*Free Boundary Problems for Parabolic Equations I. Melting of Solids**

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Introduction. Free boundary problems for the heat equation have been considered for over a century (for references prior to 1929 see BRILLOUIN [1]). In a few special cases the solutions have been found explicitly, but existence theorems for general cases were first proved about ten years ago. The well known one-dimensional Stefan problem, concerning the melting of ice when heated at the boundary with a prescribed temperature, was considered by RUBINSTEIN [22], [23], [24], [25], [26], [27] and DACEV [2], [3], [4], [5] (see also DATZEFF [6]). They used different methods whose basic features can be found in their papers [22] and [2] respectively.

Another Stefan problem where the flow of heat is given at the boundary was considered by EVANS [11] (see also [10]), J. DOUGLAS & GALLIE [7], SESTINI [28], [29], MIRANKER [20], J. DOUGLAS [8] and KYNER [19]. In the last two papers this problem (existence and uniqueness for all future times) was completely solved, regardless of whether both ice and water or just ice alone was present at the beginning.

The papers of DACEV on the Stefan problem are of heuristic nature. Actually he did not give any precise proof of his assertions about existence and uniqueness. It seems that it should be quite involved to rigorize his method. On the other hand, RUBINSTEIN [22] proved existence and uniqueness for small intervals of time except for a few incomplete details. In [25] he stated that he can prove existence and uniqueness for any interval of time with the aid of certain lemmas on "superparabolic" functions.

A method for solving some free boundary problems was developed by KOLODNER [18]. This method was used by MIRANKER [20] to solve a Stefan problem mentioned above. It was earlier applied by KELLER, KOLODNER & RITGER [14], KOLODNER [15], [16] and KOLODNER & RITGER [17] to study problems of evaporation and condensation of liquid drops.

In this work we refine the method of RUBINSTEIN [22] and use it to solve

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Stefan problems, problems of evaporation and condensation of drops and a problem of dissolution of gas bubbles in liquid, for *any* interval of time.

In Part I we consider only Stefan problems and we are essentially concerned with the case when the temperature at the boundary is prescribed and water is present at the beginning. In §1 we formulate the main result. In §3 we reduce the original problem to the problem of finding a unique solution of a certain nonlinear integral equation. In proving this reduction we use an auxiliary lemma proved in §2. In §4 we prove that the ice-water line, where it exists, is an increasing function of time. In §5 we prove the existence and uniqueness of the solution of the integral equation for small intervals of time. Our method is a modification of the method of successive approximations (both on $v(t)$ and $s(t)$) used by RUBINSTEIN [22] which makes it more suitable for the purpose of continuation of the solution into the future. In §6 we prove existence and uniqueness for all future times. The decisive step is an *a priori* estimate on the x -derivative of the solution $u(x, t)$ defined for $0 \leq t < t_0$, the bound being independent of t . In §7 we furnish the tools for the extension of the previous results to general parabolic equations with smooth coefficients. We conclude this section with a few open questions.

Added in proof. In Part II, which will appear shortly in this journal, we consider the problem of one drop surrounded by totally supersaturated or totally undersaturated vapor of its own substance. Existence and uniqueness are proved for all times as well as the analyticity of the solution with respect to the small parameter α which appears in the problem. We also derive the asymptotic behavior of the solution as time approaches ∞ . In Part III, to appear still later on, we develop a similar theory for one gas bubble which dissolves in a liquid of its own substance.

PART I: A STEFAN PROBLEM

1. Statement of the Main Result.

In this Part I we shall essentially be concerned with the following Stefan-type problem: Find $s(t) > 0$ and $u(x, t)$ such that

$$(1.1) \quad u_{xx} = u_t \quad \text{for } 0 < x < s(t), \quad t > 0,$$

$$(1.2) \quad u(0, t) = f(t) \quad \text{where } f(t) \geq 0 \quad \text{and } t > 0,$$

$$(1.3) \quad u(x, 0) = \varphi(x) \quad \text{where } \varphi(x) \geq 0, \quad 0 < x \leq b, \quad \text{and } \varphi(b) = 0, \quad b > 0,$$

$$(1.4) \quad u(s(t), t) = 0 \quad \text{for } t > 0 \quad \text{and } s(0) = b,$$

$$(1.5) \quad u_x(s(t), t) = -\dot{s}(t) = -\frac{ds(t)}{dt} \quad \text{for } t > 0.$$

$x = s(t)$ is the free boundary (for instance, the water-ice line) which is not known and is to be found together with $u(x, t)$. The conditions (1.2), (1.3), (1.4) are the usually given data whereas the additional condition (1.5) (the equation

of heat balance) is a condition on the free boundary $x = s(t)$. The assumptions $f \geq 0$, $\varphi \geq 0$ result from the fact that the temperature of water is non-negative.

Definition. We say that $u(x, t)$, $s(t)$ form a solution of the system (1.1)–(1.5) for all $t < \sigma$ ($0 < \sigma \leq \infty$) if (i) $\partial^2 u / \partial x^2$ and $\partial u / \partial t$ are continuous for $0 < x < s(t)$, $0 < t < \sigma$; (ii) u and $\partial u / \partial x$ are continuous for $0 \leq x \leq s(t)$, $0 < t < \sigma$; (iii) $u(x, t)$ is continuous, also for $t = 0$, $0 < x \leq b$ and $0 \leq \liminf u(x, t) \leq \limsup u(x, t) < \infty$ as $t \rightarrow 0$, $x \rightarrow 0$; (iv) $s(t)$ ($0 \leq t < \sigma$) is continuously differentiable; and, finally, (v) the equations (1.1)–(1.5) are satisfied. The main result of this Part I is the following theorem.

Theorem 1. Assume that $f(t)$ ($0 \leq t < \infty$) and $\varphi(x)$ ($0 \leq x \leq b$) are continuously differentiable functions. Then there exists one and only one solution $u(x, t)$, $s(t)$ of the system (1.1)–(1.5) for all $t < \infty$. Furthermore, the function $x = s(t)$ is monotone nondecreasing in t .

The proof of Theorem 1 is given in §§2–6. In §2 we prove an auxiliary lemma (a stronger form of a lemma stated in [18]). In §3 we prove that for any possible solution in $0 \leq t < \sigma$, $s(t)$ is monotone nondecreasing in t . It is also shown that if either $\varphi(x) \equiv 0$ or $f(t) \equiv 0$ in any interval $0 < t < \epsilon$ ($\epsilon > 0$) then $s(t)$ ($0 \leq t < \sigma$) is a strictly increasing function of t . In §4 we reduce the original problem (1.1)–(1.5) to a problem of solving a certain nonlinear integral equation. In this proof we make use of the lemma proved in §2. In §5 we prove existence and uniqueness of solutions for the nonlinear integral equation for a small interval of time t . Finally in §6 (using the results of §§3–5) we prove that the solution can be continued, and uniquely, to all future times. In §7 we give various generalizations of Theorem 1.

