The Holonomy Group

II. The Lie Group Induced by a Tensor*

V. HLAVATÝ

Summary. This paper is a continuation of [2]. It deals with the Lie group induced by an arbitrary tensor. The first part describes the construction of the induced Lie group. The second part deals with the application to the curvature tensor. The Lie group induced by the curvature tensor is the holonomy group. The holonomy group is a very convenient tool for solving partial differential equations for covariant constant tensors.** The third part introduces some basic notions and auxiliary results which will be used in the next paper of this series.***

I. THE LIE GROUP INDUCED BY A TENSOR

1. The basic notion. Denote by L_n an n -dimensional space endowed with a symmetric connection $\Gamma_{\lambda}^{\nu\mu}$ and by ∇_{μ} the symbol of the covariant derivative with respect to $\Gamma_{\lambda}^{\nu\mu}$. The local tangential flat space to L_n at a generic point P will be denoted by A_n or $A_n(P)$.

Let $T_{\psi_1 \dots \psi_q \omega_1 \dots \omega_s}^{\varphi_1 \dots \varphi_p \lambda_1 \dots \lambda_u}$ be a given tensor in L_n . We shall assume that it is neither symmetric nor skew symmetric in $\lambda_1 \dots \lambda_u$, $\omega_1 \dots \omega_s$ and that $quv \neq 0$. Moreover, we shall use capital Greek indices Φ_p , Ψ_q , Λ_u , Ω_s instead of $\varphi_1 \dots \varphi_p$, $\psi_1 \dots \psi_q$, $\lambda_1 \dots \lambda_u$, $\omega_1 \dots \omega_s$, so that we shall write $T_{\Psi_q \Omega_s}^{\Phi_p \Lambda_u}$ instead of $T_{\psi_1 \dots \psi_q \omega_1 \dots \omega_s}^{\varphi_1 \dots \varphi_p \lambda_1 \dots \lambda_u}$. The s^{th} covariant derivative $\nabla_{\xi_s} \dots \nabla_{\xi_1} T_{\Psi_q \Omega_s}^{\Phi_p \Lambda_u}$ will be denoted by $T_{\Psi_{q+s} \Omega_s}^{\Phi_p \Lambda_u}$ and, in particular, $T_{\Psi_{q+s} \Omega_s}^{\Phi_p \Lambda_u} \stackrel{\text{def}}{=} T_{\Psi_q \Omega_s}^{\Phi_p \Lambda_u}$. Here this abridged notation has only a formal significance, later it will help us to refine certain basic notions.

By the symbol $\{T_{\Psi_{q+s} \Omega_s}^{\Phi_p \Lambda_u}\}$ we denote the set of all tensors $T_{\Psi_{q+s} \Omega_s}^{\Phi_p \Lambda_u}$ for $s = 0, 1, 2, \dots$. Let $U_{\Phi_p}^{\Psi_{q+s}}$ be any arbitrary tensor with $q + s$ contravariant and p covariant indices. Put

$$(1.1) \quad L_{\Omega_s}^{\Lambda_u}(U) \stackrel{\text{def}}{=} T_{\Psi_{q+s} \Omega_s}^{\Phi_p \Lambda_u} U_{\Phi_p}^{\Psi_{q+s}}$$

*This paper was prepared with the support of Aeronautical Research Laboratory, Wright Air Development Center, Contract No. AF 33(616)-5695.

**These tensors, invariant with respect to the holonomy group, will be investigated in the third paper of this series.

***Small Greek indices run over the range of the first n Roman numerals.

and denote by $((T_s))$ the matrices

$$(1.2a) \quad ((T_s)) \stackrel{\text{def}}{=} ((\cdots L_{\Omega_s}^{\Lambda_u}(U) \cdots)) \quad (s = 0, 1, 2, \cdots).$$

Each row of this infinite matrix contains one arbitrary tensor U . Next consider the matrix

$$(1.2b) \quad ((T)) \stackrel{\text{def}}{=} \begin{bmatrix} ((T_0)) \\ ((T_1)) \\ ((T_2)) \\ \vdots \end{bmatrix}.$$

This matrix gives rise to the following

Definition (1.1). The Ω_s^u -rank of $\{T_{\Psi_{q+}, \Omega_s}^{\Phi_p \Lambda_u}\}$ is the rank r of the matrix $((T))$. The Ω_s^u -domain of $\{T_{\Psi_{q+}, \Omega_s}^{\Phi_p \Lambda_u}\}$ is the set of all tensors (1.1).

Remark. The rank of a quadratic tensor is by definition the rank of the $n \times n$ matrix of its components.

Lemma (1.1). We have

$$r \leq N \stackrel{\text{def}}{=} n^{u+v}.$$

Proof. Each row of $((T))$ consists of all components of one of the tensors (1.1). There are at most n^{u+v} linearly independent tensors (1.1). Hence there are at most n^{u+v} linearly independent rows of $((T))$.

Definition (1.2). The Ω_s^u -support of $\{T_{\Psi_{q+}, \Omega_s}^{\Phi_p \Lambda_u}\}$ is the union of two spaces:

- (1) The space spanned by all tensors $L_{\Omega_s}^{\Lambda_u}(U)V^{\Omega_v}$ for all U 's and V 's.
- (2) The space common to all tensors $L_{\Omega_s}^{\Lambda_u}(U)V_{\Lambda_u}$ for all U 's and V 's.

In the next theorem, as well as in all subsequent considerations, German indices run from 1 to r . They follow also the usual summation agreement.

Theorem (1.1). If the Ω_s^u -rank of $\{T_{\Psi_{q+}, \Omega_s}^{\Phi_p \Lambda_u}\}$ is r then there is a set of r tensors $L_{a\Omega_s}^{\Lambda_u}$ with the following properties:

- (1) They are linearly independent* and known up to the transformations

$$(1.3) \quad \begin{aligned} L_{a' \Omega_s}^{\Lambda_u} &= A_a^{\epsilon'} L_{\epsilon \Omega_s}^{\Lambda_u}, \\ L_{\epsilon \Omega_s}^{\Lambda_u} &= A_a^{\epsilon'} L_{a' \Omega_s}^{\Lambda_u}, \end{aligned}$$

where the A 's are arbitrary except for the restriction

$$(1.4a) \quad A_a^{\epsilon'} A_{\epsilon'}^a = \delta_a^{\epsilon'}.$$

*We understand here the linear independence in the usual geometrical sense. The matrix of r rows, the a^{th} row of which contains all components of $L_{a\Omega_s}^{\Lambda_u}$, is of rank r .

equivalent to

$$(1.4b) \quad A_a^{\epsilon'} A_\epsilon^a = \delta_a^\epsilon.$$

(2) They span the Λ_v^u -domain of $\{T_{\Psi_{q+s}, \Omega_v}^{\Phi_p \Lambda_u}\}$.

(3) There is at least one set of r tensors $L_{\Lambda_u}^{a\Omega_v}$ such that

$$(1.5) \quad L_{\Lambda_u}^{a\Omega_v} L_{\epsilon\Omega_v}^{\Lambda_u} = \delta_\epsilon^a.$$

(4) If

$$(1.6a) \quad M^{a\Phi_p}_{\Psi_{q+s}} \stackrel{\text{def}}{=} T_{\Psi_{q+s}, \Omega_v}^{\Phi_p \Lambda_u} L_{\Lambda_u}^{a\Omega_v}$$

then

$$(1.6b) \quad T_{\Psi_{q+s}, \Omega_v}^{\Phi_p \Lambda_u} = M^{\epsilon\Phi_p}_{\Psi_{q+s}} L_{\epsilon\Omega_v}^{\Lambda_u}.$$

Proof. The Λ_v^u -rank of $\{T_{\Psi_{q+s}, \Omega_v}^{\Phi_p \Lambda_u}\}$ is by Definition (1.1) the rank of the matrix $((T))$. If this rank is r , all rows of $((T))$ may be expressed as linear combinations of a certain set of r rows of $((T))$. These r rows constitute a matrix of rank r . Therefore there exists a set of r tensors $L_{\Lambda_u}^{a\Omega_v}$ with the properties (1) and (2). Consequently there is at least one system of r tensors $L_{\Lambda_u}^{a\Omega_v}$ which satisfies (1.5). If $r = N$, then the system of $L_{\Lambda_u}^{a\Omega_v}$ is unique. It is clear from these results that there must be a set of tensors $M^{a\Phi_p}_{\Psi_{q+s}}$ such that (1.6b) holds. Multiplying this equation by $L_{\Lambda_u}^{a\Omega_v}$ one obtains (1.6a).

Remark. The $L_{\Lambda_u}^{a\Omega_v}$ may be looked upon as intermediate components of a "unit" tensor which maps an arbitrary vector v^a of an auxiliary flat affine space A_r into a tensor $V_{\Omega_v}^{\Lambda_u}$ of the tangential flat affine space of L_n at P :

$$(1.7) \quad V_{\Omega_v}^{\Lambda_u} = v^a L_{a\Omega_v}^{\Lambda_u}.$$

Here, of course, the vector v^a of A_r is subject to the transformation rule

$$(1.8) \quad v^{a'} = A_\epsilon^{a'} v^\epsilon, \quad v^\epsilon = A_a^\epsilon v^{a'}.$$

Definition (1.3). Let $W_{\Lambda_u}^{\Omega_v}$ and $U_{\Phi_p}^{\Psi_{q+s}}$ be arbitrary tensors. Put

$$(1.9) \quad \begin{aligned} (a) \quad L_a(W) &\stackrel{\text{def}}{=} L_{a\Omega_v}^{\Lambda_u} W_{\Lambda_u}^{\Omega_v}, \\ (b) \quad M^a(U) &\stackrel{\text{def}}{=} M^{a\Phi_p}_{\Psi_{q+s}} U_{\Phi_p}^{\Psi_{q+s}}. \end{aligned}$$

The α -rank of $L_{\Lambda_u}^{a\Omega_v}$ is the rank of the infinite matrix

$$(1.10a) \quad ((L)) \stackrel{\text{def}}{=} ((L_1(W), L_2(W), \dots, L_r(W)))$$

for all W 's and the α -domain of $L_{\Lambda_u}^{a\Omega_v}$ in A_r is the set of all vectors $L_a(W)$ (for

all W 's) in A_r . The α -rank of $M^{\alpha\Phi_p}_{\Psi_{q+s}}$ is the rank of the infinite matrix

$$(1.10b) \quad ((M_s)) \stackrel{\text{def}}{=} ((M^1_s(U), M^2_s(U), \dots, M^r_s(U)))$$

for all U 's and the α -domain of $M^{\alpha\Phi_p}_{\Psi_{q+s}}$ in A_r is the set of all vectors $M^{\alpha}_s(U)$ (for all U 's) in A_r .

This definition enables us to state the following two theorems which will be proved simultaneously.

Theorem (1.2a). The Λ_u -rank of $\{T^{\Phi_p\Lambda_u}_{\Psi_{q+s}\Omega_v}\}$ and the α -rank of $L_{\alpha\Omega_v}^{\Lambda_u}$ have the same value r .

