A Non-linear Random Model of Gravity Waves I

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1. Introduction. The equations describing gravity waves are non-linear. The common procedure used to solve the equations is to linearize them using the theory of small departures about some equilibrium condition. The random model for gravity waves derived by PIERSON [3] was based on such a linearization. Certain shortcomings of this linear model were indicated by measurements made in a towing tank. The indications were (a) coherencies less than unity between the measurements made at two points on a line parallel to the direction of propagation of the waves and (b) a small second peak at about, but not quite, twice the frequency of the primary peak. It is the purpose of this paper to derive an improvement to the linear model for gravity waves (in the infinite crested case) using the quadratic terms in the perturbation expansions of the relevant differential equations.

2. Equations of motion. In the infinite crested, infinite depth case the equations of motion for an irrotational fluid (neglecting surface tension) are

$$(1) \quad \phi_{xx} + \phi_{zz} = 0,$$

$$(2) \quad \phi_z \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty,$$

$$(3) \quad \frac{p}{\rho} = gz - \phi_t - \frac{1}{2}(\phi_x^2 + \phi_z^2)$$

where $\phi = \phi(x, y, t)$ = velocity potential, p = pressure, ρ = density, the axes are at the undisturbed surface of the fluid and z is positive downward. We assume the atmospheric pressure is zero. The free surface is given by the curve $\eta = \eta(x, t)$ which is such as to make the pressure in the fluid, at the surface, equal to zero, *i.e.*

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$$(4) \quad g\eta - \phi_t - \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \quad \text{at } z = \eta.$$

The free surface must also satisfy the kinematic boundary condition

$$(5) \quad \eta_t + \phi_x \eta_x = \phi_z \quad \text{at } z = \eta.$$

Let us make the expansion

$$(6) \quad \phi(x, z, t) = \phi(x, 0, t) + \phi_z(x, 0, t)z + \cdots.$$

Substituting (6) into equations (4) and (5) and neglecting terms of order higher than the second in ϕ or ϕ and η mixed, we obtain the following equations for the free surface:

$$(7) \quad g\eta - \phi_t - \phi_{tz}\eta - \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0, \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \text{at } z = 0.$$

$$(8) \quad \eta_t + \phi_x \eta_x = \phi_z + \phi_{zz}\eta$$

Solving (7) for η and neglecting third and higher order terms again, we obtain

$$(9) \quad \eta = \frac{1}{g} \left[\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + \frac{1}{g} \phi_{tz} \phi_t \right] \quad \text{at } z = 0.$$

Substituting (9) into (8) and again neglecting the higher order term, we obtain an equation in ϕ alone:

$$(10) \quad (\phi_{tt} - g\phi_z) = \phi_{zz}\phi_t - 2\phi_x\phi_{tx} - \phi_z\phi_{zt} - \frac{1}{g}(\phi_{tz}\phi_{tt} + \phi_{tzt}\phi_t) \quad \text{at } z = 0.$$

A more complete expansion to higher order can be found in STOKER [4].

3. Linear approximation. The plan used here is the usual perturbation procedure. That is, a solution of the system (1), (2), (10), assuming the quadratic terms in (10) are zero, is obtained. Call this $\phi^{(1)}$. $\phi^{(1)}$ is then substituted into the right hand side of (10) and (1), (2) and (10) are solved for, say, $\phi^{(2)}$. $\phi = \phi^{(1)} + \phi^{(2)}$ is then a solution of (1), (2) and (10) correct to the second order. η is then derived from (9) by using ϕ and neglecting terms higher than second order.

It is assumed that $\eta(x, t)$ is a stationary random process with respect to x and t . Since ϕ is related to η in a time and space invariant way we desire ϕ to have this same stationary property.