2. Auxiliary Lemma.

Define

$$K(x, t; \xi, \tau) = \frac{1}{2\pi^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(x - \xi)^2}{4(t - \tau)} \right\}.$$

Lemma 1. Let $\rho(t)$ ($0 \leq t \leq \sigma$) be a continuous function and let $s(t)$ ($0 \leq t \leq \sigma$) satisfy a Lipschitz condition. Then for every $0 < t \leq \sigma$

$$\begin{aligned} (2.1) \quad \lim_{x \rightarrow s(t)-0} \frac{\partial}{\partial x} \int_0^t \rho(\tau) K(x, t; s(\tau), \tau) d\tau \\ = \frac{1}{2} \rho(t) + \int_0^t \rho(\tau) \left[\frac{\partial}{\partial x} K(x, t; s(\tau), \tau) \right]_{x=s(t)} d\tau. \end{aligned}$$

Proof. We shall first prove that for any fixed positive $\delta < t$

$$\begin{aligned} (2.2) \quad I \equiv \int_{t-\delta}^t \frac{x - s(\tau)}{2(t - \tau)} K(x, t; s(\tau), \tau) d\tau \\ - \int_{t-\delta}^t \frac{s(t) - s(\tau)}{2(t - \tau)} K(s(t), t; s(\tau), \tau) d\tau = -\frac{1}{2} + O(\delta^{\frac{1}{2}}) \end{aligned}$$

as $x \rightarrow s(t) = 0$, and $|O(\delta^{\frac{1}{2}})| \leq A_0 \delta^{\frac{1}{2}}$ where A_0 is a constant independent of x , t , δ . In the sequel we shall denote by A_i appropriate constants independent of x , t , δ .

To prove (2.2) we write $I = I_1 + I_2$ where

$$(2.3) \quad I_1 = \int_{t-\delta}^t \frac{x - s(t)}{2(t - \tau)} K(x, t; s(\tau), \tau) d\tau,$$

$$(2.4) \quad I_2 = \int_{t-\delta}^t \frac{s(t) - s(\tau)}{2(t - \tau)} [K(x, t; s(\tau), \tau) - K(s(t), t; s(\tau), \tau)] d\tau.$$

Since by assumption $|s(t) - s(\tau)| \leq A_1(t - \tau)$ we immediately get

$$(2.5) \quad |I_2| \leq A_1 \int_{t-\delta}^t \frac{d\tau}{(t - \tau)^{\frac{1}{2}}} \leq 2A_1 \delta^{\frac{1}{2}}.$$

To evaluate I_1 denote

$$(2.6) \quad J_1 = \int_{t-\delta}^t \frac{x - s(t)}{2(t - \tau)} K(x, t; s(t), \tau) d\tau.$$

Then

$$(2.7) \quad J_1 - I_1 = \int_{t-\delta}^t \frac{x - s(t)}{2(t - \tau)} K(x, t; s(t), \tau) \cdot \left[1 - \exp \left\{ - \frac{(x - s(\tau))^2 - (x - s(t))^2}{4(t - \tau)} \right\} \right] d\tau.$$

The expression in the braces in the second exponent is bounded by

$$\begin{aligned} \frac{1}{4(t - \tau)} |s(t) - s(\tau)| (|x - s(t)| + |x - s(\tau)|) \\ \leq A_2 (|x - s(t)| + |s(t) - s(\tau)|). \end{aligned}$$

Since we may assume that δ is small and hence that the right side of the last inequality is smaller than 1, we conclude that the expression in the brackets on the right side of (2.7) is bounded by

$$A_3 (|x - s(t)| + |s(t) - s(\tau)|).$$

Substituting this in (2.7) we find (using $ye^{-y} < \text{const.}$ for $y \geq 0$)

$$(2.8) \quad |J_1 - I_1| \leq A_4 \int_{t-\delta}^t \frac{d\tau}{(t - \tau)^{\frac{1}{2}}} + A_4 |x - s(t)| \int_{t-\delta}^t \frac{d\tau}{(t - \tau)^{\frac{1}{2}}} \leq A_5 \delta^{\frac{1}{2}}.$$

Now, as to J_1 , substitute in the integral (2.6)

$$z = \frac{t - \tau}{(x - s(t))^2}.$$

Noting that $x - s(t) < 0$, we get

$$(2.9) \quad J_1 = -\frac{1}{4\pi^{\frac{1}{2}}} \int_0^{\delta'} \frac{1}{z^{\frac{1}{2}}} \exp \left\{ -\frac{1}{4z} \right\} dz \quad \text{where} \quad \delta' = \frac{\delta}{(x - s(t))^2}.$$

As $x \rightarrow s(t)$, $\delta' \rightarrow \infty$ and hence $J_1 \rightarrow -\frac{1}{2}$.

Combining the last result with (2.8) and (2.5) and using the definition $I = I_1 + I_2$ it follows that

$$(2.10) \quad \limsup_{x \rightarrow s(t) - 0} |I + \frac{1}{2}| \leq A_6 \delta^{\frac{1}{2}},$$

which is the statement (2.2).

From the above evaluation of I_1 it also follows that

$$(2.11) \quad |I_1| \leq A_7$$

where A_7 depends on σ .

Having proved (2.2) we proceed to prove that

$$(2.12) \quad K_0 \equiv \int_{t-\delta}^t \frac{|x - s(\tau)|}{2(t - \tau)} K(x, t; s(\tau), \tau) d\tau \leq A_8,$$

$$(2.13) \quad K_1 \equiv \int_{t-\delta}^t \frac{|s(t) - s(\tau)|}{2(t - \tau)} K(s(t), t; s(\tau), \tau) d\tau \leq A_9.$$

The proof of (2.13) follows immediately by using the Lipschitz continuity of $s(t)$. The proof of (2.12) follows from the boundedness of I_1 (see (2.11)) and that of

$$K_2 = \int_{t-\delta}^t \frac{|s(t) - s(\tau)|}{2(t - \tau)} K(x, t; s(\tau), \tau) d\tau,$$

noting that $K_0 \leq |I_1| + K_2$.

With the aid of (2.2), (2.12), (2.13) we proceed to prove (2.1). Putting

$$(2.14) \quad \begin{aligned} L_1 = \int_{t-\delta}^t \rho(\tau) \frac{x - s(\tau)}{2(t - \tau)} K(x, t; s(\tau), \tau) d\tau \\ - \int_{t-\delta}^t \rho(\tau) \frac{s(t) - s(\tau)}{2(t - \tau)} K(s(t), t; s(\tau), \tau) d\tau \end{aligned}$$

we claim that

$$(2.15) \quad \limsup_{x \rightarrow s(t) - 0} |L_1 + \frac{1}{2}\rho(t)| \leq A_{10}(\delta^{\frac{1}{2}} + \text{l.u.b.}_{t-\delta \leq \tau \leq t} |\rho(t) - \rho(\tau)|).$$

Indeed, upon writing in (2.14) $\rho(\tau) = \rho(t) + (\rho(\tau) - \rho(t))$ and using (2.2), (2.12) and (2.13), the proof of (2.15) follows.