Theorem (1.2b). If and only if the Λ_u -rank of $T^{\Phi_p\Lambda_u}_{\Psi_q\Omega_v}$ is r ,* then the α -rank of $M^{\alpha\Phi_p}_{\Psi_q}$ is also r .

Proof. There are at most $N = n^{u+v}$ linearly independent tensors $W_{\Lambda_u}^{\Omega_v}$, which we denote by $W_{\Lambda_u}^{\Omega_v}_1, \dots, W_{\Lambda_u}^{\Omega_v}_N$. We may assume without loss of generality that the k th tensor $W_{\Lambda_u}^{\Omega_v}_k$ has the k th component equal to 1, the remaining components being zero, $k = 1, \dots, N$. Then every tensor $W_{\Lambda_u}^{\Omega_v}$ is a linear combination of $W_{\Lambda_u}^{\Omega_v}_1, \dots, W_{\Lambda_u}^{\Omega_v}_N$ and the rank of the infinite matrix (1.10a) is the same as the rank of the $r \times N$ matrix of the rows belonging to $W_{\Lambda_u}^{\Omega_v}_1, \dots, W_{\Lambda_u}^{\Omega_v}_N$, i.e.,

$$(1.11) \quad \left[\begin{array}{cccc} \vdots & \vdots & \vdots & \vdots \\ L_{1\Omega_v}^{\Lambda_u} & L_{2\Omega_v}^{\Lambda_u} & \dots & L_{r\Omega_v}^{\Lambda_u} \\ \vdots & \vdots & \vdots & \vdots \end{array} \right] \quad (N \text{ rows}).$$

According to the statement (1) of Theorem (1.1), the rank of (1.11) is equal to r . This proves Theorem (1.2a).

Using (1.1), (1.6b) and (1.9b) we have

$$(1.12a) \quad L_{\Omega_v}^{\Lambda_u}(U) = M^{\alpha}_0(U) L_{\alpha\Omega_v}^{\Lambda_u}$$

or in matrix notation

$$(1.12b) \quad ((T_0)) = ((M_0))((L_{\Omega_v}^{\Lambda_u}))^*$$

where $((L_{\Omega_v}^{\Lambda_u}))^*$ is the transposed matrix of (1.11). According to (1.10b) the rank of $((M_0))$ is not greater than r . We see from (1.12b) that if, and only if, $((T_0))$ has the rank r , $((M_0))$ cannot have rank less than r , because the rank of $((L_{\Omega_v}^{\Lambda_u}))^*$ is obviously equal to r .

*I.e. if the rank of the matrix $((T_0))$, defined by (1.2a) for $s = 0$, is r .

Theorem (1.3).* If the α -rank of $M^{a\Phi_p}_{\Psi_q}$ is r , we have

$$(1.13a) \quad \nabla_\mu L_{a\Omega_v}^{\Lambda u} = \Psi_a^{\epsilon_\mu} L_{\epsilon\Omega_v}^{\Lambda u}$$

where

$$(1.13b) \quad \Psi_a^{\epsilon_\mu} = L^{\epsilon\Omega_v}_{\Lambda u}^{-1} \nabla_\mu L_{a\Omega_v}^{\Lambda u}.$$

Proof. From (1.6b) for $s = 0, 1$ we have

$$(1.14) \quad \begin{aligned} (a) \quad T_{\Psi_q\Omega_v}^{\Phi_p\Lambda\phi} &= M^{a\Phi_p}_{\Psi_q} L_{a\Omega_v}^{\Lambda u}, \\ (b) \quad \nabla_\xi T_{\Psi_q\Omega_v}^{\Phi_p\Lambda\phi} &= M^{a\Phi_p}_{\Psi_q\xi} L_{a\Omega_v}^{\Lambda u} \end{aligned}$$

and (1.14a) yields

$$(1.14c) \quad \nabla_\xi T_{\Psi_q\Omega_v}^{\Phi_p\Lambda u} = (\nabla_\xi M^{a\Phi_p}_{\Psi_q}) L_{a\Omega_v}^{\Lambda u} + M^{a\Phi_p}_{\Psi_q} \nabla_\xi L_{a\Omega_v}^{\Lambda u}.$$

Comparing (1.14b) and (1.14c) and remembering that $M^{a\Phi_p}_{\Psi_q}$ is of α -rank r , we obtain (1.13a) and

$$(1.15) \quad M^{a\Phi_p}_{\Psi_q\xi} = \nabla_\xi M^{a\Phi_p}_{\Psi_q} + \Psi_a^{\epsilon_\xi} M^{\epsilon\Phi_p}_{\Psi_q},$$

while (1.5) and (1.13a) lead to (1.13b).

In the next section we shall investigate the result of this theorem from another point of view.

2. The connection $\Gamma_a^{\epsilon_\mu}$. The results of the previous section are valid for a generic point P of the space under consideration and therefore they are valid for a domain of L_n where the Λ_v -rank of $\{T_{\Psi_q\Omega_v}^{\Phi_p\Lambda u}\}$ does not change. We shall denote by $\delta \stackrel{\text{def}}{=} dx^\mu \nabla_\mu$ the symbol of the covariant differential in an arbitrary direction dx^μ . From the mapping (1.7) we obtain

$$(2.1) \quad \delta V_{\Omega_v}^{\Lambda u} = (dv^a) L_{a\Omega_v}^{\Lambda u} + v^a \delta L_{a\Omega_v}^{\Lambda u},$$

where, of course, a is to be considered as a dead index. If we symbolize (1.7) by $V_{\Omega_v}^{\Lambda u} \rightarrow v^a$, we see from (2.1) that we do not have $\delta V_{\Omega_v}^{\Lambda u} \rightarrow \delta v^a$, where δv^a is a vector in the auxiliary A_r . Moreover, $\delta V_{\Omega_v}^{\Lambda u}$ is independent of the A 's appearing in (1.3), (1.4) and (1.8), while $\delta L_{a\Omega_v}^{\Lambda u}$ depends on these coefficients. We could dispose of these difficulties by introducing an auxiliary connection $\Gamma_a^{\epsilon_\mu}$ with the transformation rule

$$(2.2) \quad \Gamma_a^{\epsilon'_\mu} = \frac{\partial x^\lambda}{\partial x^{\mu'}} (A_a^{\mathfrak{g}} \Gamma_a^{\mathfrak{g}\lambda} + \partial_\lambda A_a^{\mathfrak{g}}) A_{\mathfrak{g}}^{\epsilon'}.$$

We shall see that if $r = N$, there is one connection $\Gamma_a^{\epsilon_\mu}$ of this type, which is uniquely determined by our problem. In order to find it we denote by D_μ the symbol of the covariant derivative which operates on $\left\{ \begin{array}{l} \text{Greek} \\ \text{German} \end{array} \right.$ indices by means of $\left\{ \begin{array}{l} \Gamma_{\lambda}^{\nu\mu} \\ \Gamma_a^{\epsilon_\mu} \end{array} \right.$. Thus, for instance,

*See Corollary (2.1).

$$(2.3) \quad D_\mu L_{a\Omega_\bullet}^{\Lambda u} = \nabla_\mu L_{\Omega_\bullet}^{\Lambda u} - \Gamma_a^\epsilon{}_\mu L_{\epsilon\Omega_\bullet}^{\Lambda u} \cdot *$$

Theorem (2.1). *Let $L_{a\mu\Omega_\bullet}^{\Lambda u}$ be an a priori given tensor. There is at least one connection $\Gamma_a^\epsilon{}_\mu$ for which*

$$(2.4) \quad D_\mu L_{a\Omega_\bullet}^{\Lambda u} = L_{\mu a\Omega_\bullet}^{\Lambda u},$$

namely

$$(2.5a) \quad \Gamma_a^\epsilon{}_\mu \stackrel{\text{def}}{=} L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1}(\nabla_\mu L_{\Omega_\bullet}^{\Lambda u} - L_{a\mu\Omega_\bullet}^{\Lambda u}).$$

This connection is unique if $r = N$.

Proof. Multiplying (2.4) by $L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1}$ we obtain, by virtue of (2.3) and (1.5),

$$L_{\mu a\Omega_\bullet}^{\Lambda u} L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1} = L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1} \nabla_\mu L_{\Omega_\bullet}^{\Lambda u} - \Gamma_a^\epsilon{}_\mu,$$

and this equation is equivalent to (2.5a). If $r = N$, there is only one set of $L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1}$ and therefore $\Gamma_a^\epsilon{}_\mu$, defined by (2.5a), is unique.

Remark. The tensor $L_{\mu a\Omega_\bullet}^{\Lambda u}$ is completely arbitrary. It can be defined in such a way that the mapping (1.7), i.e., $V_{\Omega_\bullet}^{\Lambda u} \rightarrow v^a$, is not distorted by the operator D , so that we have also $DV_{\Omega_\bullet}^{\Lambda u} \rightarrow \delta v^a$ or, more generally,

$$(2.6a) \quad D_\mu V_{\Omega_\bullet}^{\Lambda u} = (D_\mu v^a) L_{a\Omega_\bullet}^{\Lambda u} \cdot **$$

If we put

$$X_a^\epsilon{}_\mu \stackrel{\text{def}}{=} L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1} L_{a\mu\Omega_\bullet}^{\Lambda u},$$

$$\Psi_a^\epsilon{}_\mu \stackrel{\text{def}}{=} L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1} \nabla_\mu L_{a\Omega_\bullet}^{\Lambda u}, \dagger$$

we obtain from (2.5a)

$$(2.5b) \quad \Gamma_a^\epsilon{}_\mu = \Psi_a^\epsilon{}_\mu - X_a^\epsilon{}_\mu$$

and, according to (1.7),

$$(2.6b) \quad L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1} D_\mu V_{\Omega_\bullet}^{\Lambda u} = D_\mu v^a + X_a^\epsilon{}_\mu v^a = L^{\epsilon\Omega_\bullet}{}_{\Lambda u}{}^{-1} \nabla_\mu V_{\Omega_\bullet}^{\Lambda u}$$

for any choice of the connection $\Gamma_a^\epsilon{}_\mu$. This fact suggests the following simplification.

*We write $\nabla_\mu L_{\Omega_\bullet}^{\Lambda u}$ instead of $\nabla_\mu L_{a\Omega_\bullet}^{\Lambda u}$ to stress that ∇_μ does not operate on German indices.

**According to our definition of D_μ we have

$$D_\mu V_{\Omega_\bullet}^{\Lambda u} = \nabla_\mu V_{\Omega_\bullet}^{\Lambda u}.$$

†See (1.13b).

