\tau, \tau + z_0 + rh)$, we obtain the convolution equation

$$\begin{aligned} \sum_{s=0}^r g_s e^{-\lambda(r-s)h} \frac{\{\lambda(r-s)h\}^{r-s}}{(r-s)!} &= \sum_{s=0}^r g_{r-s} e^{-\lambda sh} \frac{(\lambda sh)^s}{s!} \\ (15) \qquad \qquad \qquad &= e^{-\lambda(z_0+rh)} \frac{\{\lambda(z_0+rh)\}^r}{r!} \quad (r = 0, 1, \dots) \end{aligned}$$

for the $\{g_s\}$. By analogy with the first part of the equations (11) *et seq.*, it is readily seen that the distribution

$$\left\{ e^{-\lambda(z_0+rh)} \frac{\{\lambda(z_0+rh)\}^r}{r!} \right\} \quad (r = 0, 1, \dots)$$

has the Laplace-Stieltjes transform

$$G_1(\theta) = e^{\{\alpha_0(\theta) - \lambda - \theta\}z_0} \{1 - \lambda h e^{\{\alpha_0(\theta) - \lambda - \theta\}h}\}^{-1} \quad (\theta > 0),$$

while the distribution

$$\left\{ \frac{e^{-\lambda sh} (\lambda sh)^s}{s!} \right\} \quad (s = 0, 1, \dots)$$

of the same type, in which $z_0 = 0$, will have the transform

$$H(\theta) = \{1 - \lambda h e^{\{\alpha_0(\theta) - \lambda - \theta\}h}\}^{-1} \quad (\theta > 0).$$

Thus, if $G(\theta)$ is the transform of the distribution $\{g_s\}$, it follows from the convolution (15) that

$$G(\theta) = \frac{G_1(\theta)}{H(\theta)} = e^{\{\alpha_0(\theta) - \lambda - \theta\}z_0} \quad (\theta > 0)$$

as in (13), so that the probabilities $\{g_r\}$ are clearly of the form (9).

4. The probability of emptiness $F(0, t)$. Consider now the probability of emptiness $F(0, t)$ of the dam; this is

$$F(0, t) = \Pr \{Z(\tau + t) = 0 \mid Z(\tau) = z_0\},$$

the probability that the dam with initial content z_0 is empty after the interval $t \geq z_0$, though not necessarily for the first time. Suppose $[(t - z_0)/h]$

$= r$ ($r = 0, 1, 2, \dots$); then for the dam to be empty after time t , we may have at most r inputs during the interval $(\tau, \tau + t)$. Let us take the case where there are $j \leq r$ inputs during $(\tau, \tau + t)$; then for $Z(\tau + t) = 0$ there must be

$$\left\{ \begin{array}{ll} n_0 & \text{inputs in } (\tau, z_0 + [r-j]h \leq \tau + t - jh < z_0 + [r-j+1]h), \text{ where } n_0 \geq 1; \\ n_1 & \text{inputs in } (\tau + t - jh, \tau + t - [j-1]h), \text{ where } n_0 + n_1 \geq 2; \\ \cdot & \dots\dots\dots \\ n_{j-1} & \text{inputs in } (\tau + t - 2h, \tau + t - h), \text{ where } \sum_{i=0}^{j-1} n_i = j; \\ n_j & \text{inputs in } (\tau + t - h, \tau + t), \text{ where } n_i = 0. \end{array} \right.$$

This is precisely the same formulation as for the first emptiness problem (E) in Section 3, with $t - jh$ substituted for z_0 ; the only difference is that the total number of inputs j may now no longer be sufficient to prevent emptiness from occurring between the times τ and $\tau + t$, as well as at $\tau + t$. We conclude that the probability of emptiness after the time t , when there are j inputs into the dam, will be

$$(16) \quad \Pr \{Z(\tau + t) = 0 \mid Z(\tau) = z_0, X(t) = jh\} = e^{-\lambda t} \frac{\lambda^j}{j!} (t - jh)t^{j-1}.$$