We next observe that the function

$$(2.16) \quad \begin{aligned} L_2 = \int_0^{t-\delta} \rho(\tau) \frac{x - s(\tau)}{2(t - \tau)} K(x, t; s(\tau), \tau) d\tau \\ - \int_0^{t-\delta} \rho(\tau) \frac{s(t) - s(\tau)}{2(t - \tau)} K(s(t), t; s(\tau), \tau) d\tau \quad (\delta > 0) \end{aligned}$$

satisfies

$$(2.17) \quad \lim_{x \rightarrow s(t)} L_2 = 0.$$

Combining this remark with (2.15) we get

$$(2.18) \quad \limsup_{x \rightarrow s(t)-0} |(L_1 + L_2) + \frac{1}{2}\rho(t)| \leq A_{10}(\delta^{\frac{1}{2}} + \text{l.u.b.}_{t-\delta \leq \tau \leq t} |\rho(t) - \rho(\tau)|).$$

Since the left side of (2.18) is independent of δ we conclude, on taking $\delta \rightarrow 0$, that it is zero, and the proof of the lemma is completed.

3. Monotonicity of $s(t)$.

Suppose $u(x, t)$, $s(t)$ form a solution of the system (1.1)–(1.5) for $0 \leq t < \sigma$. By the maximum principle [21; Theorem 2] $u(x, t)$ is nonnegative for $0 < x < s(t)$, $0 \leq t < \sigma$. Since $u = 0$ on $x = s(t)$, $u_x \leq 0$ and hence, by (1.5), $\dot{s}(t) \geq 0$. This proves that $s(t)$ is nondecreasing in t for $0 \leq t < \sigma$.

We shall now prove that if either $\varphi(x) \equiv 0$ or $f(t) \equiv 0$ in any interval $0 \leq t \leq \epsilon$, then $s(t)$, if it exists, is strictly increasing for all $0 \leq t < \sigma$. For in the contrary case there exist two points $t' < t''$ ($t'' < \sigma$) such that $s(t) \equiv s(t')$ for all $t' \leq t \leq t''$. Therefore, by (1.5), $u_x = -\dot{s} = 0$ on the straight segment $x = s(t)$, $t' \leq t \leq t''$. We now use the strong maximum principle [21; Theorem 5] to show that $u(x, t) > 0$ for $0 < x < s(t)$, $0 < t < \sigma$. Since $u(s(t), t) = 0$ we use [12; Theorem 2] to conclude that u_x is negative at any point $(s(t), t)$, $t' < t < t''$ which is a contradiction.

4. The Integral Equation.

We introduce Green's function for the half plane $x > 0$:

$$(4.1) \quad \begin{aligned} G(x, t; \xi, \tau) &= \frac{1}{2\pi^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(x-\xi)^2}{4(t-\tau)} \right\} \\ &- \frac{1}{2\pi^{\frac{1}{2}}(t-\tau)} \exp \left\{ -\frac{(x+\xi)^2}{4(t-\tau)} \right\} = K(x, t; \xi, \tau) - K(-x, t; \xi, \tau). \end{aligned}$$

Suppose $u(x, t)$, $s(t)$ form a solution of the system (1.1)–(1.5). Integrating Green's identity

$$\frac{\partial}{\partial \xi} \left(G \frac{\partial u}{\partial \xi} - u \frac{\partial G}{\partial \xi} \right) - \frac{\partial}{\partial \tau} (Gu) = 0$$

over the domain $0 < \xi < s(\tau)$, $0 < \epsilon < \tau < t - \epsilon$ and letting $\epsilon \rightarrow 0$ we get, on using (1.2), (1.3), (1.4),

$$(4.2) \quad \begin{aligned} u(x, t) &= \int_0^t u_\xi(s(\tau), \tau) G(x, t; s(\tau), \tau) d\tau + \int_0^t f(\tau) G_\xi(x, t; 0, \tau) d\tau \\ &+ \int_0^b \varphi(\xi) G(x, t; \xi, 0) d\xi = M_1 + M_2 + M_3 \end{aligned}$$

where M_i denotes the i^{th} integral. Denoting

$$(4.3) \quad v(\tau) = u_\xi(s(\tau), \tau)$$

we proceed to differentiate both sides of (4.2) with respect to x and let $x \rightarrow s(t) - 0$. Using Lemma 1 we have

$$(4.4) \quad \lim_{x \rightarrow s(t) - 0} \frac{\partial M_1}{\partial x} = \frac{1}{2}v(t) + \int_0^t v(\tau) G_x(s(t), t; s(\tau), \tau) d\tau.$$

Here we used the fact that the second term of G is a regular function since $x + s(\tau) \geq b > 0$ (since $s(t)$ is nondecreasing in t).

We proceed to evaluate $\partial M_i / \partial x$, $i = 2, 3$. Denote by

$$(4.5) \quad \begin{aligned} N(x, t; \xi, \tau) &= \frac{1}{2\pi^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(x - \xi)^2}{4(t - \tau)} \right\} \\ &+ \frac{1}{2\pi^{\frac{1}{2}}(t - \tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(x + \xi)^2}{4(t - \tau)} \right\} = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau) \end{aligned}$$

the Neumann function for the half space $x > 0$. It satisfies $G_x = -N_\xi$. We now have

$$(4.6) \quad \begin{aligned} \frac{\partial M_2}{\partial x} &= \int_0^t f(\tau) G_{x\xi}(x, t; 0, \tau) d\tau = \int_0^t f(\tau) N_\tau(x, t; 0, \tau) d\tau \\ &= -f(0)N(x, t; 0, 0) - \int_0^t \dot{f}(\tau)N(x, t; 0, \tau) d\tau. \end{aligned}$$

Similarly

$$(4.7) \quad \frac{\partial M_3}{\partial x} = \int_0^b \varphi(\xi) G_x(x, t; \xi, 0) d\xi = \varphi(0)N(x, t; 0, 0) + \int_0^b \dot{\varphi}(\xi)N(x, t; \xi, 0) d\xi.$$

Using (4.4), (4.6), (4.7) we obtain, on differentiating (4.2) with respect to x and taking $x \rightarrow s(t) - 0$,

$$(4.8) \quad \begin{aligned} v(t) &= 2[\varphi(0) - f(0)]N(s(t), t; 0, 0) + 2 \int_0^b \dot{\varphi}(\xi)N(s(t), t; \xi, 0) d\xi \\ &- 2 \int_0^t \dot{f}(\tau)N(s(t), t; 0, \tau) d\tau + 2 \int_0^t v(\tau) G_x(s(t), t; s(\tau), \tau) d\tau \end{aligned}$$

where, by (1.5) and (4.3),

$$(4.9) \quad s(t) = b - \int_0^t v(\tau) d\tau.$$

We have thus proved that for every solution u, s of the system (1.1)–(1.5), v (defined by (4.3)) must satisfy the integral equation (4.8), where $s(t)$ is defined by (4.9).