Theorem (2.2). *There is at least one connection $\Gamma_a^a{}_\mu$ for which the tensor $D_\mu V_{\Omega_a}{}^{\Lambda_u}$ (independent of $\Gamma_a^a{}_\mu$) acquires the form (2.6a). It is the connection (2.5) for $L_{a\Omega_a}{}^{\Lambda_u} = 0$,*

$$(2.7a) \quad \Gamma_a^a{}_\mu = L_{\Omega_a}{}^{\Lambda_u} \nabla_\mu L_{\Omega_a}{}^{\Lambda_u} = \Psi_a^a{}_\mu ,$$

so that

$$(2.7b) \quad D_\mu L_{a\Omega_a}{}^{\Lambda_u} = 0.$$

This connection is uniquely determined if $r = N$.

The proof, which is almost trivial, follows from

$$D_\mu V_{\Omega_a}{}^{\Lambda_u} = (D_\mu v^a) L_{a\Omega_a}{}^{\Lambda_u} + v^a D_\mu L_{a\Omega_a}{}^{\Lambda_u}.$$

Remark. From now on we shall always use the connection $\Gamma_a^a{}_\mu$ defined by (2.7), so that the following corollary holds.

Corollary (2.1). *Theorem (1.3) holds regardless of the α -rank of $M^{a\Phi_p}{}_{\Psi_a}$.*

Theorem (2.3). *We have*

$$(2.8) \quad \begin{aligned} (a) \quad & T_{\Psi_a\Omega_a}^{\Phi_p\Lambda_u} = M^{a\Phi_p}{}_{\Psi_a} L_{a\Omega_a}{}^{\Lambda_u}, \\ (b) \quad & D_{\xi_s} \cdots D_{\xi_1} T_{\Psi_a\Omega_a}^{\Phi_p\Lambda_u} = M^{a\Phi_p}{}_{\Psi_a\xi_s \cdots \xi_1} L_{a\Omega_a}{}^{\Lambda_u} \end{aligned}$$

where

$$(2.9) \quad M^{a\Phi_p}{}_{\Psi_a\xi_s \cdots \xi_1} \stackrel{\text{def}}{=} D_{\xi_s} \cdots D_{\xi_1} M^{a\Phi_p}{}_{\Psi_a}, \quad s = 1, 2, \dots$$

Proof. We have obviously

$$\nabla_{\xi_s} \cdots \nabla_{\xi_1} T_{\Psi_a\Omega_a}^{\Phi_p\Lambda_u} = D_{\xi_s} \cdots D_{\xi_1} T_{\Psi_a\Omega_a}^{\Phi_p\Lambda_u},$$

so that (2.8) is the explicit form of (1.6b). The equation (2.9) is an immediate consequence of (2.8) and (2.7b).

3. The Lie group induced by $T_{\Psi_a\Omega_a}^{\Phi_p\Lambda_u}$. In this section we shall confine ourselves again to the tangential flat space A_n at a generic point P of L_n . Hence all tensors are considered at P and therefore have constant components.

Moreover, we assume $u = v \neq 0$ and write simply $T_{\Psi_a\Omega_a}^{\Phi_p\Lambda_u}$, $L_{a\Omega_a}{}^{\Lambda_u}$ instead of $T_{\Psi_a\Omega_a}^{\Phi_p\Lambda_u}$, $L_{a\Omega_a}{}^{\Lambda_u}$. A tensor $L_{\Omega_a}{}^{\Lambda_u}$ may be considered either as a tensor of valence $2u$ in the tangent space A_n or as a quadratic tensor in an auxiliary A_m , $m = n^u$. This auxiliary A_m will be referred to a system of coordinates y^a . These coordinates represent in A_n a tensor $y^{\omega_1 \cdots \omega_u}$ and therefore the transformation for the y^a is linear homogeneous with the coefficients

$$A_{\Lambda'}^{\Omega'} = \frac{\partial x^{\omega'_1}}{\partial x^{\lambda'_1}} \cdots \frac{\partial x^{\omega'_u}}{\partial x^{\lambda'_u}}, \quad A_{\Lambda'}^{\Omega} = \frac{\partial x^{\omega_1}}{\partial x^{\lambda'_1}} \cdots \frac{\partial x^{\omega_u}}{\partial x^{\lambda'_u}} \quad \text{at } P. \quad *$$

*Of course we have

$$A_{\Lambda'}^{\Omega} A_{\Phi}^{\Lambda'} = \delta_{\Phi}^{\Omega}$$

where δ_{Φ}^{Ω} stands for $\delta_{\varphi_1}^{\omega_1} \cdots \delta_{\varphi_u}^{\omega_u}$.

Hence the coordinates y^{Ω} in A_m are *cartesian coordinates*. On the other hand, the $L_{\alpha\Omega}^{\Lambda}$ at P are constant and are not functions of the y^{Ω} 's.

Lemma (3.1). *The quantities*

$$(3.1) \quad C_{\alpha\beta}^{\epsilon} \stackrel{\text{def}}{=} 2 \overset{-1}{L}{}^{\Lambda}_{\Omega} L_{[\alpha|\Lambda|}^{\Phi} L_{\beta]\Phi}^{\Omega}$$

have the following properties:

(1) *They are components of a tensor (skew symmetric in $\alpha\beta$) in the auxiliary space A_r .*

(2) *The tensor $C_{\alpha\beta}^{\epsilon}$ is constant in the sense that its components are not functions of the y^{Ω} 's.**

(3) *The tensor $C_{\alpha\beta}^{\epsilon}$ is covariant constant for $r = n^{2u}$:***

$$(3.2) \quad D_{\mu} C_{\alpha\beta}^{\epsilon} = 0.$$

(4) *The tensor $C_{\alpha\beta}^{\epsilon}$ is uniquely determined by (3.1) even if $r < n^{2u}$.*

Proof. The first statement follows from (1.3) and

$$(3.3) \quad \overset{-1}{L}{}^{\Lambda}_{\Omega} = A_{\alpha}^{\epsilon'} \overset{-1}{L}{}^{\Lambda}_{\Omega}.$$

The second statement follows from (3.1) and from the fact that the L 's do not contain y^{Ω} .

Using (2.7b) we obtain

$$(3.4) \quad \overset{-1}{L}{}^{\Omega}_{\Lambda} D_{\mu} L_{\epsilon\Omega}^{\Lambda} = 0,$$

and this equation is, by virtue of (1.5), equivalent to

$$(3.5) \quad L_{\epsilon\Omega}^{\Lambda} D_{\mu} \overset{-1}{L}{}^{\Omega}_{\Lambda} = 0.$$

Hence if $r = n^{2u}$, we obtain from (3.5)

$$(3.6) \quad D_{\mu} \overset{-1}{L}{}^{\Omega}_{\Lambda} = 0,$$

and (3.2) follows from (3.1), (2.7b), and (3.6).

The fourth statement will be proved in the remark following the proof of the next lemma.

Agreement (3.1). *In subsequent considerations we shall assume that $L_{[\alpha|\Lambda|}^{\Phi} L_{\beta]\Phi}^{\Omega}$ is in the $\overset{\Omega}{\Lambda}$ -support of $\{T_{\Psi\epsilon+\epsilon\Lambda}^{\Phi\Omega}\}$.*

Lemma (3.2). *We have*

$$(3.7) \quad 2L_{[\alpha|\Lambda|}^{\Phi} L_{\beta]\Phi}^{\Omega} = C_{\alpha\beta}^{\epsilon} L_{\epsilon\Lambda}^{\Omega}.$$

*Of course $C_{\alpha\beta}^{\epsilon} = C_{\alpha\beta}^{\epsilon}(P)$ are functions of position of P in L_n (and as such they are also constant at P).

**Cf. Corollary (3.1).

Proof. According to Agreement (3.1) we have

$$2L_{[\alpha|\Lambda]}^{\Phi} L_{\Phi]}^{\Omega} = x_{\alpha\Phi}{}^{\flat} L_{\flat\Lambda}^{\Omega}.$$

Multiplying this equation by $L^{\Lambda}{}_{\Omega}^{-1}$ we obtain (3.7), by virtue of (3.1).

Remark. Remembering that every one of the possible systems of tensors $L^{\alpha\Omega}{}_{\Lambda}$ has to satisfy (1.5), we see at once from (3.7) that there is only one system of components $C_{\alpha\Phi}{}^{\epsilon}$ defined by (3.1). This completes the proof of the previous lemma.

Corollary (3.1). *The equation (3.2) holds even for $r < n^{2u}$.*

Proof. Applying the operator D_{μ} to (3.7a) and remembering (2.7b) one obtains (3.2).

Lemma (3.3). *We have*

$$(3.8) \quad C_{[\alpha\epsilon}{}^{\flat} C_{\flat]}{}^{\epsilon} = 0.$$

Proof. Consider the identities

$$\begin{aligned} (3.9) \quad & 6L_{[\alpha|\Lambda]}^{\Phi} L_{\epsilon|\Phi]}^{\Omega} L_{\Omega]}^{\Psi} \\ &= 2(L_{\alpha\Lambda}^{\Phi} L_{[\epsilon|\Phi]}^{\Omega} L_{\Omega]}^{\Psi} + L_{\epsilon\Lambda}^{\Phi} L_{[\Phi]}^{\Omega} L_{\alpha]\Omega}^{\Psi} + L_{\Phi\Lambda}^{\Phi} L_{[\alpha|\Phi]}^{\Omega} L_{\epsilon]\Omega}^{\Psi}) \\ &= 2(L_{[\alpha|\Lambda]}^{\Phi} L_{\epsilon|\Phi]}^{\Omega} L_{\Omega]}^{\Psi} + L_{[\epsilon|\Lambda]}^{\Phi} L_{\Phi]}^{\Omega} L_{\alpha\Omega}^{\Psi} + L_{[\Phi|\Lambda]}^{\Phi} L_{\alpha]\Phi]}^{\Omega} L_{\epsilon\Omega}^{\Psi}). \end{aligned}$$

Substituting into the middle term and into the last term from (3.7a) we obtain

$$C_{[\alpha\epsilon}{}^{\flat} C_{\flat]}{}^{\epsilon} L_{\epsilon\Lambda}^{\Psi} = 0,$$

and this equation is obviously equivalent to (3.8).

Lemma (3.4). *Put*

$$(3.10) \quad X_{\alpha} \stackrel{\text{def}}{=} L_{\alpha\Lambda}^{\Omega} y^{\Lambda} \frac{\partial}{\partial y^{\Omega}}.$$

The r operators X_{α} have the following properties:

(1) *If C^{α} is a system of constants and if*

$$(3.11a) \quad C^{\alpha} X_{\alpha} = 0,$$

then

$$(3.11b) \quad C^{\alpha} = 0.$$

(2) *The following relation holds:*

$$(3.12a) \quad 2X_{[a}X_{e]} = C_{ae}{}^b X_b ,$$

and therefore

$$(3.12b) \quad X_{[a}X_eX_{g]} = 0.$$

Proof. The tensors $L_{a\Lambda}{}^\Omega$ are linearly independent. Therefore

$$(3.13a) \quad C^a L_{a\Lambda}{}^\Omega = 0$$

implies

$$(3.13b) \quad C^a = 0.$$

On the other hand, by virtue of (3.10),

$$C^a X_a = C^a L_{a\Lambda}{}^\Omega y^\Lambda \frac{\partial}{\partial y^\Omega}.$$

Hence (3.13a) can be satisfied only if (3.11a) is satisfied. Therefore, if C^a are constants, this latter relation is satisfied only if (3.13b) is satisfied.