Since the dam may empty in time t when there are any number of inputs $j = 0, 1, \dots, [(t - z_0)/h]$, it follows that

$$(17) \quad F(0, t) = e^{-\lambda t} \sum_{j=0}^{[(t-z_0)/h]} \frac{\lambda^j}{j!} (t - jh)t^{j-1}.$$

In the special case where $z_0 = 0$, this becomes

$$F_0(0, t) = e^{-\lambda t} \sum_{j=0}^{[t/h]} \frac{\lambda^j}{j!} (t - jh)t^{j-1},$$

and it may readily be verified that

$$F(0, t) = \sum_{j=0}^{[(t-z_0)/h]} g(z_0, z_0 + jh) F_0(0, t - z_0 - jh),$$

since the probability of emptiness of the dam after time t starting with a content z_0 is the sum of the probabilities of first emptiness after an interval $z_0 + jh$ ($j = 0, 1, \dots, [(t - z_0)/h]$) and of further emptiness after the remaining interval $t - z_0 - jh$.

It is possible here to obtain the transform of $F(0, t)$ with respect to t directly; for $s > 0$ we have

$$\begin{aligned}
F^*(s) &= \int_0^\infty e^{-st} F(0, t) dt = \int_{z_0}^\infty e^{-(s+\lambda)t} \sum_{j=0}^{\lfloor (t-z_0)/h \rfloor} \frac{\lambda^j}{j!} (t-jh)t^{j-1} dt \\
&= \sum_{r=0}^\infty \int_{z_0+rh}^\infty e^{-(s+\lambda)t} \frac{\lambda^r}{r!} (t-rh)t^{r-1} dt \\
(18) \quad &= \sum_{r=0}^\infty \int_{z_0}^\infty e^{-(s+\lambda)(u+rh)} \frac{\lambda^r}{r!} u(u+rh)^{r-1} du \\
&= \int_{z_0}^\infty du \sum_{r=0}^\infty e^{-(s+\lambda)(u+rh)} g(u, u+rh) \\
&= \int_{z_0}^\infty e^{\lfloor \alpha_0(s) - \lambda - s \rfloor u} du = \frac{e^{\lfloor \alpha_0(s) - \lambda - s \rfloor z_0}}{s + \lambda - \alpha_0(s)},
\end{aligned}$$

where $\alpha_0(s) \leq \lambda$ is the smaller root of $\alpha e^{-\alpha h} = \lambda e^{-(\lambda+s)h}$ ($s > 0$), exactly as in (12). This is of the general form given by BENEŠ [5], reducing to the special case derived by TAKÁCS [4] when $z_0 = 0$; the explicit form (17) for $F(0, t)$ has not, however, been obtained previously.

5. The general solution $F(z, t)$. We are now in a position to solve the difference-differential equation (2) for the d.f. $F(z, t)$ by direct methods. In order to simplify this equation, we first perform a change of variables from z, t to $u = z + t \geq t, t$. Let $F(z, t) = G(u, t)$; it is clear from the properties of the d.f. $F(z, t)$ that $G(u, t)$ is zero for all $u < t$ ($0 \leq t < \infty$), satisfies the condition that $0 < G(u, t) \leq 1$ for all $u \geq t$ ($0 \leq t < \infty$) and has the limit $\lim_{u \rightarrow \infty} G(u, t) = 1$ as $u \rightarrow \infty$, so that it is itself a d.f. Further, $G(u, t)$ will be everywhere continuous in $t < u < \infty$, but will have a discontinuity $G(t, t) = F(0, t)$ at $u = t$; its derivative $\partial G(u, t)/\partial u$ exists for all $t < u < t + h$ and $t + h < u < \infty$, but has only a right-hand value at $u = t$, while at $u = t + h$ the right and left-hand derivatives are distinct and satisfy the equation

$$\frac{\partial}{\partial u} G(t + h + 0, t) - \frac{\partial}{\partial u} G(t + h - 0, t) = -\lambda G(t, t).$$

After the change in variables, due attention being paid to the appropriate interpretation of the derivative $\partial G/\partial u$, the equation (2) may be written as

$$(19) \quad \frac{\partial}{\partial t} G(u, t) = -\lambda \{G(u, t) - G(u - h, t)\} \quad (t \leq u < \infty).$$

We can now proceed to a step by step solution of this in the ranges $t + jh \leq u < t + (j + 1)h$ ($j = 0, 1, 2, \dots$).

