*An Existence and Uniqueness Theorem for a Nonlinear Stefan Problem**

W. T. KYNER

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1. Introduction. Stefan problems are associated with the physical processes of change of state, *e.g.*, melting of ice, recrystallization of metals, evaporation and condensation, flow through porous media. These processes and their corresponding parabolic boundary value problems are characterized by the presence of a free boundary, *e.g.* the surface of the melting ice, which is not known in advance, but is to be determined as part of the problem.

J. DOUGLAS has recently published a uniqueness proof for a typical Stefan problem [1]. The original purpose of the present paper was an existence proof for DOUGLAS' problem. However, after slight modifications, our original proof gives both uniqueness and existence of the solution of a Stefan problem in which the density and thermal conductivity (see below) can vary with time, position, and temperature. This problem specializes to that treated by DOUGLAS.

EVANS, DATZEFF, SESTINI, and MIRANKER [2], [3], [4], [10] have proved existence and uniqueness of the solutions of Stefan problems for the classical heat equation. The theorem given here is believed to be the first existence theorem for a Stefan problem referring to a nonlinear equation.

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2. The Physical Problem. We will begin with the statement of a physical problem in the recrystallization of metals which can serve as the basis for the mathematical model.

If heat is uniformly applied over the face of a semi-infinite slab of iron which is at a temperature of 1643°F, the crystals of the metal will first change form before an increase in temperature will occur. At the interface between the new and the old crystals (see Figure 1), an equation of heat balance is satisfied,

$$(2.1) \quad \rho H \Delta x = -p u_x \Delta t,$$

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where

- ρ = density of the recrystallized metal,
- p = thermal conductivity of the recrystallized metal,
- u = temperature (the solution of the heat equation),
- H = latent heat of recrystallization,
- $-pu_x$ = flux of heat.

The equation of heat balance couples the two unknowns, the temperature and the position of the recrystallization front.

FIGURE 1.
The Physical Model

3. The Mathematical Problem. Two functions $u(x, t)$ and $h(t)$ are to be determined by the following requirements:

$$\begin{aligned}
 & \text{(a) } [p(x, t, u, u_x)u_x]_x = f(x, t, u)u_t \quad \text{for } 0 < t, \quad 0 < x < h(t), \\
 & \text{(b) } p_0(t)u_x(0, t) = -a(t), \quad 0 < t, \\
 & \text{(3.1) (c) } u(h(t), t) = 0, \quad 0 < t, \\
 & \text{(d) } h(0) = 0, \\
 & \text{(e) } \frac{dh}{dt}(t) = b - p(h(t), t, 0, u_x(h(t), t))u_x(h(t), t), \quad 0 < t,
 \end{aligned}$$

where b is a non-negative constant and the functions a , f , and p have continuous second derivatives. Furthermore,

- (f) $0 < a_0 \leq a(t)/p_0(t) \equiv b(t) \leq a^0$,
- (g) $0 < f_0 \leq f(x, t, u)$,
- (h) $0 < p_1 \leq p(x, t, u, u_x) = p_0(t)$ plus terms which vanish at $x = 0$.

It is convenient to require that

$$(i) \quad p_x \geq 0, \quad p_u \leq 0, \quad p_x \equiv p_{u_x} \leq 0.$$

Since it will be shown below that u_x is non-positive, these inequalities imply that the differential equation (3.1,a) can be written as a linear differential equation with non-negative coefficients, *i.e.*,

$$(3.2) \quad (p + p_x u_x) u_{xx} + (p_x + p_u u_x) u_x = f u_t.$$

The problem studied by DOUGLAS can be obtained from (3.1) by setting

$$a(t) = a_0, \quad p(x, t, u, u_x) = p_1, \quad f(x, t, u) = f_0.$$

In the recrystallization problem, f and p are constant, b is zero, and the variable u corresponds to the deviation from the critical temperature, 1643°F.

4. The Integral Equation. In general, it is easier to solve an ordinary boundary value problem than a free boundary problem. If the boundary curve $h(t)$ is given, and if the free boundary condition (3.1,e) is not imposed, we have an *almost* classical boundary value problem. We say *almost*, since the sharp corner of the domain in the (x, t) plane introduces complications. This is discussed below.

However, let us assume that we can solve this (reduced) boundary value problem if the boundary curve is any member of some reasonable class of curves. Then we can try to find some method of selecting a particular curve so that the free boundary condition (3.1, e) is satisfied.