Using (3.10) and (3.7) we obtain

$$\begin{aligned} 2X_{[a}X_{e]} &= 2L_{[a|\Lambda|}{}^\Phi y^\Lambda \frac{\partial}{\partial y^{|\Phi|}} L_{e] \Psi}{}^\Omega y^\Psi \frac{\partial}{\partial y^\Omega} = 2L_{[a|\Lambda|}{}^\Phi L_{e] \Phi}{}^\Omega y^\Lambda \frac{\partial}{\partial y^\Omega} \\ &= C_{ae}{}^b L_{b\Lambda}{}^\Omega y^\Lambda \frac{\partial}{\partial y^\Omega} = C_{ae}{}^b X_b . \end{aligned}$$

This equation proves (3.12a). The equation (3.12b) follows from (3.12a) and (3.8).

Theorem (3.1). *The X_a defined by (3.10) are r linearly independent (in the sense of the theory of Lie groups*) infinitesimal operators which define a Lie group G_r (of r parameters), the structural constants of which are $C_{ae}{}^b$, defined by (3.1) (and satisfying (3.8)).*

Proof. According to the second fundamental theorem of Lie, the linearly independent* infinitesimal operators X_a (defined by (3.10) and) satisfying (3.12a) define a Lie group G_r of r parameters.

Definition (3.1). *The Lie group of Theorem (3.1) will be referred to as the Lie group $G_r(T)$ induced by the tensor $T_{\Psi_a\Omega}^{\Phi_p\Lambda}$.*

Remark. The equation (1.7), which reduces in our case to

$$(3.14a) \quad V_\Omega{}^\Lambda = v^a L_{a\Omega}{}^\Lambda ,$$

may be interpreted in terms of the Lie group $G_r(T)$. It gives rise to the relation

$$(3.14b) \quad V_\Omega{}^\Lambda y^\Omega \frac{\partial}{\partial y^\Lambda} = v^a X_a .$$

*This kind of linear independence is defined by the statement (1) of Lemma (3.4).

Hence $V_{\alpha}^{\Lambda} y^{\alpha} \frac{\partial}{\partial y^{\Lambda}}$ is the infinitesimal operator of a transformation of $G_r(T)$ in A_m , $m = n^u$, which corresponds in the auxiliary space A_r to the vector v^a . This justifies the following

Definition (3.2). *The auxiliary space A_r will be referred to as the group space G_r of the Lie group. Any vector $v^a \subset G_r$ will be referred to as the symbol of the infinitesimal transformation (3.14b) in A_m , $m = n^u$.*

Remark. The equation (2.6a) simplifies in our case to

$$(3.15a) \quad D_{\mu} V_{\alpha}^{\Lambda} = (D_{\mu} v^a) L_{a\alpha}^{\Lambda},$$

so that

$$(3.15b) \quad (D_{\mu} V_{\alpha}^{\Lambda}) y^{\alpha} \frac{\partial}{\partial y^{\Lambda}} = (D_{\mu} v^a) X_a.$$

This equation permits us to extend our consideration at a generic point to its neighborhood. In fact, we have from (3.15b)

$$(3.16) \quad (V_{\alpha}^{\Lambda} + D V_{\alpha}^{\Lambda}) y^{\alpha} \frac{\partial}{\partial y^{\Lambda}} = (v^a + D v^a) X_a.$$

II. THE HOLONOMY GROUP*

4. The holonomy group of an L_n . Let

$$R_{\omega\mu\lambda}{}^{\nu} \stackrel{\text{def}}{=} 2(\partial_{[\mu} \Gamma_{\omega]\lambda}{}^{\nu} + \Gamma_{\alpha}{}^{\nu}{}_{[\mu} \Gamma_{\omega]}{}^{\alpha}{}_{\lambda})$$

be the curvature tensor of the L_n under consideration and $\Delta s^{\omega\mu}$ an infinitesimal simple bivector. Denote by P a generic point of its boundary and displace a vector $v^{\nu}(P)$, by parallelism along this boundary, back to P . At this point the change Δv^{ν} of the vector $v^{\nu}(P)$ is given by

$$(4.1) \quad \Delta v^{\nu} = \frac{1}{2} R_{\omega\mu\lambda}{}^{\nu} v^{\lambda} \Delta s^{\omega\mu}.$$

This is an infinitesimal transformation of the holonomy group of the L_n under consideration.

Definition (4.1). *Put $p = 0$, $q = 2$, $u = v = 1$ in Definition (3.1), so that $T_{\omega\mu\lambda}{}^{\nu}$ may be identified with the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$. The Lie group $G_r(R)$ of r parameters induced by the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$ is termed the holonomy group of $R_{\omega\mu\lambda}{}^{\nu}$ (or of L_n).*

Remark I. Using the notation of the first part of this paper we shall sometimes write $R_{\Psi\alpha\lambda}{}^{\nu}$ or $R_{\Psi\alpha+\alpha\lambda}{}^{\nu}$ instead of $R_{\omega\mu\lambda}{}^{\nu}$ and $R_{\Psi\alpha+\alpha\lambda}{}^{\nu}$ instead of

*Some results exhibited in Sections 4 and 5 can be found in [3], pp. 361–370. Here these results are presented as a special case of the more general theory dealt with in Part I of this paper, inspired by (3).

$\nabla_{\xi_s} \dots \nabla_{\xi_1} R_{\omega\mu\lambda}{}^{\nu}$. The set of all tensors $R_{\Psi_{s+s}\lambda}{}^{\nu}$, $s = 0, 1, 2, \dots$, will be denoted by $\{R_{\Psi_{s+s}\lambda}{}^{\nu}\}$.

Remark II. According to Lemma (1.1) we have here

$$r \leq n^2$$

because $u = v = 1$.

Remark III. Both groups $G(T)$ and $G(R)$ will be needed in the fourth and fifth papers of this series.

Theorem (4.1). Let r be the λ -rank of $\{R_{\Psi_{s+s}\lambda}{}^{\nu}\}$. The λ -domain of $\{R_{\Psi_{s+s}\lambda}{}^{\nu}\}$ is described by r linearly independent tensors $L_{a\lambda}{}^{\nu}$ known up to transformations

$$(4.2a) \quad L_{a'\lambda}{}^{\nu} = A_a^{\iota'} L_{\iota\lambda}{}^{\nu}, \quad L_{a\lambda}{}^{\nu} = A_a^{\iota'} L_{\iota'\lambda}{}^{\nu},$$

where the A 's are arbitrary except for the restrictions (1.4).

Theorem (4.2). There is at least one set of tensors $\bar{L}^{a\lambda}{}_{\nu}$, satisfying the conditions

$$(4.2b) \quad L_{a\lambda}{}^{\nu} \bar{L}^{a\lambda}{}_{\nu} = \delta_a^{\iota}.$$

Theorem (4.3a). The λ -rank of $\{R_{\Psi_{s+s}\lambda}{}^{\nu}\}$ and the α -rank of $L_{a\lambda}{}^{\nu}$ have the same value r .

Theorem (4.3b). Put

$$(4.3) \quad M^a{}_{\omega\mu} \stackrel{\text{def}}{=} R_{\omega\mu\lambda}{}^{\nu} \bar{L}^{a\lambda}{}_{\nu}.$$

Then

$$(4.4) \quad R_{\omega\mu\lambda}{}^{\nu} = M^a{}_{\omega\mu} L_{a\lambda}{}^{\nu}.$$

If and only if the λ -rank of $R_{\omega\mu\lambda}{}^{\nu}$ is r , then the α -rank of $M^a{}_{\omega\mu}$ is also r .

Proof. In these theorems we applied Definitions (1.1), (1.2), (1.3) to the tensor $T_{\Psi_2\Omega_1}{}^{\Lambda_1} = R_{\alpha\beta\omega}{}^{\nu}$. Then all the statements of these theorems are special cases of Theorems (1.1), (1.2).

Theorem (4.4). The tensors $M^a{}_{\omega\mu}$, $L_{a\lambda}{}^{\nu}$ are related by the following conditions:

$$(4.5) \quad \begin{aligned} (a) \quad & M^a{}_{[\omega\mu} L_{|\alpha|\lambda]}{}^{\nu} = 0, \\ (b) \quad & 2M^a{}_{\alpha[\omega} L_{|\lambda|\mu]}{}^{\nu} + M^a{}_{\omega\mu} L_{a\lambda}{}^{\nu} = 0. \end{aligned}$$

Proof. These equations follow from (4.4) and from the well known algebraic identities

$$(4.6) \quad \begin{aligned} (a) \quad & R_{[\omega\mu\lambda]}{}^{\nu} = 0, \\ (b) \quad & 2R_{\alpha[\omega\mu]}{}^{\nu} + R_{\omega\mu\lambda}{}^{\nu} = 0. \end{aligned}$$

Theorem (4.5). *There is at least one connection $\Gamma_a^a{}_\mu$ for which*

$$(4.7) \quad D_\mu L_{a\lambda}{}^\nu = 0,$$

namely

$$(4.8) \quad \Gamma_a^a{}_\mu \stackrel{\text{def}}{=} L^{\alpha\lambda}{}_\nu \nabla_\mu L_{\alpha\lambda}{}^\nu.$$

This connection is unique if $r = n^2$.

Proof. This is a special case of Theorem (2.2).

Theorem (4.6). *The tensors $M^a{}_{\omega\mu}$ satisfy the condition*

$$(4.9a) \quad D_{[\xi} M^a{}_{\omega\mu]} = 0,$$

equivalent to

$$(4.9b) \quad \partial_{[\xi} M^a{}_{\omega\mu]} + \Gamma_a^a{}_{[\xi} M^a{}_{\omega\mu]} = 0.$$

Proof. Applying Bianchi's identity

$$(4.10) \quad \nabla_{[\xi} R_{\omega\mu]\lambda}{}^\nu = 0$$

to the equations (4.4) and (4.7), we obtain

$$0 = \nabla_{[\xi} R_{\omega\mu]\lambda}{}^\nu = D_{[\xi} R_{\omega\mu]\lambda}{}^\nu = (D_{[\xi} M^a{}_{\omega\mu]}) L_{a\lambda}{}^\nu,$$

and this equation is, by virtue of (4.2b), equivalent to (4.9a). The equation (4.9b) follows immediately from (4.9a) and from

$$(4.11) \quad D_{\xi} M^a{}_{\omega\mu} = \nabla_{\xi} M^a{}_{\omega\mu} + \Gamma_a^a{}_{\xi} M^a{}_{\omega\mu}.$$

Theorem (4.7). *We have*

$$(4.12a) \quad R_{\omega\mu\lambda}{}^\nu = M^a{}_{\omega\mu} L_{a\lambda}{}^\nu,$$

$$\nabla_{\xi_s} \cdots \nabla_{\xi_1} R_{\omega\mu\lambda}{}^\nu = D_{\xi_s} \cdots D_{\xi_1} R_{\omega\mu\lambda}{}^\nu = M^a{}_{\omega\mu\xi_s \cdots \xi_1} L_{a\lambda}{}^\nu$$

where

$$(4.13a) \quad M^a{}_{\omega\mu\xi_s \cdots \xi_1} \stackrel{\text{def}}{=} D_{\xi_s} \cdots D_{\xi_1} M^a{}_{\omega\mu}, \quad s = 1, 2, \dots$$

Proof. This is a special case of Theorem (2.3).