We begin with the first range $t \leq u < t + h$ in which the equation may be more simply expressed as

$$\frac{\partial G}{\partial t} = -\lambda G(u, t) \quad (t \leq u < t + h).$$

This readily yields the result

$$G(u, t) = e^{-\lambda t} A_1(u) \quad (t \leq u < t + h),$$

and $A_1(u)$ may be evaluated by putting $u = t$ when we find

$$G(t, t) = e^{-\lambda t} A_1(t) = F(0, t) \quad (0 \leq t < \infty).$$

It follows that $A_1(u) = e^{\lambda u} F(0, u)$, so that we finally obtain the d.f. $G(u, t)$ as

$$(20) \quad G(u, t) = e^{-\lambda(t-u)} F(0, u) \quad (t \leq u < t + h).$$

In the second range $t + h \leq u < t + 2h$, the equation (19), in which the expression for $G(u - h, t)$ lying in the first range has been substituted, will result in

$$\frac{\partial G}{\partial t} + \lambda G(u, t) = \lambda e^{-\lambda(t+h-u)} F(0, u - h) \quad (t + h \leq u < t + 2h).$$

By direct integration, it follows that

$$(21) \quad G(u, t) = \lambda(t + h - u) e^{-\lambda(t+h-u)} F(0, u - h) + e^{-\lambda t} A_2(u).$$

The function $A_2(u)$ may also be evaluated by making use of the continuity of $G(u, t)$ at $u = t + h$; from (20) and (21) we obtain that

$$G(t + h, t) = e^{\lambda h} F(0, t + h) = e^{-\lambda t} A_2(t + h),$$

so that $A_2(u) = e^{\lambda u} F(0, u)$. We thus have that the d.f. is

$$(22) \quad G(u, t) = \lambda(t + h - u) e^{-\lambda(t+h-u)} F(0, u - h) + e^{-\lambda(t-u)} F(0, u) \\ (t + h \leq u < t + 2h).$$

By recursion, it may be shown that in the $(n + 1)^{\text{th}}$ range, $t + nh \leq u < t + (n + 1)h$, the d.f. $G(u, t)$ may be written as

$$(23) \quad G(u, t) = \sum_{r=0}^n \frac{\{\lambda(t + rh - u)\}^r}{r!} e^{-\lambda(t+rh-u)} F(0, u - rh),$$

and this may be easily verified by induction. For, accepting (23) as the d.f. in $t + nh \leq u < t + (n + 1)h$, the difference-differential equation (19) for the range $t + (n + 1)h \leq u < t + (n + 2)h$ yields the relation

$$\frac{\partial G}{\partial t} + \lambda G(u, t) = \sum_{r=0}^n \frac{\lambda^{r+1}}{r!} \{t + (r + 1)h - u\}^r e^{-\lambda(t+(r+1)h-u)} F(0, u - rh - h).$$

By direct integration, we obtain after some manipulation that

$$(24) \quad e^{\lambda t} G(u, t) = \sum_{r=1}^{n+1} \frac{\{\lambda(t + rh - u)\}^r}{r!} e^{-\lambda(u-rh)} F(0, u - rh) + A_{n+2}(u),$$

and the continuity of $G(u, t)$ at $u = t + (n + 1)h$ gives

$$\begin{aligned} \sum_{r=1}^{n+1} \frac{\{\lambda(r-n-1)h\}^r}{r!} e^{\lambda(t+nh+h-rh)} F(0, u-rh) + A_{n+2}(t+nh+h) \\ = \sum_{r=0}^n \frac{\{\lambda(r-n-1)h\}^r}{r!} e^{\lambda(t+nh+h-rh)} F(0, t+nh+h-rh). \end{aligned}$$