Conditions (3.1,d) and (3.1,e) can be replaced by an equivalent integral equation

$$(4.1) \quad h(t) = bt - \int_0^t p(h(s), s, 0, u_x(h(s), s)) u_x(h(s), s) ds.$$

This equation could be used in a fixed point argument, but such an argument would be delicate, since the line integral of the flux along the boundary is a sensitive functional of the domain.

The integral equation which we use in our fixed point proof is obtained by integrating the differential equation (3.1,a) over the region in the (x, t) plane given by

$$0 \leq t \leq t', \quad 0 \leq x \leq h(t)$$

and using the boundary conditions. In the resulting integral equation, the solution of the reduced problem is integrated with respect to x . It is therefore a less sensitive functional of the domain than (4.1).

In order to derive the required integral equation, introduce the notation

$$(4.2) \quad F(x, t, u) = \int_0^u f(x, t, s) ds, \quad H(x, t, u) = - \int_0^u f_t(x, t, s) ds.$$

The differential equation (3.1,a) can now be written

$$(4.3) \quad [p(x, t, u, u_x)u_x]_x - [F(x, t, u)]_t = H(x, t, u).$$

We integrate (4.3) over the domain

$$(4.4) \quad \int_0^{t'} \int_0^{h(t)} \{ [p(x, t, u(x, t), u_x(x, t))u_x(x, t)]_x - [F(x, t, u(x, t))]_t \} dx dt \\ = \int_0^{t'} \int_0^{h(t)} H(x, t, u(x, t)) dx dt,$$

or

$$(4.5) \quad \int_0^{t'} \int_0^{h(t)} [p(x, t, u, u_x)u_x]_x dx dt - \int_0^{h(t')} \int_{h^{-1}(x)}^{t'} [F(x, t, u)]_t dt dx \\ = \int_0^{t'} \int_0^{h(t)} H(x, t, u) dx dt.$$

In changing the order of integration, we use the monotonicity of the boundary curve and replace $x = h(t)$ by $t = h^{-1}(x)$. This will be justified later.

If we now integrate and use the boundary conditions (3.1,b-e), we get (omitting primes)

$$(4.6) \quad h(t) = c(t) - \int_0^{h(t)} F(x, t, u(x, t)) dx - \int_0^t \int_0^{h(s)} H(x, s, u(x, s)) dx ds,$$

where

$$(4.7) \quad c(t) = \int_0^t [b + a(s)] ds.$$

This integral equation replaces the differential boundary condition (3.1,e). We use it to define a transformation on a class of boundary curves. The unique fixed point of this transformation is the moving boundary curve.

5. The Reduced Problem, I. The function u is the solution of the reduced boundary value problem if

$$(5.1) \quad \begin{aligned} (a) \quad & [p(x, t, u, u_x)u_x]_x = f(x, t, u)u_t \quad \text{for } 0 < x < h(t), \quad 0 < t, \\ (b) \quad & p_0(t)u_x(0, t) = -a(t), \quad 0 < t, \\ (c) \quad & u(h(t), t) = 0, \quad 0 < t, \end{aligned}$$

where $h(t)$ is a *given* continuous monotonic function which vanishes at $t = 0$.

In order to emphasize the dependence of the solution of the reduced problem on the given boundary curve, we write $u = u(x, t; h)$.

The estimates of Lemmas 1 and 2, which are derived in Section 8 below, are similar to those used by DOUGLAS in [1].

Lemma 1. *If $u(x, t; h)$ is a solution of the reduced problem, then there exist numbers a^0 and b^0 , independent of h , such that*

$$(5.2) \quad \begin{aligned} (a) \quad & 0 \leq u(x, t; h) \leq a^0(h(t) - x) \quad \text{for } 0 \leq x \leq h(t), \quad 0 < t, \\ (b) \quad & -b^0 \leq p(x, t, u, u_x)u_x \leq 0. \end{aligned}$$

Lemma 2. *If h and g are two boundary functions, if $u(x, t; h)$ and $u(x, t; g)$ are the corresponding solutions of (5.1) and if*

$$(5.3) \quad j(t) = \min(h(t), g(t)),$$

then

$$(5.4) \quad \begin{aligned} (a) \quad & |u(j(t), t; h) - u(j(t), t; g)| \leq a^0 |h(t) - g(t)|, \quad 0 < t, \\ (b) \quad & |u(x, t; h) - u(x, t; g)| \leq a^0 \max_s |h(s) - g(s)| \\ & \quad \text{for } 0 \leq x \leq j(t), \quad 0 < s \leq t. \end{aligned}$$

In Section 8 below, we prove

Theorem 1. *There exists a unique solution of the reduced problem.*

6. The Existence Proof. Let T be an arbitrary positive number. We will prove that there exists a pair of functions $h^0(t)$, $u(x, t; h^0)$, which satisfies the differential equation and the boundary conditions of (3.1) on the interval $[0, T]$. This will be done by using the integral relation (4.6) to define a mapping of a suitably chosen set of boundary curves into itself. We will prove that the mapping is continuous on this set. Using a standard theorem of functional analysis, we conclude that there is at least one boundary curve which is invariant under the transformation. Finally, by a separate argument, we prove that this boundary curve is unique and that it is the curve $x = h^0(t)$.