The following two theorems will be proved simultaneously.

Theorem (4.8a). *A necessary and sufficient condition that the L_n under consideration be recurrent is: There is a vector M_ξ such that*

$$(4.13b) \quad D_{\xi} M^a{}_{\omega\mu} = M_\xi M^a{}_{\omega\mu}.$$

If this condition is satisfied then the α -rank of $M^a{}_{\omega\mu}$ is r and if $M_\xi \neq 0$ the bivectors $M^a{}_{\omega\mu}$ are simple:

$$(4.13c) \quad M^a{}_{\omega\mu} = M_{[\omega} M^a{}_{\mu]}.$$

Theorem (4.8b). *A necessary and sufficient condition for the L_n under consideration to be symmetric is: The equations (4.12a) are satisfied for*

$$(4.13d) \quad M_{\xi} = 0.$$

Proof. An L_n is by definition recurrent if

$$(4.12b) \quad \nabla_{\xi} R_{\omega\mu\lambda}{}^{\nu} = M_{\xi} R_{\omega\mu\lambda}{}^{\nu}.$$

It is by definition symmetric if this equation is satisfied by $M_{\xi} = 0$. Applying to these facts (4.12a) and (4.13a) for $s = 1$, we obtain the main statements of our theorems. From (4.12b) and (4.13a) we see that the ν -rank of $R_{\omega\mu\lambda}{}^{\nu}$ and $\{R_{\Psi_2+\alpha\lambda}{}^{\nu}\}$ is r so that, according to Theorem (4.3b), $M^{\alpha}{}_{\omega\mu}$ must have α -rank equal to r . The equation (4.13c) follows from (4.13b) and (4.9a).

It is interesting to consider normal tensors of L_n expressed in terms of $M^{\alpha}{}_{\omega\mu}$ and $L_{\alpha\lambda}{}^{\nu}$. In the next theorem we shall confine ourselves to the first normal tensor.

Theorem (4.9). *We have*

$$(4.14a) \quad 3N_{\omega\mu\lambda}{}^{\nu} = -2M^{\alpha}{}_{\mu(\omega} L_{|\alpha|\lambda)}{}^{\nu}.$$

Proof. The first normal tensor $N_{\omega\mu\lambda}{}^{\nu}$ may be expressed by means of $R_{\omega\mu\lambda}{}^{\nu}$ as follows:

$$(4.14b) \quad 3N_{\omega\mu\lambda}{}^{\nu} = 2R_{(\omega|\mu|\lambda)}{}^{\nu} \cdot *$$

Substituting into (4.14b) from (4.4), we obtain (4.14a).

Theorem (4.10). *The r linearly independent infinitesimal operators of the holonomy group $G_r(R)$ are given by*

$$(4.15) \quad X_{\alpha} \stackrel{\text{def}}{=} L_{\alpha\lambda}{}^{\nu} y^{\lambda} \frac{\partial}{\partial y^{\nu}}$$

where y^{ν} are Cartesian coordinates describing the tangent flat space A_n of L_n at P . These operators satisfy the condition

$$(4.16) \quad 2X_{[\alpha} X_{\epsilon]} = C_{\alpha\epsilon}{}^{\flat} X_{\flat}$$

where the structural constants $C_{\alpha\epsilon}{}^{\flat}$ of $G_r(R)$ are given by

$$(4.17) \quad C_{\alpha\epsilon}{}^{\flat} \stackrel{\text{def}}{=} \bar{L}^{\lambda}{}_{\nu} L_{[\alpha|\lambda]}{}^{\alpha} L_{\epsilon]\alpha}{}^{\nu}.$$

These structural constants satisfy also the equations

$$(4.18) \quad 2L_{[\alpha|\lambda]}{}^{\alpha} L_{\epsilon]\alpha}{}^{\nu} = C_{\alpha\epsilon}{}^{\flat} L_{\epsilon\lambda}{}^{\nu},$$

$$(4.19) \quad C_{[\alpha\epsilon}{}^{\flat} C_{\flat]\beta}{}^{\sharp} = 0,$$

$$(4.20) \quad D_{\mu} C_{\alpha\epsilon}{}^{\flat} = 0.$$

*See, for instance, [4], p. 131 in a slightly different notation.

Proof. These statements describe special cases of Lemmas (3.1), (3.2), (3.3), (3.4), Theorem (3.1) and Corollary (3.1). Agreement (3.1) is here automatically satisfied.

Remark. The infinitesimal transformation (4.1) of $G_r(R)$ may be written

$$w^s X_a$$

where

$$2w^a = M^a_{\omega\mu} \Delta s^{\omega\mu}.$$

Theorem (4.11). *Let $P_{\lambda}^{\nu\cdots}$ be a tensor in L_n . The complete set of integrability conditions of*

$$(4.21) \quad \nabla_\mu P_{\lambda}^{\nu\cdots} = P_{\lambda}^{\nu\cdots} \partial_\mu \sigma$$

reduces to

$$(4.22) \quad L_{a\lambda}{}^\alpha P_{\alpha}^{\nu\cdots} + \cdots - L_{a\alpha}{}^\nu P_{\lambda}^{\alpha\cdots} - \cdots = 0.$$

Proof. The first set of integrability conditions is

$$R_{\omega\mu\lambda}{}^\alpha P_{\alpha}^{\nu\cdots} + \cdots - R_{\omega\mu\alpha}{}^\nu P_{\lambda}^{\alpha\cdots} - \cdots = 0$$

or

$$(4.23a) \quad M^a_{\omega\mu} (L_{a\lambda}{}^\alpha P_{\alpha}^{\nu\cdots} + \cdots - L_{a\alpha}{}^\nu P_{\lambda}^{\alpha\cdots} - \cdots) = 0.$$

In order to obtain the second set of integrability conditions we have to take the derivative of (4.23a) and to eliminate the unknowns by means of (4.21). In this way we obtain, by virtue of (4.23a),

$$(4.23b) \quad D_{\xi} M^a_{\omega\mu} (L_{a\lambda}{}^\alpha P_{\alpha}^{\nu\cdots} + \cdots - L_{a\alpha}{}^\nu P_{\lambda}^{\alpha\cdots} - \cdots) = 0.$$

Proceeding in this way, we obtain finally the complete set of integrability conditions in the form

$$(4.24) \quad M^a_{\Psi_{2+s}} L_{a\beta}{}^\alpha Q_{\alpha\lambda}^{\beta\nu\cdots} = 0,$$

where $M^a_{\Psi_{2+s}}$ stands for $M^a_{\omega\mu}$ and $M^a_{\omega\mu\xi_s\cdots\xi_1}$ ($s = 1, 2, \cdots \text{inf.}$) defined by (4.13a) and

$$Q_{\alpha\lambda}^{\beta\nu\cdots} = \delta_{\lambda}^{\beta} P_{\alpha}^{\nu\cdots} + \cdots - \delta_{\alpha}^{\nu} P_{\lambda}^{\beta\cdots} - \cdots.$$

The ${}^\alpha$ -rank of $\{M^a_{\Psi_{2+s}} L_{a\beta}{}^\alpha\}$, $s = 0, 1, 2, \cdots \text{inf.}$, is r , and $L_{a\beta}{}^\alpha$ are r linearly independent tensors describing the ${}^\alpha$ -support of $\{M^a_{\Psi_{2+s}} L_{a\beta}{}^\alpha\}$, $s = 0, 1, 2, \cdots \text{inf.}$ Therefore (4.24) is equivalent to

$$L_{a\beta}{}^\alpha Q_{\alpha\lambda}^{\beta\nu\cdots} = 0,$$

and this equation is equivalent to (4.22).

Remark. This theorem will be the starting point of the next paper.

5. The holonomy group of a V_n . An L_n is termed a Riemann space and

denoted by V_n if it is endowed with a covariant constant tensor $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n :

$$(5.1a) \quad \nabla_\mu g_{\lambda\nu} = 0.$$

This tensor is the metric tensor of the V_n under consideration. The local tangential space is euclidean and will be denoted by E_n . The inverse of $g_{\lambda\mu}$ will be denoted by $g^{\lambda\nu} = g^{\nu\lambda}$. We shall use $g_{\lambda\mu}$ and $g^{\lambda\nu}$ for lowering and raising the indices. The coefficients of the connection $\Gamma_\lambda{}^\nu{}_\mu$ reduce to the Christoffel symbols $\{\lambda{}^\nu{}_\mu\}$ of $g_{\lambda\mu}$:

$$(5.1b) \quad \Gamma_\lambda{}^\nu{}_\mu = \{\lambda{}^\nu{}_\mu\} \stackrel{\text{def}}{=} \frac{1}{2} g^{\nu\alpha} (\partial_\lambda g_{\alpha\mu} + \partial_\mu g_{\alpha\lambda} - \partial_\alpha g_{\lambda\mu}),$$

and the curvature tensor $R_{\omega\mu\lambda}{}^\nu$ reduces to the Riemann Christoffel tensor

$$(5.1c) \quad K_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} 2[\partial_{[\mu} \{\omega\}{}^\nu{}_{\lambda]} + \{\alpha{}^\nu{}_{[\mu} \{\omega\}{}^\alpha{}_{\lambda]}\}].$$

The holonomy group $G_r(R)$ reduces to the holonomy group $G_r(K)$.

Theorem (5.1). *All results about $G_r(R)$ exhibited in previous sections hold also for $G_r(K)$.*

Proof. All results of previous sections were formulated in such a way that they hold for any symmetric connection $\Gamma_\lambda{}^\nu{}_\mu$. Therefore they hold also for the connection $\{\lambda{}^\nu{}_\mu\}$ given by (5.1b).

In this section we shall discuss only some special features of $G_r(K)$ which are due to the existence of $g_{\lambda\mu}$ and therefore do not exist for a $G_r(R)$.