Suppose conversely that for some $\sigma > 0$, $v(t)$ ($0 \leq t < \sigma$) is a continuous solution of the integral equation (4.8) where $s(t)$ is defined by (4.9) and $s(t) > 0$.

We prove that $u(x, t)$, $s(t)$ (where $u(x, t)$ is defined by (4.2) with $u_\xi(s(\tau), \tau)$ replaced by $v(\tau)$) form a solution of the system (1.1)–(1.5).

First of all one can easily verify (1.1)–(1.3). We now differentiate $u(x, t)$ with respect to x and take $x \rightarrow s(t) - 0$. Using Lemma 1, the previous calculations of $\partial M_i / \partial x$ ($i = 2, 3$) and the integral equation (4.8) we find that $u_x(s(t), t) = v(t)$. Since, by (4.9), $v(t) = -\dot{s}(t)$, (1.5) follows. We remark that $u(x, t)$ and $s(t)$ have all the regularity properties in order to form a solution in the sense of §1. Thus it remains to prove that $u(s(t), t) = 0$.

Integrate Green's identity (with G and u) in the domain $0 < \xi < s(\tau)$, $0 < \epsilon < \tau < t - \epsilon$ and let $\epsilon \rightarrow 0$. Comparing the integral representation for $u(x, t)$ thus obtained with (4.2) we conclude that

$$(4.10) \quad \int_0^t u(s(\tau), \tau) G_\xi(x, t; s(\tau), \tau) d\tau = 0 \quad \text{if} \quad 0 < x < s(t), \quad 0 < t < \sigma.$$

Taking $x \rightarrow s(t) - 0$ and using Lemma 1 we conclude that $\varphi(t) \equiv u(s(t), t)$ satisfies the integral equation

$$\varphi(t) = -2 \int_0^t \varphi(\tau) G_\xi(s(t), t; s(\tau), \tau) d\tau.$$

Since

$$|G_\xi(s(t), t; s(\tau), \tau)| \leq \frac{\text{const.}}{(t - \tau)^{\frac{1}{2}}}$$

we easily conclude that $u(s(t), t) \equiv \varphi(t) \equiv 0$.

We have thus proved that *the problem (1.1)–(1.5) is equivalent to the problem of finding a continuous solution of the integral equation (4.8) where $s(t)$ is defined by (4.9) and is a positive function.*

We remark that it cannot happen that, for some $t_0 > 0$, $s(t) > 0$ for all $t < t_0$ and $s(t) \rightarrow 0$ as $t \rightarrow t_0$. (Here $s(t)$ is defined by (4.9) and $v(t)$ is a solution of (4.8).) Indeed, by what we have already proved $u(x, t)$, $s(t)$ form a solution of (1.1)–(1.5) and hence, by §3, $s(t)$ is a nondecreasing function of t ; in particular, $s(t) \geq b > 0$ for all $t < t_0$.

5. Existence and Uniqueness for Small Times.

In this section we shall prove that the integral equation (4.8) has a unique solution for $0 \leq t < \sigma$, σ being a sufficiently small number. We denote by C_σ the Banach space of functions $v(t)$ defined and continuous for $0 \leq t \leq \sigma$ with the uniform norm, namely $\|v\| = \text{l.u.b.}_{0 \leq t \leq \sigma} |v(t)|$. We denote by $C_{\sigma, M}$ the closed sphere $\|v\| \leq M$. On the set $C_{\sigma, M}$ we define a transformation $w = Tv$ in the following way:

$$(5.1) \quad \begin{aligned} w(t) = & 2[\varphi(0) - f(0)]N(s(t), t; 0, 0) + 2 \int_0^b \dot{\varphi}(\xi)N(s(t), t; \xi, 0) d\xi \\ & - 2 \int_0^t \dot{f}(\tau)N(s(t), t; 0, \tau) d\tau \end{aligned}$$

$$+ 2 \int_0^t v(\tau) G_x(s(t), t; s(\tau), t) d\tau \quad (0 \leq t \leq \sigma)$$

where

$$(5.2) \quad s(t) = b + \int_0^t v(\tau) d\tau.$$

We shall prove a few properties of T .

5.1. T maps $C_{\sigma, M}$ into itself. From (5.2) it follows that

$$(5.3) \quad |s(t) - s(\tau)| \leq M(t - \tau).$$

Denote by B_i appropriate positive constants depending (continuously) on b only. We take σ to satisfy

$$(5.4) \quad 2M\sigma \leq b, \quad \sigma \leq 1$$

so that

$$(5.5) \quad \frac{1}{2}b \leq s(t) \leq \frac{3}{2}b \quad (0 \leq t \leq \sigma).$$

With the aid of (5.3), (5.5) we easily find that

$$(5.6) \quad \|w\| \leq 4 \|\dot{\varphi}\| + B_1(\varphi(0) + f(0) + \|f\|)\sigma + B_2 M^2 \sigma^{\frac{1}{2}}.$$

Defining

$$(5.7) \quad M = 4 \|\dot{\varphi}\| + 1$$

and taking σ to satisfy, besides (5.4),

$$(5.8) \quad B_1(\varphi(0) + f(0) + \|f\|)\sigma + B_2(4 \|\dot{\varphi}\| + 1)^2 \sigma^{\frac{1}{2}} \leq 1$$

we conclude that $\|w\| \leq M$, that is, T maps $C_{\sigma, M}$ into itself. Using (5.7) we can replace the conditions (5.4) by

$$(5.9) \quad 2(4 \|\dot{\varphi}\| + 1)\sigma \leq b, \quad \sigma \leq 1.$$

5.2. T is a contraction. Let $w = Tv$, $w' = Tv'$ and denote

$$(5.10) \quad \|v - v'\| = \epsilon.$$

If $s'(t)$ corresponds to $v'(t)$ by (5.2), then we have the following inequalities:

$$(5.11) \quad |s(t) - s'(t)| \leq t\epsilon, \quad \|\dot{s} - \dot{s}'\| \leq \epsilon.$$

Since $\|v\| \leq M$, $\|v'\| \leq M$ we also have

$$(5.12) \quad |s(t) - s(\tau)| \leq M(t - \tau), \quad |s'(t) - s'(\tau)| \leq M(t - \tau), \quad \epsilon \leq 2M.$$

Finally, taking σ to satisfy (5.9) we have

$$(5.13) \quad \frac{1}{2}b \leq s(t) \leq \frac{3}{2}b, \quad \frac{1}{2}b \leq s'(t) \leq \frac{3}{2}b \quad \text{for } 0 \leq t \leq \sigma.$$

With the aid of (5.11), (5.12), (5.13) we now proceed to prove that, for small σ , T is a contraction.