The equation

$$(6.1) \quad \begin{aligned} g(t) = c(t) - \int_0^{h(t)} F(x, t, u(x, t; h)) \, dx \\ - \int_0^t \int_0^{h(s)} H(x, s, u(x, s; h)) \, dx \, ds, \end{aligned}$$

defines a mapping R of the function h onto the function g , i.e.,

$$(6.2) \quad g = R(h).$$

R is defined on the set of real-valued, continuous, monotonic functions which are defined on the interval $[0, T]$ and which vanish at $t = 0$. Actually, we only need differentiable functions whose derivatives are not less than the constant b and not greater than the constant $b + b^0$.

This restriction is natural to our problem, since if h is any differentiable boundary curve and g its image under the mapping R , then

$$(6.3) \quad \frac{dg}{dt}(t) = b + a(t) - \frac{dh}{dt} F(h(t), t, u(h(t), t; h)) \\ - \int_0^{h(t)} \{ [F(x, t, u(x, t; h))]_t + H(x, t, u(x, t; h)) \} dx.$$

The function u and hence also the function F vanish at $x = h(t)$. The integrand of equation (6.3) is equal to $[pu_x]_x$. Integrating and using the left boundary condition (5.1,b), we get

$$(6.4) \quad \frac{dg}{dt} = b - p(h(t), t, 0, u_x)u_x(h(t), t; h).$$

Since $-pu_x$ is greater than zero, but not greater than b^0 ,

$$(6.5) \quad b \leq \frac{dg}{dt} \leq b + b^0.$$

The function g is a differentiable, monotonic function which vanishes at the origin, and it can therefore serve as a boundary curve for the reduced problem.

We have proved

Lemma 3. *If S is the set of continuously differentiable functions defined on $[0, T]$, which vanish at the origin and whose derivatives satisfy (6.5), and if R is the transformation given by equation (6.1), then R is defined on S and maps S into itself.*

The inequality (6.5) implies that the set S is convex and equi-continuous. S is convex since, if h and g are two differentiable functions whose derivatives satisfy (6.5), the derivative of $rh + (1 - r)g$ also satisfies (6.5), where r is any number between zero and one. S is equi-continuous, since there is a uniform bound on the derivatives of its elements.

The set S can be considered to be a subset of the real Banach space of continuous functions defined on $[0, T]$ under the uniform norm

$$(6.6) \quad ||h|| = \max_t |h(t)|, \quad 0 \leq t \leq T.$$

$\bar{S}$, the closure of S , is also convex. The functions in $\bar{S}$ are monotonic, uniformly Lipschitzian, and uniformly bounded. By ASCOLI's theorem [6] every infinite subset of $\bar{S}$ has an accumulation point in $\bar{S}$, i.e., $\bar{S}$ is compact.

We have shown that R maps S into itself. We now prove

Lemma 4. *R is continuous on $\bar{S}$, the closure of S .*

Proof. We must show that if g is in $\bar{S}$ and if $\{g_n\}$ is a sequence of elements of S such that

$$(6.7) \quad \lim_{n \rightarrow \infty} ||g - g_n|| = 0,$$

then

$$(6.8) \quad \lim_{n \rightarrow \infty} ||R(g) - R(g_n)|| = 0.$$

The functions $u(x, t; g)$ and $u(x, t; g_n)$ have, in general, different domains of definition in the (x, t) plane. We define

$$(6.9) \quad \begin{aligned} j_n(t) &= \min (g_n(t), g(t)), \\ k_n(t) &= \max (g_n(t), g(t)). \end{aligned}$$

Clearly, j_n bounds the common domain of $u(x, t; g)$ and $u(x, t; g_n)$. In this region we use the estimates (5.4,b) and (5.2,a) to get

$$(6.10) \quad \begin{aligned} (a) \quad &|u(x, t; g) - u(x, t; g_n)| \leq a^0 ||g - g_n||, \quad 0 \leq x \leq j_n, \quad 0 < t, \\ (b) \quad &|u(x, t; g) - u(x, t; g_n)| \leq a^0(k_n(t) - x), \quad j_n \leq x \leq k_n, \quad 0 < t. \end{aligned}$$

We have adopted the convention that a solution function is zero outside its original domain of definition, *e.g.*, $u(x, t; g) = 0, x \geq g(t)$.