Theorem (5.2). *The tensor*

$$(5.2a) \quad L_{a\lambda\nu} \stackrel{\text{def}}{=} L_{a\lambda}{}^\alpha g_{\alpha\nu}$$

is a bivector,

$$(5.2b) \quad L_{a(\lambda\nu)} = 0,$$

$$(5.2c) \quad L_{a\lambda}{}^\nu = -L_a{}^\nu{}_\lambda.$$

The ranks of the matrices

$$(5.3a) \quad ((L_\lambda{}^\nu)) \stackrel{\text{def}}{=} \left[\begin{pmatrix} \cdots & L_{1\lambda}{}^\nu & \cdots \\ & \vdots & \\ & & \vdots & \\ \cdots & L_{r\lambda}{}^\nu & \cdots \end{pmatrix} \right],$$

$$(5.3b) \quad ((L_{\lambda\nu})) \stackrel{\text{def}}{=} \left[\begin{pmatrix} \cdots & L_{1\lambda\nu} & \cdots \\ & \vdots & \\ & & \vdots & \\ \cdots & L_{r\lambda\nu} & \cdots \end{pmatrix} \right]$$

are equal to $r \leq \frac{1}{2}n(n-1)$.

Proof. The curvature tensor $K_{\omega\mu\lambda\nu}$ is skew symmetric in $\lambda\nu$:

$$(5.4a) \quad K_{\omega\mu(\lambda\nu)} = 0, \quad \nabla_{\xi_s} \cdots \nabla_{\xi_1} K_{\omega\mu(\lambda\nu)} = 0, \quad s = 1, 2, \dots$$

Applying to these identities the equations (4.12a) and (4.13a), we obtain

$$(5.4b) \quad M^a_{\omega\mu} L_{a(\lambda\nu)} = 0, \quad (D_{\xi_s} \cdots D_{\xi_1} M^a_{\omega\mu}) L_{a(\lambda\nu)} = 0,$$

and these equations yield (5.2b). The equation (5.2c) follows easily from (5.2b). The rank of the matrix $((L_\lambda^\nu))$ is r and the ranks of the matrices (5.3) are arithmetic invariants (*i.e.*, independent of the coordinate system). Therefore we may find the ranks in a particular system at a generic point P . We choose a coordinate system for which

$$(5.5) \quad g_{\lambda\mu} = g^{\lambda\mu} = \begin{cases} 0 & \text{for } \lambda \neq \mu \\ \epsilon_\lambda & \text{for } \lambda = \mu \end{cases} \text{ at } P, \quad \epsilon_\lambda = \pm 1. *$$

Then we have

$$(5.6) \quad L_{a\lambda\mu} = L_{a\lambda}{}^\alpha g_{\alpha\mu} = L_{a\lambda}{}^\mu \epsilon_\mu \quad (\text{no sum}).$$

Hence

$$((L_{\lambda\mu})) = \begin{bmatrix} \left[\begin{array}{ccc} \cdots & L_{1\lambda}{}^\mu \epsilon_\mu & \cdots \\ & \vdots & \\ \cdots & L_{r\lambda}{}^\mu \epsilon_\mu & \cdots \end{array} \right] \\ \vdots \\ \left[\begin{array}{ccc} \cdots & L_{1\lambda}{}^\mu \epsilon_\mu & \cdots \\ & \vdots & \\ \cdots & L_{r\lambda}{}^\mu \epsilon_\mu & \cdots \end{array} \right] \end{bmatrix} \quad (\text{no sum}).$$

Therefore the corresponding columns of the matrices (5.3) differ only by the multiplicative factor ± 1 . This fact leads at once to the last part of our theorem.

Corollary (5.1). *The λ -rank of $K_{\omega\mu\lambda}{}^\nu$ and the $\lambda\nu$ -rank of $K_{\omega\mu\lambda}{}^\nu$ are the same.***

The proof follows at once from the previous theorem.

Using the metric tensor we may redefine the λ -support of $\{K_{\Psi_{s+s}\lambda}{}^\nu\}$.

Definition (5.1). *By the λ -support of $\{K_{\Psi_{s+s}\lambda}{}^\nu\}$ we understand the space spanned by the tensors $L_a^{\lambda\nu}$, *i.e.*, the space described by*

$$K_{\Psi_{s+s}}{}^{\lambda\nu} U^{\Psi_{s+s}}, \quad s = 0, 1, \dots,$$

for all U 's.

Theorem (5.3). *Put*

$$(5.7a) \quad h^{ae} \stackrel{\text{def}}{=} M^a_{\omega\mu} L^{\omega\mu e}.$$

Then the set h^{ae} has the following properties:

- (a) h^{ae} is a tensor in the group space G_r .

The equality sign $\stackrel{}{=}$ stresses the fact that this equation depends on the coordinate system.

**This rank need not be r . It must be $\leq r$.

- (b) *It is symmetric.*
 (c) *It is of rank r if and only if the ν -rank of $K_{\omega\mu\lambda}{}^\nu$ is r .*
 (d) *It satisfies the relations*

$$(5.7b) \quad h^{a\epsilon} = K_{\omega\mu\lambda\nu} L^{\epsilon\omega\mu} L^{a\lambda\nu}$$

and

$$(5.8a) \quad M^a{}_{\omega\mu} = h^{a\epsilon} L_{\epsilon\omega\mu} ,$$

$$(5.8b) \quad K_{\omega\mu\lambda\nu} = h^{a\epsilon} L_{a\omega\mu} L_{\epsilon\lambda\nu} .$$

Proof. Applying the equation (4.4) to the algebraic identity

$$(5.9a) \quad K_{\omega\mu\lambda\nu} = K_{\lambda\nu\omega\mu} ,$$

we obtain

$$(5.9b) \quad M^a{}_{\omega\mu} L_{a\lambda\nu} = M^a{}_{\lambda\nu} L_{a\omega\mu} .$$

Multiplying this equation by $L^{\epsilon\omega\mu}$ and taking into account (5.7a), we obtain (5.8a). Hence (5.8b) holds and (5.7b) follows. This latter equation, together with (5.9a), shows also the symmetry of $h^{a\epsilon}$. Its tensorial character (in G_r) follows from (5.7b) in conjunction with (4.2). The α -rank of $L_{a\lambda}{}^\nu$ is obviously the same as the α -rank of $L_{a\lambda\nu}$, namely r . The α -rank of $M^a{}_{\omega\mu}$ is r if and only if the ν -rank of $K_{\omega\mu\lambda}{}^\nu$ is r (Theorem 4.3b). Therefore in this case there is a set of r linearly independent vectors M^a and a set of linearly independent vectors L_a which satisfy, by virtue of (5.8a), the relation

$$(5.10) \quad M^a = h^{a\epsilon} L_\epsilon .$$

Hence $h^{a\epsilon}$ is of rank r .

Theorem (5.4). *We have*

$$(5.11) \quad \begin{aligned} (a) \quad & h^{a\epsilon} L_{a[\omega\mu} L_{|\epsilon|\lambda]}{}^\nu = 0, \\ (b) \quad & h^{a\epsilon} L_{a\alpha[\omega} L_{|\epsilon|\mu]}{}^\alpha = 0. \end{aligned}$$

Proof. (5.11a) follows from (4.5a) and (5.8a). (5.11b) follows from (4.5b), (5.8a) and from (5.2c).

Theorem (5.5). *We have*

$$(5.12) \quad \begin{aligned} K_{\omega\mu\lambda}{}^\nu &= M^a{}_{\omega\mu} L_{a\lambda}{}^\nu, \\ \nabla_{\xi_s} \cdots \nabla_{\xi_1} K_{\omega\mu\lambda}{}^\nu &= M^a{}_{\omega\mu\xi_s \cdots \xi_1} L_{a\lambda}{}^\nu, \quad s = 1, 2, \dots, \end{aligned}$$

where

$$(5.13) \quad \begin{aligned} M^a{}_{\omega\mu} &= h^{a\epsilon} L_{\epsilon\omega\mu} , \\ M^a{}_{\omega\mu\xi_s \cdots \xi_1} &= (D_{\xi_s} \cdots D_{\xi_1} h^{a\epsilon}) L_{\epsilon\omega\mu} . \end{aligned}$$

The proof follows easily from Theorem (4.7) and equations (4.7) and (5.8a).

Theorem (5.6). *The complete set of integrability conditions of (5.1a) reduces to (5.2b).*

Proof. According to Theorem (4.11) the complete set of integrability conditions of (5.1a) reduces to

$$(5.14) \quad L_{a\lambda}{}^\alpha g_{\alpha\nu} + L_{a\nu}{}^\alpha g_{\lambda\alpha} = 0,$$

and this equation is equivalent to (5.2b).

Remark. The equation (5.14) will play an important role in the next paper of this series.

III. AUXILIARY RESULTS CONCERNING $G_r(R)$ AND $G_r(K)$

In this part of the paper we shall discuss some auxiliary results needed in the next papers.

6. Basic definitions.* Let $L_\mu{}^\nu$ be an arbitrary tensor and

$$(6.1) \quad \text{Det} ((\lambda \delta_\mu^\nu - L_\mu{}^\nu)) = 0$$

its corresponding secular equation. The roots λ of (6.1) are called the eigenvalues of $L_\lambda{}^\nu$.

Definition (6.1). *The tensor $L_\lambda{}^\nu$ is termed a complete tensor if it has n different eigenvalues λ_a , all different from zero.*

Remark I. If the space under consideration is a V_n , $n = 2m + 1$, and $L_\lambda{}^\nu$ is a bivector, then $L_\lambda{}^\nu$ never can be a complete tensor, for at least one of the eigenvalues must be equal to zero.

Remark II. If $L_\lambda{}^\nu$ is a complete tensor, then it has exactly n linearly independent eigenvectors $a_i{}^\nu$. These eigenvectors are solutions of

$$(6.2) \quad \lambda_a a_x{}^\nu = L_\lambda{}^\nu a_x{}^\lambda.$$

The eigenvector $a_x{}^\nu$, belonging to λ_a , is defined by (6.2) up to an arbitrary multiplicative factor. If $\dot{a}_\lambda^i$ is the inverse set:

$$(6.3) \quad a_i{}^\nu \dot{a}_\nu^i = \delta_i^i, \quad a_i{}^\nu \dot{a}_\lambda^i = \delta_\lambda^\nu,$$

then $\dot{a}_\lambda^i$ will be termed also an eigenvector of $L_\lambda{}^\nu$ belonging to λ . Using the

*Unless otherwise stated, small Latin indices run from 1 to n . The indices x, y, z, t do not follow the usual summation agreement. Summing for these indices will always be indicated by the symbol $\sum$.

nonholonomic frame $a_{\dot{i}}^{\nu}, \dot{a}_{\lambda}^i$, we may define the nonholonomic components $L_{\dot{i}}^i$ of L_{λ}^{ν} by

$$(6.4) \quad \begin{aligned} L_{\dot{i}}^i &= a_{\dot{i}}^{\lambda} \dot{a}_{\lambda}^i L_{\lambda}^{\nu}, \\ L_{\lambda}^{\nu} &= \dot{a}_{\lambda}^i a_{\dot{i}}^{\nu} L_{\dot{i}}^i. \end{aligned}$$

From (6.2) and (6.3) we have, in particular,

$$(6.5) \quad \begin{aligned} (a) \quad L_x^i &= \lambda \delta_x^i, \\ (b) \quad L_y^x &= \lambda \delta_y^x. \end{aligned}$$

Definition (6.2). A holonomy group $G_r(R)$ of an L_n (or the holonomy group $G_r(K)$ of a V_n) is termed complete if it is possible to choose the $L_{a\lambda}^{\nu}$ in such a way that at least one of these tensors, say $L_{1\lambda}^{\nu}$, is complete. Then the nonholonomic frame $a_{\dot{i}}^{\nu}, \dot{a}_{\lambda}^i$ belonging to $L_{1\lambda}^{\nu}$ will be termed a proper nonholonomic frame.