Write

$$(5.14) \quad w - w' = V_1 + V_2 + V_3 + V_4$$

where

$$(5.15) \quad V_1 = 2[\varphi(0) - f(0)][N(s(t), t; 0, 0) - N(s'(t), t; 0, 0)],$$

$$(5.16) \quad V_2 = 2 \int_0^b \dot{\varphi}(\xi)[N(s(t), t; \xi, 0) - N(s'(t), t; \xi, 0)] d\xi,$$

$$(5.17) \quad V_3 = -2 \int_0^t \dot{f}(\tau)[N(s(t), t; 0, \tau) - N(s'(t), t; 0, \tau)] d\tau,$$

$$(5.18) \quad \begin{aligned} -V_4 = & \int_0^t v(\tau) \frac{s(t) - s(\tau)}{t - \tau} K(s(t), t; s(\tau), \tau) d\tau \\ & - \int_0^t v'(\tau) \frac{s'(t) - s'(\tau)}{t - \tau} K(s'(t), t; s'(\tau), \tau) d\tau - V'_4. \end{aligned}$$

Here V'_4 denotes the sum of the first two terms on the right side of (5.18) but with $s(\tau)$ and $s'(\tau)$ replaced by $-s(\tau)$ and $-s'(\tau)$ respectively.

The estimations of V_1 and V_3 are quite simple and are derived by using the mean value theorem. In estimating V_2 we also make use of the mean value theorem but we have to be more careful because the integrand would be singular if $\xi = s(t)$ or $\xi = s'(t)$. In the estimation of V_4 and, in particular, in that of W_3 (see (5.24), (5.27)) it is necessary to make an even more careful analysis, and the direct use of the mean value theorem is not enough.

Using the mean value theorem, (5.11) and (5.13) we easily get

$$(5.19) \quad |V_1| \leq B_3[\varphi(0) + f(0)]\sigma\epsilon.$$

To estimate V_2 we first estimate

$$(5.20) \quad V'_2 = \int_0^b 2\dot{\varphi}(\xi)[K(s(t), t; \xi, 0) - K(s'(t), t; \xi, 0)] d\xi.$$

We may assume that $s'(t) > s(t)$. In case $s'(t) < b$ we divide the integral into three parts:

$$V'_2 = \int_0^b = \int_0^{s(t)} + \int_{s(t)}^{s'(t)} + \int_{s'(t)}^b = Y_1 + Y_2 + Y_3.$$

In case $s(t) < b \leq s'(t)$ we divide it only into two parts, namely,

$$V'_2 = \int_0^b = \int_0^{s(t)} + \int_{s(t)}^b.$$

Finally, in case $b \leq s(t)$ we do not divide it at all. In what follows we shall

estimate V'_2 only in the first case, the other cases can be treated in a similar manner. To estimate Y_1 we use the mean value theorem and note that if $s(t) < \xi < s'(t)$ then

$$\exp \left\{ -\frac{(\xi - \xi)^2}{\alpha t} \right\} \leq \exp \left\{ -\frac{(s(t) - \xi)^2}{\alpha t} \right\} \quad (\alpha > 0).$$

Hence we get, in view of (5.11),

$$|Y_1| \leq B_4 \|\dot{\varphi}\| \int_0^b \frac{|s(t) - s'(t)|}{t} \exp \left\{ -\frac{(s(t) - \xi)^2}{8t} \right\} d\xi \leq B_4 \|\dot{\varphi}\| \sigma^{\frac{1}{2}} \epsilon.$$

Y_3 can be estimated in a similar manner. Finally,

$$|Y_2| \leq |s'(t) - s(t)| \|\dot{\varphi}\| \frac{1}{t^{\frac{1}{2}}} \leq \|\dot{\varphi}\| \sigma^{\frac{1}{2}} \epsilon.$$

Combining the estimates of the Y_i we get

$$(5.21) \quad |V'_2| \leq B_5 \|\dot{\varphi}\| \sigma^{\frac{1}{2}} \epsilon.$$

We can similarly estimate $V_2 - V'_2$ which is equal to the right side of (5.20) with ξ replaced by $-\xi$. We obtain

$$(5.22) \quad |V_2| \leq B_6 \|\dot{\varphi}\| \sigma^{\frac{1}{2}} \epsilon.$$

To estimate V_3 we use the mean value theorem, (5.11) and (5.13). We conclude that

$$(5.23) \quad |V_3| \leq B_7 \|\dot{f}\| \sigma \epsilon.$$

To estimate V_4 we write

$$(5.24) \quad V_4 = W_1 + W_2 + W_3 - V'_4$$

where

$$(5.25) \quad W_1 = \int_0^t [v(\tau) - v'(\tau)] \frac{s(t) - s(\tau)}{t - \tau} K(s(t), t; s(\tau), \tau) d\tau,$$

$$(5.26) \quad W_2 = \int_0^t v'(\tau) \left[\frac{s(t) - s(\tau)}{t - \tau} - \frac{s'(t) - s'(\tau)}{t - \tau} \right] K(s(t), t; s(\tau), \tau) d\tau,$$

$$(5.27) \quad W_3 = \int_0^t v'(\tau) \frac{s'(t) - s'(\tau)}{t - \tau} K(s(t), t; s(\tau), \tau) \cdot \left[1 - \exp \left\{ -\frac{(s'(t) - s'(\tau))^2 - (s(t) - s(\tau))^2}{4(t - \tau)} \right\} \right] d\tau.$$

Using (5.10), (5.12) we get

$$(5.28) \quad |W_1| \leq M\epsilon \int_0^t \frac{d\tau}{(t - \tau)^{\frac{1}{2}}} \leq B_8 M \sigma^{\frac{1}{2}} \epsilon.$$

To estimate W_2 we note that the expression in the brackets on the right side of (5.26) is equal to

$$\frac{[s(t) - s'(t)] - [s(\tau) - s'(\tau)]}{t - \tau} = \dot{s}(\tilde{\tau}) - \dot{s}'(\tilde{\tau}) \quad (\tau < \tilde{\tau} < t).$$

Hence, using (5.11) we obtain

$$(5.29) \quad |W_2| \leq B_9 M \sigma^{\frac{1}{2}} \epsilon.$$

To estimate W_3 we note that the expression in the second braces on the right side of (5.27) is bounded by

$$(5.30) \quad \frac{1}{4(t - \tau)} | [s'(t) - s(t)] - [s'(\tau) - s(\tau)] |$$

$$| [s'(t) - s'(\tau)] + [s(t) - s(\tau)] | \leq \frac{1}{2} M \epsilon (t - \tau).$$