The spectral representation theorem for a stationary process [1] then states that

$$(11) \quad \phi(x, z, t) = \iint_{-\infty}^{\infty} e^{i(\alpha x + \beta t)} d\xi_1(\alpha, \beta; z) + M$$

where $d\xi_1(\alpha, \beta; z)$ is the differential of a two-dimensional random process of uncorrelated increments which has the property

$$(12) \quad E[d\xi_1(\alpha, \beta; z) d\xi_1(\alpha', \beta'; z)] = \begin{cases} dS(\alpha, \beta; z) & \text{if } \alpha = \alpha', \quad \beta = \beta' \\ 0 & \text{otherwise.} \end{cases}$$

$S(\alpha, \beta; z)$ is the spectral distribution function, and M is the mean which we assume to be zero. It is also necessary to assume that moments of $S(\alpha, \beta; z)$ exist so that the relevant derivatives are defined.

Since ϕ must satisfy (1) and (2) it is well known that

$$(13) \quad d\xi_1(\alpha, \beta; z) = d\xi_2(\alpha, \beta) e^{-|\alpha|z}.$$

With the quadratic terms set equal to zero, (10) becomes

$$(14) \quad \phi_{it}^{(1)} - g\phi_z^{(1)} = 0, \quad z = 0.$$

Substituting (11) and (13) into (14), we obtain

$$(15) \quad \int_{-\infty}^{\infty} e^{i(\alpha x + \beta t)} (-\beta^2 + g|\alpha|) d\xi_2^{(1)}(\alpha, \beta) = 0.$$

In order for the integral in (15) to be zero it is necessary for the spectral distribution of the process defined by the left hand side of (15) to be zero, *i.e.*

$$(16) \quad (-\beta^2 + g|\alpha|)^2 dS^{(1)}(\alpha, \beta) \equiv 0.$$

Aside from the trivial solution $S^{(1)}(\alpha, \beta) \equiv 0$, the solution of (16) is $S^{(1)}(\alpha, \beta)$ constant except for a discontinuity along $-\beta^2 + g|\alpha| = 0$. The double integral in (15) can thus be written as the sum of four single integrals

$$(17) \quad \begin{aligned} \phi^{(1)} = & \int_0^{\infty} [e^{i[(\omega^2/g)x + \omega t]} d\xi_3^{(1)}(\omega) + e^{i[(\omega^2/g)x - \omega t]} d\xi_3^{(1)}(\omega)] \\ & + \int_{-\infty}^0 [e^{i[(-\omega^2/g)x + \omega t]} d\xi_3^{(1)}(\omega) + e^{i[(-\omega^2/g)x - \omega t]} d\xi_3^{(1)}(\omega)] \quad \text{at } z = 0. \end{aligned}$$

The first and third integrals represent waves traveling in the $-x$ direction. We neglect them* and obtain

$$(18) \quad \phi^{(1)}(x, z, t) = \int_{-\infty}^{\infty} e^{i[\omega|\omega x/g - \omega t]} e^{-\omega^2 z/g} d\xi_3^{(1)}(\omega).$$

Here $d\xi_3^{(1)}(\omega)$ is a one-dimensional process of uncorrelated increments. Substituting (18) into (9) and neglecting quadratic terms, we obtain the *linear* η :

$$(19) \quad \eta^{(1)}(x, t) = -i \int_{-\infty}^{\infty} \frac{\omega}{g} e^{i[\omega|\omega x/g - \omega t]} d\xi_3^{(1)}(\omega).$$

We can make (18) simpler by absorbing $-i\omega/g$ into the random differential:

$$(18a) \quad \phi^{(1)} = \int_{-\infty}^{\infty} e^{i[\omega|\omega x/g - \omega t]} e^{\omega^2 z/g} \left(\frac{g}{\omega} i \right) d\xi(\omega),$$

$$(19a) \quad \eta^{(1)} = \int_{-\infty}^{\infty} e^{i[\omega|\omega x/g - \omega t]} d\xi(\omega).$$

We have dropped the (1) superscript on the differential.

*If these are not neglected, the possibility of standing waves exists.

We invoke the *additional* assumption that ϕ , and thus η , is Gaussian. With this further restriction (18a) is fully equivalent to the model given by PIERSON in the long crested case.