Illustration (6.1). Assume $n = 4, r = 2$; denote by $a_{\dot{i}}^{\nu}, \dot{a}_{\lambda}^i$ an arbitrary set of vectors in L_4 satisfying (6.3) and by $\lambda_i \neq 0$ four different scalars. Put

$$(6.6a) \quad \begin{aligned} L_{1\lambda}^{\nu} &= \lambda \dot{a}_{\lambda}^1 a_{\dot{1}}^{\nu} + \lambda \dot{a}_{\lambda}^2 a_{\dot{2}}^{\nu} + \lambda \dot{a}_{\lambda}^3 a_{\dot{3}}^{\nu}, \\ L_{2\lambda}^{\nu} &= \lambda \dot{a}_{\lambda}^4 a_{\dot{4}}^{\nu}. \end{aligned}$$

The group defined by infinitesimal operators (4.15), (6.6a) is obviously abelian. If we apply to (6.6a) the transformations (4.2a) with

$$\left[\begin{pmatrix} \dot{A}_{\dot{1}}^1 & \dot{A}_{\dot{1}}^2 \\ \dot{A}_{\dot{2}}^1 & \dot{A}_{\dot{2}}^2 \end{pmatrix} \right] = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right],$$

we obtain

$$(6.6b) \quad \begin{aligned} L_{1\lambda}^{\nu} &= \Sigma_x \lambda \dot{a}_{\lambda}^x a_{\dot{x}}^{\nu}, \\ L_{2\lambda}^{\nu} &= L_{2\lambda}^{\nu} = \lambda \dot{a}_{\lambda}^4 a_{\dot{4}}^{\nu}. \end{aligned}$$

Hence the group under consideration is complete, the eigenvalues of $L_{1\lambda}^{\nu}$ are λ_i and the proper nonholonomic frame is $a_{\dot{i}}^{\nu}, \dot{a}_{\lambda}^i$.

*A nonholonomic frame of this kind has been used for solving EINSTEIN's equations for the connection of the unified field theory. See [1], Chapter II.

**See (4.2a).

Definition (6.3). The holonomy group $G_r(R)$ ($G_r(K)$) is termed general if it has the maximal number of parameters.

Theorem (6.1). The general holonomy group $G_r(K)$ has $r = \frac{1}{2}n(n-1)$ parameters and the local space spanned by the bivectors $L_{a\lambda\nu}$, i.e., the $\lambda\nu$ -support of $\{K_{\Psi_{a+\lambda\nu}}\}$, is n -dimensional, i.e., is E_n .

The proof follows from the fact that the maximal number of linearly independent bivectors is $r = \frac{1}{2}n(n-1)$.

Definition (6.4). The holonomy group $G_r(K)$ is termed degenerate if

- (1) it has $r = \frac{1}{2}q(q-1)$ parameters, $2 \leq q < n$;
- (2) the (local) space spanned by the bivectors $L_{a\lambda\nu}$ is q -dimensional, $E_q \subset E_n$;
- (3) E_q is not tangent to the (local) cone

$$(6.7) \quad g_{\lambda\mu}v^\lambda v^\mu = 0.$$

Illustration (6.2). Assume that $g_{\lambda\mu}$ in V_4 is positive definite, denote by u_i^ν four mutually perpendicular unit vectors and consider the bivectors

$$(6.8) \quad \begin{aligned} L_{\dot{1}}^{\lambda\nu} &= 2C_{\dot{1}}u_{\dot{1}}^{\lambda}u_{\dot{2}}^{\nu}, \\ L_{\dot{2}}^{\lambda\nu} &= 2C_{\dot{2}}u_{\dot{2}}^{\lambda}u_{\dot{3}}^{\nu}, \\ L_{\dot{3}}^{\lambda\nu} &= 2C_{\dot{3}}u_{\dot{3}}^{\lambda}u_{\dot{1}}^{\nu} \end{aligned}$$

where C_a are functions of position, constant with respect to the local Cartesian coordinate y^ν of E_4 . The group defined by the infinitesimal operators (4.15), (6.8) has the structural constants

$$(6.9a) \quad C_{a\epsilon}^{\epsilon} = -C_a C_\epsilon \quad (a, \epsilon, g \text{ any cyclic permutation of } \dot{1}, \dot{2}, \dot{3}),$$

$$(6.9b) \quad \text{the remaining } C_{\dot{1}\dot{1}}^{\dot{1}} = 0.$$

Hence we have $r = q = \frac{1}{2}q(q-1) = 3$, i.e., a three parameter group, and the bivectors (6.7) span the space E_3 of the simple trivector $u_{\dot{1}\dot{2}\dot{3}}^{\nu\lambda\mu}$ in V_4 . This trivector is obviously not tangent to the (local) cone (6.7).

7. Auxiliary results concerning $G_r(K)$. In this section we derive some results concerning $G_r(K)$ which we shall need in the next paper.

The metric tensor $g_{\lambda\mu}$ may always be written in the form

$$(7.1) \quad g_{\lambda\mu} = \Sigma_x \epsilon_x \tilde{u}_\lambda^x \tilde{u}_\mu^x$$

where $\epsilon_x = \pm 1$ and $\tilde{u}_\lambda^x$ are n mutually perpendicular unit vectors. Their contravariant components are

$$(7.2a) \quad \tilde{u}^\nu = g^{\lambda\nu} \tilde{u}_\lambda^x.$$

Hence if we put

$$(7.2b) \quad u^{\nu} \stackrel{\text{def}}{=} \epsilon_x^{\nu} u^x$$

we obtain a nonholonomic frame u^{ν}_i, u^i_{λ} satisfying the usual conditions

$$(7.3) \quad u^{\nu}_i u^i_{\nu} = \delta^i_i, \quad u^{\nu}_i u^i_{\lambda} = \delta^{\nu}_{\lambda}.$$

The nonholonomic components (with respect to this frame) of a tensor $P^{\nu \dots \star}_{\lambda \dots \star}$ are defined by

$$(7.4) \quad \begin{aligned} P^{i \dots \star}_{j \dots \star} &= u^i_{\nu} u^{\nu}_{\lambda} \dots P^{\nu \dots \star}_{\lambda \dots \star}, \\ P^{\nu \dots \star}_{\lambda \dots \star} &= u^{\nu}_i u^i_{\lambda} \dots P^{i \dots \star}_{j \dots \star}. \end{aligned}$$

In particular, the nonholonomic components of $g_{\lambda\mu}, g^{\lambda\nu}$ are

$$(7.5) \quad g^{ij} = g_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ \epsilon_j & i = j \end{cases}.$$

The components (7.5) will be used to lower and raise the indices of nonholonomic components of a tensor. Thus if we put, according to (7.4),

$$(7.6a) \quad L_{a i}^i = u^{\lambda}_i u^i_{\nu} L_{a \lambda}^{\nu}$$

we have

$$(7.6b) \quad L_{a y x} = L_{a y}^x \epsilon_x = -L_{a x}^y \epsilon_y.$$

If $G_r(K)$ is degenerate, we assume that the first q vectors $u^{\nu}_1, \dots, u^{\nu}_q$ span the (local) space E_q of the bivectors $L_a^{\lambda\nu}$. If we denote these vectors by $u^{\nu}_{a^*}$ and the remaining set by $u^{\nu}_{a^{**}}$, we have

$$(7.7) \quad \begin{aligned} L_{a a^* b^*} &= L_{a \lambda \nu} u^{\lambda}_{a^*} u^{\nu}_{b^*}, \\ L_{a a^* b^{**}} &= L_{a a^{**} b^*} = L_{a a^{**} b^{**}} = 0, \\ L_{a \lambda \nu} &= L_{a a^* b^*} u^{\lambda}_{a^*} u^{\nu}_{b^*}. \end{aligned}$$

In subsequent considerations *we shall deal exclusively with a general or degenerate holonomy group $G_r(K)$* . In the first theorem we shall compare the ranks of the matrices (5.3) and

$$(7.8a) \quad ((L)) \stackrel{\text{def}}{=} ((L_{a x_1 y_1}, L_{a x_2 y_2}, \dots, L_{a x_r y_r}))^*$$

where a labels the r rows, $x_u y_u$ labels the r columns and

*This matrix must not be confused with the matrix defined by (1.10a).

$$(7.8) \quad (b) \quad x_u, y_u = 1, \dots, M = \begin{cases} n \\ q \end{cases} \text{ for a } \begin{cases} \text{general} \\ \text{degenerate} \end{cases} G_r(K),$$

$$(c) \quad x_u < y_u, \quad u = 1, 2, \dots, r.$$

Of course, if $G_r(K)$ is general, then $r = \frac{1}{2}n(n-1)$, while for a degenerate $G_r(K)$ we have $r = \frac{1}{2}q(q-1)$, $1 < q < n$.

Theorem (7.1). *The ranks of the matrices (5.3) and (7.8) are r .*

Proof. The ranks of the matrices (5.3) are the same, namely r . (Cf. Theorem (5.2)). Let us introduce a coordinate system for which

$$u^{\nu} \stackrel{*}{=} \delta_i^{\nu} \quad \text{at } P.$$

Then we obtain for (5.3) at P

$$(7.9) \quad (a) \quad ((L_i^i)) = \left[\begin{pmatrix} \cdots & L_{1i}^i & \cdots \\ & \vdots & \\ \cdots & L_{ri}^i & \cdots \end{pmatrix} \right], \quad (b) \quad ((L_{ji})) = \left[\begin{pmatrix} \cdots & L_{1ji} & \cdots \\ & \vdots & \\ \cdots & L_{rji} & \cdots \end{pmatrix} \right],$$

and these matrices have also the rank r . Consider one particular column L_{azt} of (7.9b). We have

$$(7.10) \quad (a) \quad L_{azt} = -L_{atz}, \quad (b) \quad L_{azz} = 0.$$

Assume that $G_r(K)$ is general and let us leave out of (7.9b) all columns with elements (7.10b) and all columns

$$(7.10c) \quad L_{atz} \quad \text{for } t > z.$$

The resulting matrix is $((L))$, defined by (7.8a). It has obviously the same rank as $((L_{ji}))$, namely r . If $G_r(K)$ is degenerate then we have, besides (7.10a,b),

$$(7.10d) \quad L_{azt} = 0 \quad \text{for } z > q \quad \text{and/or } t > q.$$

Leaving out of $((L_{ji}))$ the columns of elements (7.10b,c,d), we do not change the rank of this matrix, and we obtain the matrix $((L))$ defined by (7.8).