If we take σ to satisfy

$$(5.31) \quad M^2 \sigma \leq 1$$

then the right side of (5.30) is bounded by 1. We then conclude that the expression in the brackets on the right side of (5.27) is bounded by $B_{10} M \epsilon (t - \tau)$. Making use of this last remark and of (5.12) and (5.31) we get

$$(5.32) \quad |W_3| \leq B_{10} M^3 \epsilon \int_0^t (t - \tau)^{\frac{1}{2}} d\tau \leq \frac{2}{3} B_{10} M^3 \sigma^{\frac{1}{2}} \epsilon \leq \frac{2}{3} B_{10} M \sigma^{\frac{1}{2}}.$$

Combining the estimates (5.28), (5.29), (5.32) with (5.24) we get

$$(5.33) \quad |W_1 + W_2 + W_3| \leq B_{11} M \sigma^{\frac{1}{2}} \epsilon.$$

The estimation of V_4 follows immediately by using the mean value theorem and noting that $s(t) + s(\tau) \geq b > 0$, $s'(t) + s'(\tau) \geq b > 0$. Combining this remark with (5.33) we get

$$(5.34) \quad |V_4| \leq B_{12} M \sigma^{\frac{1}{2}} \epsilon.$$

Combining the estimates (5.19), (5.22), (5.23), (5.34) with (5.14) we get

$$(5.35) \quad \frac{||w - w'||}{\epsilon} \leq B_3 [\varphi(0) + f(0)] \sigma$$

$$+ B_6 ||\dot{\varphi}|| \sigma^{\frac{1}{2}} + B_7 ||\dot{f}|| \sigma + B_{12} M \sigma^{\frac{1}{2}} \equiv A \sigma^{\frac{1}{2}}$$

provided σ satisfies (5.9) and (5.31). In view of (5.7), (5.31) becomes

$$(5.36) \quad (4 ||\dot{\varphi}|| + 1)^2 \sigma \leq 1.$$

We conclude that if σ satisfies (5.9), (5.36) and

$$(5.37) \quad A\sigma^{\frac{1}{2}} < 1$$

then T is a contraction operator on $C_{\sigma, M}$.

5.3. Completion of the proof. We have proved that if σ satisfies (5.8), (5.9) then T maps $C_{\sigma, M}$ into itself and that if σ satisfies (5.9), (5.36) and (5.37) then T is a contraction of $C_{\sigma, M}$. Thus, taking σ to satisfy all the above conditions, it follows that there exists a unique fixed point $v(t)$ of T in $C_{\sigma, M}$.

Having proved the existence of a solution $v(t)$ of the integral equation (4.8) (with $s(t)$ defined by (4.9)) we proceed to prove that every solution of the integral equation (4.8) (regardless whether it is bounded by the M defined in (5.7) or not) must coincide with $v(t)$ in their common interval of existence. Suppose $v'(t)$, $0 \leq t < \bar{\sigma}$ is another solution of (4.8). We may assume that $\bar{\sigma} \leq \sigma$ and we then have to prove that $v(t) \equiv v'(t)$ for all $0 \leq t < \bar{\sigma}$. Replace, in the considerations of 5.1, 5.2, M by

$$M' = \max(M, \text{l.u.b. } |v'(t)| \text{ for } 0 \leq t \leq \bar{\sigma}).$$

Then instead of σ we get σ' ($\sigma' < \bar{\sigma}$) which satisfies the inequalities which guarantee that T is a contraction and that T maps, $C_{\sigma', M'}$ into itself. Since $v(t)$ and $v'(t)$ are both fixed points in $C_{\sigma', M'}$ of the transformation T we have $v(t) \equiv v'(t)$ ($0 \leq t \leq \sigma'$).

Now let $\sigma_1 < \bar{\sigma}$ be any number such that $v(t) \equiv v'(t)$, $0 \leq t < \sigma_1$. Then $v(\sigma_1) = v'(\sigma_1)$. If we show that, for some $\epsilon > 0$, $v(t) \equiv v'(t)$, $0 \leq t < \sigma_1 + \epsilon$ then the proof of the uniqueness is easily completed.

Let $u(x, t)$, $u'(x, t)$ be the solutions of (1.1)–(1.5) corresponding to $v(t)$, $v'(t)$ respectively. If we apply Green's formula as in §4 we obtain integral equations for $v(t)$ and $v'(t)$ analogous to (4.8) in the domains $\sigma_1 \leq t < \sigma$ and $\sigma_1 \leq t < \bar{\sigma}$ respectively and with $\varphi(\xi)$ replaced by $u(\xi, \sigma_1)$ and $u'(\xi, \sigma_1)$ respectively. We recall that $u(\xi, \sigma_1) \equiv u'(\xi, \sigma_1)$. Proceeding essentially as before (in the case $\sigma_1 = 0$) we conclude that $v(t) \equiv v'(t)$ for $\sigma_1 \leq t < \sigma_1 + \epsilon$ for some $\epsilon > 0$. We have thus completed the proof of existence and uniqueness of solutions for small intervals of time.

6. Existence and Uniqueness for All Times.

We now apply the proof of §5 step by step. To show that such a process can be carried out so as to obtain a solution $u(x, t)$, $s(t)$ (or $v(t)$) for all times we have to prove the following statement: *There exists $\epsilon > 0$ such that if the continuous solution $v(t)$ of the integral equation (4.8) (or the solution $u(x, t)$, $s(t)$ of (1.1)–(1.5)) exists and is unique for $0 \leq t < t_0$, then it exists and is unique for $0 \leq t < t_0 + \epsilon$.*

We shall first show that $v(t)$ ($0 \leq t < t_0$) is a bounded function. In this proof we shall make use of the fact (proved in §4) that the problem of solving (1.1)–(1.5) is equivalent to the problem of solving (4.8) with $s(t) > 0$ and defined by (4.9), and of the fact established in §3 that $s(t)$ is nondecreasing in t . Accordingly, $s(t) \geq b$ and $v(t) = -\dot{s}(t) \leq 0$.