4. Potential function quadratic approximation. $\phi^{(1)}$ is now substituted into the right hand side of (10). Noting that $\phi_{it}^{(1)} = g\phi_z^{(1)}$ and $\phi_{itz}^{(1)} = g\phi_{zz}^{(1)}$, we obtain equation (20):

$$(20) \quad g\phi_z^{(2)} - \phi_{it}^{(2)} = 2\phi_z^{(1)}\phi_{zt}^{(1)} + 2\phi_x^{(1)}\phi_{xt}^{(1)} \quad \text{at } z = 0.$$

Since $\phi^{(2)}$ is a stationary process and must satisfy (1) and (2), it must also be of the form (11), (13). In spectral terms (20) becomes

$$(21) \quad \iint e^{i(\alpha x + \beta t)} (\beta^2 - g|\alpha|) d\xi^{(2)}(\alpha, \beta) \\ = 2i \iint e^{i[(|\omega| + |\omega'|)x/g - (\omega + \omega')t]} (\omega^2\omega' - |\omega||\omega'|) d\xi(\omega) d\xi(\omega') \quad \text{at } z = 0.$$

If we make the substitution

$$(22) \quad g\alpha = |\omega||\omega| + |\omega'||\omega', \quad \beta = -(\omega + \omega')$$

the solution of (21) becomes, on converting back to the ω, ω' system,

$$(23) \quad \phi^{(2)}(x, 0, t) = 2i \iint \exp \left\{ i \left[(|\omega||\omega| + |\omega'||\omega') \frac{x}{g} - (\omega + \omega')t \right] \right\} G(\omega, \omega') d\xi(\omega) d\xi(\omega')$$

where

$$(24) \quad G(\omega, \omega') = \frac{\omega(\omega\omega' - |\omega\omega'|)}{(\omega + \omega')^2 - ||\omega||\omega| + ||\omega'||\omega'|}.$$

$\phi^{(2)}(x, z, t)$ is, of course, (23) with an additional factor inside the integrand. This factor is $e^{-[\omega^2 + \omega'^2] z/g}$.

The properties of $G(\omega, \omega')$ can be expressed in simple terms. For this purpose it is convenient to divide the (ω, ω') plane into 45° sectors and label them R_1 through R_8 , starting from the positive ω axis and proceeding counterclockwise. Then

$$(25) \quad G(\omega, \omega') = \begin{cases} 0 & \text{in } R_1, R_2, R_5, R_6, \\ \omega\omega'/\omega + \omega' & \text{in } R_3, R_7, \\ \omega^2/\omega + \omega' & \text{in } R_4, R_8. \end{cases}$$

We will also need $\phi_t(x, 0, t)$. This is essentially (24) with $G(\omega, \omega')$ replaced by $F(\omega, \omega') = -iG(\omega, \omega')(\omega + \omega')$; ϕ is now $\phi = \phi^{(1)} + \phi^{(2)}$.

5. Free surface quadratic approximation. $\phi = \phi^{(1)} + \phi^{(2)}$ is substituted into (9). We get

$$(26) \quad \eta = \frac{1}{g} \phi_t^{(1)} - \frac{1}{g} \phi_t^{(2)} + \frac{1}{2g} (\phi_z^{(1)})^2 + \frac{1}{2g} (\phi_x^{(1)})^2 + \frac{1}{g^2} \phi_{tz}^{(1)} \phi_t^{(1)} = \eta^{(1)} + \eta^{(2)}$$

where $\eta^{(1)}$ is, by (19a), equal to $(1/g)\phi_t^{(1)}$ (at $z = 0$). $\eta^{(2)}$ is made up of second order terms and we have neglected those of higher order.