In the next theorem we shall consider the matrices of $M-1$ columns and r rows

$$(7.11a) \quad ((L_z)) \stackrel{\text{def}}{=} ((L_{az1}, L_{az2}, \dots, L_{az(z-1)}, L_{az(z+1)}, \dots, L_{azM})),$$

$$z = 1, 2, \dots, M,$$

where

$$(7.11b) \quad M = \begin{cases} n \\ q \end{cases} \text{ for a } \begin{cases} \text{general} \\ \text{degenerate} \end{cases} G_r(K).$$

* $\delta_j^{\nu} = 1$ if and only if the Latin numeral j and the Roman numeral ν describe the same number, $\delta_1^I = \delta_2^{II} = \dots = 1$. Otherwise $\delta_j^{\nu} = 0$.

We built $((L_z))$ up from $M - 1$ columns of $((L))$, taking into account the skew-symmetry of $L_{\alpha i j}$. * Moreover, we shall assume $n > 2$.

Theorem (7.2). *Every one of the M matrices $((L_z))$ has the rank $M - 1$, where M is given in (7.11b).*

Proof. This theorem is trivial for a degenerate group with $q = 2$ (and $r = 1$). Therefore in dealing with a degenerate group we assume $q > 2$. According to Theorem (7.1) the rank of the $r \times r$ matrix $((L))$ is equal to r and therefore

$$(7.12) \quad L \stackrel{\text{def}}{=} \text{Det}((L)) \neq 0.$$

Keep z fixed and denote by $((L_z(i_1, \dots, i_{M-1})))$ the $(M - 1) \times (M - 1)$ matrix containing only the $i_1^{\text{th}}, \dots, i_{M-1}^{\text{th}}$ rows of $((L_z))$.† Write $((C_z(i_M, \dots, i_r)))$ for its complementary matrix in $((L))$. Then the determinant L of $((L))$ is equal to the sum of expressions

$$\text{Det}((L_z(i_1, \dots, i_{M-1}))) \text{Det}((C_z(i_M, \dots, i_r)))$$

with appropriate signs. Hence, according to (7.12), at least one of the $(M - 1) \times (M - 1)$ matrices $((L_z(i_1, \dots, i_{M-1})))$ must be of rank $M - 1$ and therefore the $(M - 1) \times r$ matrix $((L_z))$ must be of rank $M - 1$. This result holds obviously for any $z = 1, \dots, M$.

The last theorem of this section is

Theorem (7.3). *Let i_0, j_0 be two distinct indices from the range $1, \dots, M$. There is always an index α_0 from the range $1, \dots, r$ such that*

$$(7.13) \quad L_{\alpha_0 i_0 j_0} \neq 0.$$

Proof. This theorem is trivial for a degenerate group with $q = 2$ (and $r = 1$). Therefore in dealing with a degenerate group we assume $q > 2$. Assume that

$$(7.14) \quad L_{\alpha i_0 j_0} = 0$$

for all $\alpha = 1, \dots, r$. Then there is at least one of the M matrices $((L_z))$ with a column which consists of zeros only. Consequently the rank of this matrix is less than $M - 1$ which contradicts Theorem (7.2). Hence (7.14) cannot hold for all $\alpha = 1, \dots, r$, and therefore we have at least one α_0 from the range $1, \dots, r$ for which (7.13) holds.

8. A perfect holonomy group. In this section we shall complete the statement of Theorem (4.3b).

Definition (8.1). *The holonomy group $G_r(R)$ is termed perfect if the λ -domain of the tensor set*

*I.e., we replace the column of $((L))$ with the elements $L_{\alpha t z}$, $t < z$, by $L_{\alpha z t} = -L_{\alpha t z}$ in (7.11a).

† $((L_z))$ has r rows. Because $q > 2$ and because we assume $n > 2$, we have $r > M - 1$.

$$(8.1) \quad \nabla_{\xi_s} \cdots \nabla_{\xi_1} R_{\omega\mu\lambda}{}^\nu, \quad s = 1, 2, 3, \dots,$$

is contained in, or equal to, the χ -domain of $R_{\omega\mu\lambda}{}^\nu$.

In the next theorem we shall use the equations (4.12a) and (4.13a).

Theorem (8.1). *A necessary and sufficient condition for the holonomy group $G_r(R)$ to be perfect is the fulfillment of either one of the mutually equivalent conditions:*

(1) *The χ -rank of $R_{\omega\mu\lambda}{}^\nu$ is r .*

(2) *There is a set of tensors $F_{\epsilon}{}^a{}_{\xi}$ such that*

$$(8.2) \quad M^a{}_{\omega\mu\xi} = F_{\epsilon}{}^a{}_{\xi} M^{\epsilon}{}_{\omega\mu},$$

where the $M^{\epsilon}{}_{\omega\mu}$ are taken from (4.12a), i.e.,

$$(8.3) \quad R_{\omega\mu\lambda}{}^\nu = M^{\epsilon}{}_{\omega\mu} L_{\epsilon\lambda}{}^\nu.$$

Proof. The first statement is an obvious consequence of Definition (8.1). Assume that (8.2) holds where the $M^{\epsilon}{}_{\omega\mu}$ are taken from (8.3). An easy inspection shows that

$$(8.4) \quad (M^a{}_{\omega\mu\xi_s \cdots \xi_1}) = D_{\xi_s} \cdots D_{\xi_1} M^a{}_{\omega\mu} = F_{\epsilon}{}^a{}_{\xi_s \cdots \xi_1} M^{\epsilon}{}_{\omega\mu}$$

with conveniently chosen F 's and with the $M^{\epsilon}{}_{\omega\mu}$ taken from (8.3). Hence the χ -domain of the tensor set (8.1), i.e.,

$$(8.5) \quad \nabla_{\xi_s} \cdots \nabla_{\xi_1} R_{\omega\mu\lambda}{}^\nu = D_{\xi_s} \cdots D_{\xi_1} R_{\omega\mu\lambda}{}^\nu \\ = F_{\epsilon}{}^a{}_{\xi_s \cdots \xi_1} M^{\epsilon}{}_{\omega\mu} L_{a\lambda}{}^\nu, \quad s = 1, 2, 3, \dots,$$

is contained in, or equal to, the χ -domain of $R_{\omega\mu\lambda}{}^\nu$ given by (8.3). Hence the holonomy group is perfect. Assume now that the holonomy group is perfect. Then the χ -domain of the tensor set (8.1) is contained in, or equal to, the χ -domain of $R_{\omega\mu\lambda}{}^\nu$, and therefore the tensor set (8.1) must have the form (8.5) which for $s = 1$ yields (8.2).

Remark. This theorem is of no great practical value. In fact, the first statement requires the knowledge of r , the second statement requires the knowledge of $F_{\epsilon}{}^a{}_{\xi}$. In general, this necessary knowledge can be obtained only by dealing with the set (8.1). In doing so, we obtain the χ -domain of this set and can decide without the help of Theorem (8.1) whether the group under consideration is or is not a perfect group.

The following theorem does not require the whole tensor set (8.1).

Theorem (8.2). *A necessary and sufficient condition that the holonomy group, induced by $R_{\omega\mu\lambda}{}^\nu$, be perfect is the fulfillment of either one of the mutually equivalent conditions:*

(1) *The χ -domain of $\nabla_{\xi} R_{\omega\mu\lambda}{}^\nu$ is contained in, or equal to, the χ -domain of $R_{\omega\mu\lambda}{}^\nu$.*

(2) *There is a set of coefficients $\Phi_{\epsilon}{}^a{}_{\xi}$ such that*

$$(8.6) \quad \nabla_{\xi} M^a{}_{\omega\mu} = \Phi_{\epsilon}{}^a{}_{\xi} M^{\epsilon}{}_{\omega\mu}$$

where $M^{\epsilon}{}_{\omega\mu}$ are taken from (8.3).

If either one of these conditions is satisfied, then the number r of parameters of the holonomy group under consideration

- (a) is equal to the χ -rank of $R_{\omega\mu\lambda}{}^\nu$,
- (b) is equal to the number of $M^a{}_{\omega\mu}$ (of $L_{a\lambda}{}^\nu$) in (8.3).*

Proof. We have, by virtue of (8.2),

$$(8.7) \quad \nabla_\xi R_{\omega\mu\lambda}{}^\nu = F_\epsilon{}^a{}_\xi M^e{}_{\omega\mu} L_{a\lambda}{}^\nu.$$

On the other hand, if the first statement holds, then $\nabla_\xi R_{\omega\mu\lambda}{}^\nu$ must be of the form (8.7). Hence the first statement is equivalent to (8.2). Moreover, we obtain from (8.2)

$$(8.8) \quad D_\xi M^a{}_{\omega\mu} = \nabla_\xi M^a{}_{\omega\mu} + \Gamma_\epsilon{}^a{}_\xi M^e{}_{\omega\mu} = M^a{}_{\omega\mu\xi},$$

and this equation shows that the second statements of Theorems (8.1) and (8.2) are equivalent. The remaining part of our theorem follows from the following facts, valid for a perfect group:

(a) The χ -domain of (8.1) and (8.3) is fully described by the tensors involved in (8.3).

(b) Theorem (4.3b).

Definition (8.1) and Theorems (8.1), (8.2) hold also if we replace $R_{\omega\mu\lambda}{}^\nu$ by $K_{\omega\mu\lambda}{}^\nu$, i.e., if we replace $G_r(R)$ by $G_r(K)$.

Theorem (8.3). *A necessary and sufficient condition that the holonomy group induced by the curvature tensor $K_{\omega\mu\lambda}{}^\nu$ of a given V_n be perfect is: The tensor h^{ac} involved in (5.8b) is not singular. If this condition is satisfied, then the number r of parameters of the group under consideration is equal to the rank of h^{ac} .*

Proof. This theorem is an immediate consequence of the application of (5.8a) to the previous results. (See also the statement (c) of Theorem (5.3).)

Corollary (8.1). *A holonomy group of a $L_n(V_n)$ which has only one parameter is always perfect.*

The proof is obvious.

Remark. The next paper dealing with covariant constant quadratic tensors will be based chiefly on Theorems (5.6), (7.1), (7.2), (7.3) and (8.3).

BIBLIOGRAPHY

- HLAVATÝ, V., *Geometry of Einstein's Unified Field Theory* (P. Noordhoff, Groningen, 1958; 341 pp.).
- HLAVATÝ, V., The holonomy group I. The curvature tensor. The mapping of connections, *Jour. Math. and Mech.* 8 (1959). pp. 285-307.
- SCHOUTEN, J. A., *Ricci Calculus* (Springer, Berlin, 1954; 516 pp.).
- THOMAS, T. Y., *The Differential Invariants of Generalized Spaces* (Cambridge University Press, 1934; 420 pp.).

Indiana University
Bloomington, Indiana

*The decomposition described by (8.3) requires only the knowledge of $R_{\omega\mu\lambda}{}^\nu$.