The method used to prove (4.8) can be used to prove that

$$\begin{aligned}
 v(t) &= 2[u(0, t_0 - \delta) - f(t_0 - \delta)]N(s(t), t; 0, t_0 - \delta) \\
 &\quad + 2 \int_0^{s(t_0 - \delta)} u_\xi(\xi, t_0 - \delta) N(s(t), t; \xi, t_0 - \delta) d\xi \\
 &\quad - 2 \int_{t_0 - \delta}^t f(\tau) N(s(t), t; 0, \tau) d\tau \\
 &\quad + 2 \int_{t_0 - \delta}^t v(\tau) G_x(s(t), t; s(\tau), \tau) d\tau \\
 &= T_1 + T_2 + T_3 + T_4
 \end{aligned}
 \tag{6.1}$$

where $0 < \delta < t_0$, $t < t_0$ and T_i denotes the i^{th} term on the right side of (6.1). (Note that $T_1 = 0$.) Since $v(t) \leq 0$ we only have to find a *lower* bound on $v(t)$. Denoting

$$\psi(t) = \underset{t_0 - \delta < \tau < t}{\text{g.l.b.}} \ v(\tau)
 \tag{6.2}$$

we proceed to evaluate T_4 .

$$\begin{aligned}
 T_4 &= - \int_{t_0 - \delta}^t v(\tau) \frac{s(t) - s(\tau)}{t - \tau} K(s(t), t; s(\tau), \tau) d\tau \\
 &\quad + \int_{t_0 - \delta}^t v(\tau) \frac{s(t) + s(\tau)}{t - \tau} K(s(t), t; -s(\tau), \tau) d\tau = T'_4 + T''_4.
 \end{aligned}
 \tag{6.3}$$

Since $s(t) - s(\tau) \geq 0$ and $v(\tau) \leq 0$ we have

$$T'_4 \geq 0.
 \tag{6.4}$$

Since $s(t) + s(\tau) \geq 2b$ we have

$$\begin{aligned}
 |T''_4| &\leq B_{12} |\psi(t)| \int_{t_0 - \delta}^t \frac{1}{t - \tau} \exp \left\{ -\frac{b^2}{2(t - \tau)} \right\} d\tau \\
 &\leq B_{13} |\psi(t)| \delta \leq \frac{1}{2} |\psi(t)|
 \end{aligned}
 \tag{6.5}$$

if

$$\delta \leq \frac{1}{2B_{13}}.
 \tag{6.6}$$

We now fix δ so that it satisfies (6.6). There exists a constant B' (depending on δ , but not on t) such that

$$|T_1 + T_2 + T_3| \leq B' \quad \text{for all } t, \quad t_0 - \delta \leq t < t_0.
 \tag{6.7}$$

Combining (6.4), (6.5), (6.7) with (6.1) we conclude that

$$v(t) \geq \frac{1}{2}\psi(t) - B' \quad (t_0 - \delta \leq t < t_0).
 \tag{6.8}$$

Taking the g.l.b. of both sides of (6.8) when $t_0 - \delta \leq t \leq t'$ we easily get

$$\psi(t') > -2B'$$

and the boundedness of $v(t)$ ($t_0 - \delta \leq t < t_0$) follows.

We now differentiate (4.2) with respect to x , and make use of Lemma 1, (2.12), the boundedness of $v(t)$ and (4.6), (4.7). We then find that $u_x(x, t)$ is a bounded function for $0 \leq x \leq s(t)$, $t_0 - \delta \leq t < t_0$.

From the inequalities (5.8), (5.9), (5.36), (5.37) which restrict the length σ of the time for which a solution was proved to exist it is clear that if $|u_x(x, 0)| = |\dot{\phi}(x)| \leq A$, $b \geq b_0 > 0$ then σ depends only on A , b_0 and the function $f(t)$. Thus if we proceed with the method of §5 but start from $t = t_0 - \eta$ upward (instead of $t = 0$) then the inequalities restricting σ will be *independent* of η since we have proved that $u_x(x, t)$ is bounded on the set $0 \leq x \leq s(t)$, $t_0 - \delta \leq t < t_0$ for any fixed $0 < \delta < t_0$. Thus if we start with η sufficiently small we get a continuous solution $\bar{v}(t)$ for $t_0 - \eta \leq t < t_0 + \epsilon$ for some $\epsilon > 0$. This solution coincides with $v(t)$ for $t_0 - \eta < t < t_0$, since we assumed uniqueness for $0 \leq t < t_0$.

To prove that there exists only one solution $v(t)$ for $0 \leq t < t_0 + \epsilon$ we can use a reasoning similar to the one used in 5.3. Details are omitted.

7. Generalizations.

7.1. Note on the differentiability of solutions of (1.1). In §1 we defined a solution of (1.1)–(1.5) as a pair of functions $u(x, t)$, $s(t)$ which satisfy the equations (1.1)–(1.5) and certain differentiability properties. Among these differentiability conditions only one may seem to be unnatural, namely, that $u_x(x, t)$ is continuous on $x = 0$, $t > 0$. We shall now prove that this condition is superfluous, that is, *if $f(t)$ is continuously differentiable and if $u(x, t)$ is solution of (1.1), (1.2) then $u_x(x, t)$ is continuous near and up to $x = 0$.*

Introducing Green's function

$$(7.1) \quad M(x, t; \xi, \tau) = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(x+4k\alpha-\xi)^2}{4(t-\tau)} \right\} \\ - \sum_{k=-\infty}^{\infty} \frac{1}{2\pi^{\frac{1}{2}}(t-\tau)^{\frac{1}{2}}} \exp \left\{ -\frac{(-x+4k\alpha-\xi)^2}{4(t-\tau)} \right\}$$

in the rectangle $0 \leq x \leq 2\alpha$, $0 < \beta \leq t \leq \gamma$, we have the following representation formula for $u(x, t)$:

$$(7.2) \quad u(x, t) = \int_0^{2\alpha} u(\xi, \beta) M(x, t; \xi, \beta) d\xi + \int_{\beta}^t f(\tau) \left[\frac{\partial}{\partial \xi} M(x, t; \xi, \tau) \right]_{\xi=0} d\tau \\ - \int_{\beta}^t u(2\alpha, \tau) \left[\frac{\partial}{\partial \xi} M(x, t; \xi, \tau) \right]_{\xi=2\alpha} d\tau.$$

The proof of (7.2) follows by noting that the right side of (7.2) is a solution of the heat equation with the same boundary values as $u(x, t)$. We now differentiate

(7.2) with respect to x , perform in the second integral integration by parts (compare with (4.6)) and then let $x \rightarrow 0$; the assertion then easily follows.

7.2. General parabolic equations. Theorem 1 can be generalized to parabolic equations

$$(7.3) \quad a(x, t)u_{xx} + b(x, t)u_x + c(x, t)u = u_t$$

provided $c(x, t) \leq 0$ and the functions

$$\frac{\partial a}{\partial x}, \quad \frac{\partial^2 a}{\partial x^2}, \quad \frac{\partial a}{\partial t}, \quad b, \quad \frac{\partial b}{\partial x}, \quad c$$

are Hölder continuous for $0 \leq x < \infty, t \geq 0$.