In spectral terms $\eta^{(2)}$ becomes

$$(27) \quad \eta^{(2)} = \frac{1}{g} \iint \exp [-] H(\omega, \omega') d\xi(\omega) d\xi(\omega')$$

where

$$(28) \quad \begin{aligned} H(\omega, \omega') &= 2F(\omega, \omega') - \frac{1}{2}\omega\omega' + \frac{1}{2}|\omega\omega'| - \omega^2 \\ &= \begin{cases} -\omega^2 & \text{in } R_1, R_2, R_5, R_6, \\ \omega\omega' - \omega^2 & \text{in } R_3, R_7, \\ \omega^2 - \omega\omega' & \text{in } R_4, R_8. \end{cases} \end{aligned}$$

It should be noticed that up to now we have not needed the Gaussian assumption on $\phi^{(1)}$. For the remainder of the paper we invoke the assumption that $\eta^{(1)}(t)$ is Gaussian. As is the usual case with random process analyses, this is a convenient assumption to make. However, there is considerable physical evidence in favor of such an assumption. Many predictions about the sea surface using $\eta^{(1)}(t)$, with the Gaussian assumption, as a model for gravity waves in the open ocean have been verified [3].

In a sense we are making a double perturbation: one of the differential equation, about some equilibrium condition, and the other of the probability structure, about the Gaussian "point."

6. Spectrum of the quadratic process. The spectrum of the process $\eta = \eta^{(1)} + \eta^{(2)}$ at a fixed point in space can be easily evaluated by taking the expected value of $\eta(x, t)\eta(x, t + \tau)$ and using the Wiener-Kintchine theorem. First notice that

$$(29) \quad \begin{aligned} R(\tau) &= E[\eta(x, t)\eta(x, t + \tau)] = E[\eta^{(1)}(x, t)\eta^{(1)}(x, t + \tau)] \\ &\quad + E[\eta^{(2)}(x, t)\eta^{(2)}(x, t + \tau)] = R^{(1)}(\tau) + R^{(2)}(\tau). \end{aligned}$$

The cross product term is zero as it is the expected value of an integral of triples of Gaussian random variables and this is always zero if the means of the variables are zero. It remains to evaluate terms of the form

$$(30) \quad E\{d\xi(\omega_1) d\xi(\omega_2) d\xi(\omega_3) d\xi(\omega_4)\}.$$

Since the $d\xi(\omega)$ are Gaussian random variables with zero mean, the expected value of 4-fold products can be written in terms of the expected value of all

two-fold products [2]. Thus (30) becomes

$$\begin{aligned}
 & \mathbf{E}\{d\xi(\omega_1) d\xi(\omega_2)\} \mathbf{E}\{d\xi(\omega_3) d\xi(\omega_4)\} + \mathbf{E}\{d\xi(\omega_1) d\xi(\omega_3)\} \mathbf{E}\{d\xi(\omega_2) d\xi(\omega_4)\} \\
 & \quad + \mathbf{E}\{d\xi(\omega_1) d\xi(\omega_4)\} \mathbf{E}\{d\xi(\omega_2) d\xi(\omega_3)\} \\
 & = s(\omega_1) d\omega_1 s(\omega_3) d\omega_3 \quad \text{if } \begin{cases} \omega_3 = \omega_4, & \omega_1 = \omega_2 \\ 0 & \text{otherwise} \end{cases} \\
 & + s(\omega_1) d\omega_1 s(\omega_2) d\omega_2 \quad \text{if } \begin{cases} \omega_1 = \omega_3, & \omega_2 = \omega_4 \\ 0 & \text{otherwise} \end{cases} \\
 & + s(\omega_1) d\omega_1 s(\omega_2) d\omega_3 \quad \text{if } \begin{cases} \omega_1 = \omega_4, & \omega_2 = \omega_3 \\ 0 & \text{otherwise.} \end{cases}
 \end{aligned}$$

We have used the basic result that

$$\mathbf{E}\{d\xi(\omega) \overline{d\xi(\omega')}\} = s^{(1)}(\omega) d\omega$$

if $\omega = \omega'$ and $d\xi(\omega) = \overline{d\xi(-\omega)}$.