It follows that $A_{n+2}(u) = e^{\lambda u} F(0, u)$ as before, so that from (24) we have that, in $t + (n+1)h \leq u < t + (n+2)h$,

$$G(u, t) = \sum_{r=0}^{n+1} \frac{\{\lambda(t+rh-u)\}^r}{r!} e^{-\lambda(t+rh-u)} F(0, u-rh),$$

of precisely the same form as (23).

We may now write $F(z, t)$ explicitly for the range $nh \leq z < (n+1)h$ as

$$(25) \quad F(z, t) = G(u, t) = \sum_{r=0}^n \frac{\{\lambda(rh-z)\}^r}{r!} e^{-\lambda(rh-z)} F(0, z+t-rh),$$

where $F(0, z+t-rh)$ is itself given from (17) by

$$(26) \quad \begin{aligned} F(0, z+t-rh) \\ = e^{-\lambda(z+t-rh)} \sum_{j=0}^{\lfloor (z+t-rh-z_0)/h \rfloor} \frac{\lambda^j}{j!} (z+t-rh-jh)(z+t-rh)^{j-1}. \end{aligned}$$

The Laplace-Stieltjes transform of (25), given in equation (6), may be written as

$$(27) \quad \psi(\theta, t) = e^{(\theta-\lambda(1-e^{-\theta h}))t} \left\{ e^{-\theta z_0} - \theta \int_0^t \sum_{j=0}^{\lfloor (u-z_0)/h \rfloor} \frac{\lambda^j}{j!} (u-jh)u^{j-1} e^{-(\theta+\lambda e^{-\theta h})u} du \right\};$$

it has not so far proved possible to obtain a more useful expression for this transform.

The transform $\psi^*(\theta, s)$ of $\psi(\theta, t)$ with respect to t , however, can be easily found. For $s > \theta - \lambda(1 - e^{-\theta h}) > 0$, it is readily evaluated as

$$\begin{aligned} (28) \quad \psi^*(\theta, s) &= \int_0^\infty e^{-st} \psi(\theta, t) dt = \int_0^\infty e^{-\{s-\theta+\lambda(1-e^{-\theta h})\}t} e^{-\theta z_0} dt \\ &\quad - \theta \int_0^\infty dt e^{-\{s-\theta+\lambda(1-e^{-\theta h})\}t} \int_0^t du F(0, u) e^{-(\theta+\lambda(1-e^{-\theta h}))u} \\ &= \frac{e^{-\theta z_0}}{s - \theta + \lambda(1 - e^{-\theta h})} \\ &\quad - \theta \int_0^\infty du F(0, u) e^{-(\theta+\lambda(1-e^{-\theta h}))u} \int_u^\infty dt e^{-\{s-\theta+\lambda(1-e^{-\theta h})\}t} \\ &= \frac{e^{-\theta z_0} - \theta \int_0^\infty e^{-su} F(0, u) du}{s - \theta + \lambda(1 - e^{-\theta h})} = \frac{e^{-\theta z_0} - \theta F^*(s)}{s - \theta + \lambda(1 - e^{-\theta h})} \end{aligned}$$

where $F^*(s)$ is given by the equation (18). This is precisely of the form derived by BENEŠ [5].

It is obvious that the stationary distribution $\Phi(z) = \lim_{t \rightarrow \infty} F(z, t)$ of (4) can be obtained directly from (25). For it is already known that

$$\lim_{t \rightarrow \infty} F(0, z + t - rh) = 1 - \lambda h,$$

so that the limit as $t \rightarrow \infty$ of (25) yields

$$\Phi(z) = (1 - \lambda h) \sum_{r=0}^{n=\lfloor z/h \rfloor} e^{-\lambda(rh-z)} \frac{\{\lambda(rh-z)\}^r}{r!},$$

as in (4).

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