Indeed, we have only to construct Green's function in the domain $x > 0, t > 0$ and then proceed as in the special case of the heat equation. To construct Green's function we apply the method of [13]. We first extend the coefficients of (7.3) into the whole strip $t \geq 0$ by defining

$$a(-x) = a(x), \quad c(-x) = c(x) \quad \text{for } x > 0, \quad b(-x) = -b(x) \quad \text{for } x > 0.$$

Then we can construct a fundamental solution $\Gamma(x, t; \xi, \tau)$ (see [9]) which will be continuous for $x \neq \xi, 0 \leq \xi < \infty, -\infty < x < \infty$ and continuously differentiable with respect to x and ξ for $x \neq \xi, 0 < \xi < \infty, 0 < x < \infty$ and $0 < \xi < \infty, -\infty < x < 0$. We take as Green's function the function

$$\Gamma(x, t; \xi, \tau) - \Gamma(-x, t; \xi, \tau).$$

7.3. A general Stefan problem. We shall consider the case where the temperature of the ice is not constantly zero and is kept under control at $x = c > b$. We then have the following system:

$$(7.4) \quad \frac{\partial^2 u_1}{\partial x^2} = a_1^2 \frac{\partial u_1}{\partial t} \quad \text{for } 0 < x < s(t), \quad t > 0 \quad (a_1 \neq 0),$$

$$(7.5) \quad \frac{\partial^2 u_2}{\partial x^2} = a_2^2 \frac{\partial u_2}{\partial t} \quad (a_2^2 \neq a_1^2) \quad \text{for } s(t) < x < c \quad (a_2 \neq 0),$$

$$(7.6) \quad u_1(0, t) = f_1(t), \quad f_1(t) \geq 0 \quad \text{for } t > 0,$$

$$(7.7) \quad u_1(x, 0) = \varphi_1(x), \quad \varphi_1(x) \geq 0 \quad \text{for } 0 < x \leq b, \quad \varphi_1(b) = 0 \quad (b > 0),$$

$$(7.8) \quad u_2(x, 0) = \varphi_2(x), \quad \varphi_2(x) \leq 0 \quad \text{for } b \leq x < c, \quad \varphi_2(b) = 0 \quad (c > b),$$

$$(7.9) \quad u_2(c, t) = f_2(t), \quad f_2(t) \leq 0 \quad \text{for } t > 0,$$

$$(7.10) \quad u_1(s(t), t) = u_2(s(t), t) = 0 \quad \text{for } t \geq 0, \quad s(0) = b,$$

$$(7.11) \quad \dot{s}(t) = -k_1 \frac{\partial u_1(s(t), t)}{\partial x} + k_2 \frac{\partial u_2(s(t), t)}{\partial x} \quad (k_1 > 0, k_2 > 0).$$

With the aid of Green's functions G_1 for (7.4) (in the domain $x > 0, t > 0$)

and G_2 for (7.5) (in the domain $x < c$, $t > 0$) we can reduce the problem (7.4)–(7.11) to the problem of solving a system of two nonlinear integral equations which we symbolically write in the form

$$(7.12) \quad v_i(t) = I_i(s(t), v_i(t)) \quad \text{and} \quad 0 \leq \tau \leq t \quad (i = 1, 2),$$

where $s(t)$ is defined by

$$(7.13) \quad s(t) = -k_1 \int_0^t v_1(\tau) d\tau + k_2 \int_0^t v_2(\tau) d\tau + b.$$

Using the method of §5 we then can prove that the system (7.4)–(7.11) has a solution $u_1(x, t)$, $u_2(x, t)$, $s(t)$ for small values of t . The curve $x = s(t)$ does not increase monotonically in t .

The previous method of proving Theorem 1 can also be modified to solve the system (1.1)–(1.5) with (1.2) replaced by

$$(1.2') \quad u_x(0, t) = f(t), \quad f(t) \leq 0 \quad \text{for} \quad t > 0$$

and to solve the system (7.4)–(7.11) with (7.6) replaced by

$$(7.6') \quad \frac{\partial u_1(0, t)}{\partial x} = f_1(t), \quad f_1(t) \leq 0 \quad \text{for} \quad t > 0$$

and/or with (7.9) replaced by

$$(7.9') \quad \frac{\partial u_2(c, t)}{\partial x} = f_2(t), \quad f_2(t) \geq 0 \quad \text{for} \quad t > 0.$$

Finally, we may consider also the case $c = \infty$ and then replace (7.9) or (7.9') by the condition that $u(x, t)$ and $u_x(x, t)$ remain bounded as $x \rightarrow \infty$, $0 \leq t \leq \sigma$, for any $0 < \sigma < \infty$.

7.4. Three-dimensional problems. The considerations of this paper can be generalized to the case where the boundary conditions (1.5) or (7.11) are replaced by various other conditions. Thus one can show that Theorem 1 remains true if (1.5) is replaced by

$$(1.5') \quad u_x(s(t), t) = -s(t)\dot{s}(t) \quad \text{for} \quad t > 0.$$

As an application consider the following 3-dimensional Stefan problem:

$$(7.14) \quad \Delta c(x, t) = c_t(x, t) \quad \text{for} \quad a < x < s(t), \quad t > 0 \quad \text{and} \quad a > 0,$$

$$(7.15) \quad c(a, t) = \tilde{f}(t) \quad \text{where} \quad \tilde{f}(t) \geq 0 \quad \text{and} \quad t \geq 0,$$

$$(7.16) \quad c(x, 0) = \tilde{\varphi}(x) \quad \text{where} \quad \tilde{\varphi}(x) \geq 0, \quad a < x \leq b, \quad \text{and} \quad \tilde{\varphi}(b) = 0, \quad b > a,$$

$$(7.17) \quad c(s(t), t) = 0 \quad \text{for} \quad t \geq 0 \quad \text{and} \quad s(0) = b,$$

$$(7.18) \quad c_x(s(t), t) = -\dot{s}(t) \quad \text{for} \quad t > 0$$

where x is the 3-dimensional distance from the origin and $\Delta = \partial^2/\partial x^2 + 2x^{-1} \partial/\partial x$.

Substituting $u(x, t) = xc(x, t)$ we find that $u(x, t)$, $s(t)$ satisfy the equations (1.5') and (1.1)–(1.4) with $x = 0$ replaced by $x = a$, $f(t)$ replaced by $a\tilde{f}(t)$ and $\varphi(x)$ replaced by $x\varphi(x)$. Using the remark made previously in this section one concludes that the system (7.14)–(7.18) has a unique solution $c(x, t)$, $s(t)$ for all $t < \infty$ and that $s(t)$ is a nondecreasing function of t .

7.5. Some open questions. From the contractive character of the transformation T which appears in the proof of Theorem 1 it follows that the solution $v(t)$ of (4.8) (and hence the solution $u(x, t)$, $s(t)$ of (1.1)–(1.5)) can be calculated by successive approximations for a small range of time. Using this remark one can easily show that the solution varies continuously with the data f , φ . We raise the following question: If $f' \geq f$, $\varphi' \geq \varphi$ and s' , s are the ice-water curves for the corresponding problems, is $s'(t) \geq s(t)$? In a few particular cases the answer is affirmative. A similar question can be asked for other Stefan problems.

The case $b = 0$ was excluded from our considerations. In that case RUBINSTEIN [23] proved existence for small intervals of time under some additional restrictions on f . It remains to prove existence without these restrictions. Furthermore, the question of uniqueness is still open.

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