With these results

$$\begin{aligned}
 R^{(2)}(\tau) &= \frac{1}{g^2} \iiint \exp \left\{ (|\omega_1| \omega_1 + |\omega_2| \omega_2 - |\omega_3| \omega_3 - \omega_4 |\omega_4|) \frac{x}{g} \right. \\
 & \quad \left. - (\omega_1 + \omega_2)t + (\omega_3 + \omega_4)(t + \tau) \right\} \\
 (32) \quad & \cdot H(\omega_1, \omega_2) H(\omega_3, \omega_4) \mathbf{E} d\xi(\omega_1) d\xi(\omega_2) \overline{d\xi(\omega_3)} \overline{d\xi(\omega_4)} \\
 &= \frac{1}{g^2} \iint e^{i(\omega + \omega')\tau} [H^2(\omega, \omega') + H(\omega, \omega') H(\omega', \omega)] s^{(1)}(\omega) s^{(1)}(\omega') d\omega d\omega'.
 \end{aligned}$$

Upon making the transformation $\omega + \omega' = \lambda$ the spectrum of the quadratic portion is

$$(33) \quad s^{(2)}(\lambda) = \frac{1}{g^2} \int_{-\infty}^{\infty} K(\omega; \lambda) s^{(1)}(\lambda - \omega) s^{(1)}(\omega) d\omega$$

where

$$\begin{aligned}
 K(\omega; \lambda) &= H^2(\omega, \lambda - \omega) + H(\omega, \lambda - \omega) H(\lambda - \omega, \omega) \\
 (34) \quad &= \begin{cases} \omega^2(\lambda^2 - 2\lambda\omega + 2\omega^2), & 0 < \omega < \lambda, \quad \lambda > 0 \\ (\lambda - 2\omega)^2 \omega \lambda, & \omega < 0, \quad \omega > \lambda, \quad \lambda > 0. \end{cases}
 \end{aligned}$$

Since the spectrum is symmetrical it is only necessary to have $K(\omega; \lambda)$ for $\lambda > 0$.

It is interesting to notice that $s^{(2)}(\lambda)$ will contain no energy around $\lambda = 0$. It is usually the case that a quadratic operator distributes energy around zero. Also notice that the total energy content in $s^{(2)}(\lambda)$ is small due to the $1/g^2$. The total spectrum is thus

$$(35) \quad s(\omega) = s^{(1)}(\omega) + s^{(2)}(\omega).$$

By the same procedure it is easy to evaluate the cross-spectrum. This is the Fourier transform of $E(\eta(x, t)\eta(x + \xi, t + \tau))$. This is

$$(36) \quad c(\lambda) + iq(\lambda) = e^{-i\xi\lambda^{2/3}} s^{(1)}(\lambda) + \frac{1}{g^{2/3}} \int_{-\infty}^{\infty} e^{-i\xi\{|\omega| + |\lambda - \omega|(\lambda - \omega)^{1/3}\}} K(\omega; \lambda) s^{(1)}(\lambda - \omega) s^{(1)}(\omega) d\omega,$$

where c is the co-spectrum and q is the quad-spectrum.

Careful note should be taken of a distinct difference between the nonlinear behavior of systems with continuous and discrete spectra. Most of the analyses to be found in the literature concern a single frequency in the linear spectrum. The first perturbation will generate the first harmonic. This will have the effect of changing the profile of the wave. Since the speed of propagation of the first harmonic is the same as the fundamental, the wave will propagate without change of form, and the notion of wave speed is still relevant. However, when the linear term contains a continuous number of frequencies, then energy in any frequency band of the first perturbation will have come from all frequencies in the linear spectrum. (This follows from the convolution operation in (33).) Since the speed of the "quadratic" wave depends on the frequencies of the linear waves from which it came, the notion of speed of each spectral component is lost, as far as the quadratic energy is concerned.

7. Covariance function. It is possible to write (33) in a different form so that it becomes a convolution integral. In this way the covariance function can be determined almost by observation.

Now $K(\omega; \lambda)$ can be written as $K_1(\omega; \lambda) + K_2(\omega; \lambda)$ where, for $\lambda > 0$,

$$(37) \quad \begin{aligned} K_1(\omega; \lambda) &= (\lambda - \omega)^3 \omega - (\lambda - \omega)^2 \omega^2 - (\lambda - \omega) \omega^3 + \omega^4 \\ K_2(\omega; \lambda) &= \begin{cases} \omega^3(\lambda - \omega) + 2(\lambda - \omega)^2 \omega^2 - (\lambda - \omega)^3 \omega, & 0 < \omega < \lambda \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$K_2(\omega; \lambda)$ can be written in a way such that it has one analytical definition over the entire ω range by introducing the $\text{sgn}(x)$ (sign of x) operator. In these terms

$$(38) \quad K_2(\omega; \lambda) = \frac{(1 + \text{sgn}(\lambda - \omega))(1 + \text{sgn} \omega)}{4} \cdot [\omega^3(\lambda - \omega) + 2(\lambda - \omega)^2 \omega^2 - (\lambda - \omega)^3 \omega].$$

Thus we write

$$(39) \quad \begin{aligned} s^{(2)}(\lambda) &= \frac{1}{g^{2/3}} \int_{-\infty}^{\infty} \{\omega^4 - (\lambda - \omega)\omega^3 - (\lambda - \omega)^2 \omega^2 + (\lambda - \omega)^3 \omega \\ &\quad + \frac{1}{4}[(1 + \text{sgn}(\lambda - \omega))(1 + \text{sgn} \omega)] \\ &\quad \cdot [\omega^3(\lambda - \omega) + 2(\lambda - \omega)^2 \omega^2 - (\lambda - \omega)^3 \omega]\} s^{(1)}(\lambda - \omega) s^{(1)}(\omega) d\omega. \end{aligned}$$

The covariance function of the linear part of the spectrum is

$$(40) \quad R_1(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} s^{(1)}(\omega) d\omega.$$

Also we define

$$(41) \quad Q_1(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} \operatorname{sgn} \omega s^{(1)}(\omega) d\omega.$$

It can also be shown that [5]

$$(42) \quad Q(\tau) = \frac{1}{\pi} \text{P} \int_{-\infty}^{\infty} \frac{R(t)}{\tau - t} dt,$$

where P refers to the principal value. Therefore by Parseval's theorem

$$(43) \quad s^{(2)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{e^{-i|\lambda|t} R_1(t) R_1^{(iv)}(t) - \frac{1}{2} [R_1''(t)]^2 + \frac{1}{2} [Q_1''(t)]^2 + Q_1' R_1'\} dt.$$

FIGURE 1. $S^{(2)}(\omega)$ FOR $S^{(1)}(\omega) = \frac{1.8 \times 10^4}{\omega^6}$, $|\omega| > \omega_0$.

By introducing $|\lambda|$ in the exponent, we have the form correct for all λ .

Equation (43) can be inverted, giving

$$(44) \quad R_2(\tau) = R_1^{(iv)}(\tau)R_1(\tau) - \frac{1}{2}[R_1''(\tau)]^2 + \frac{1}{2}[Q_1''(\tau)]^2 + i\mathcal{P} \int_{-\infty}^{\infty} \frac{Q_1''(t)R_1''(t)}{\pi(\tau - t)} dt.$$

A similar type of reduction can be done on (36). Here one is dealing with convolutions of the form

$$\omega^n e^{-i\xi|\omega|\omega/g} s(\omega) * \omega^m e^{-i\xi|\omega|\omega/g} s(\omega)$$

or those involving $\text{sgn } \omega$. As simple a form as (44) is not obtained due to the exponential.

8. An example. For the sake of illustration a calculation of $s^{(2)}(\omega)$ was made using

$$s^{(1)}(\omega) = \frac{1.8 \times 10^4}{\omega^6}, \quad |\omega| > \omega_0 = 2, 3, 6 \text{ rad/sec} \quad \text{and} \quad g = 10^3 \text{ cm/sec}^2.$$

The curves are given in Figure 1.

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