Let

$$(6.11) \quad \begin{aligned} \Delta F_n &= |F(x, t, u(x, t; g)) - F(x, t, u(x, t; g_n))|, \\ \Delta H_n &= |H(x, t, u(x, t; g)) - H(x, t, u(x, t; g_n))|. \end{aligned}$$

We have

$$(6.12) \quad \begin{aligned} |R(g)(t) - R(g_n)(t)| &\leq \int_0^{j_n(t)} \Delta F_n dx + \int_{j_n(t)}^{k_n(t)} \Delta F_n dx \\ &\quad + \int_0^t \left\{ \int_0^{j_n(s)} \Delta H_n dx + \int_{j_n(s)}^{k_n(s)} \Delta H_n dx \right\} ds. \end{aligned}$$

The functions F_u and H_u are both bounded by M (say). By the mean value theorem,

$$(6.13) \quad \begin{aligned} \Delta F_n(x, t) &\leq M |u(x, t; g) - u(x, t; g_n)|, \\ \Delta H_n(x, t) &\leq M |u(x, t; g) - u(x, t; g_n)|. \end{aligned}$$

We now use the inequalities (6.11)

$$(6.14) \quad \max_t |R(g)(t) - R(g_n)(t)| \leq (1 + T)Ma^0 [||j_n|| ||g - g_n|| + \tfrac{1}{2} ||g - g_n||^2],$$

and let n approach infinity. Hence we conclude (see (6.9)) that R is continuous on $\bar{S}$. This completes the proof of Lemma 4.

Corollary: R maps $\bar{S}$ into itself.

Proof. Let g be any element of $\bar{S}$ and $\{g_n\}$ a sequence of elements of S having g as its limit. Since R is continuous on $\bar{S}$, $\{R(g_n)\}$ approaches $R(g)$ in the limit. $\bar{S}$ is closed, hence $R(g)$ is an element of $\bar{S}$.

We have shown that $\bar{S}$ is a compact, convex subset of a Banach space and that R is a continuous mapping of $\bar{S}$ into itself. By SCHAUDER's theorem [7], there must be at least one element of $\bar{S}$ which is left invariant under R , that is, $h^0 = R(h^0)$, for some h^0 .

The function h^0 is therefore a solution of the integral equation (4.6) and $u(x, t; h^0)$ is the corresponding solution of the reduced problem. If h^0 is differentiable, the differential free boundary condition (3.1,e) is satisfied and we have a solution of the free boundary problem (3.1). We now prove that h^0 is, in fact, differentiable.

Lemma 5. *If h is in $\bar{S}$ and $h = R(h)$, then h is differentiable, i.e., h is in S .*

Proof.

$$\begin{aligned}
 h(t+k) - h(t) &= \int_t^{t+k} \{[b + a(s)] - H(x, s, u(x, s))\} dx \, ds \\
 (6.15) \qquad &- \int_0^{h(t)} [F(x, t+k, u(x, t+k)) - F(x, t, u(x, t))] dx \\
 &- \int_{h(t)}^{h(t+k)} F(x, t+k, u(x, t+k)) dx.
 \end{aligned}$$

Using the law of the mean and the differential equation satisfied by u , we get

$$(6.16) \quad F(x, t+k, u(x, t+k)) - F(x, t, u(x, t)) = [F]_t k = \{[pu_x]_x - H\}k,$$

where the right side of (6.16) is evaluated at t^* , $t < t^* < t+k$. We now integrate (6.16) with respect to x , use the boundary condition (5.1,b), apply the law of the mean to the two other integrals of (6.15), and solve for the difference quotient. We now have

$$(6.17) \quad \frac{h(t+k) - h(t)}{k} \doteq \frac{b - p(h(t), t^*, 0, u_x)u_x(h(t), t^*)}{1 + F(x_0, t+k, u(x_0, t+k))},$$

omitting terms which go to zero as k approaches zero. In (6.17), $h(t) < x_0 < h(t+k)$. Since all of the functions appearing in (6.17) are continuous, the limit as k approaches zero exists, and we obtain the moving boundary conditions

$$(6.18) \quad \frac{dh}{dt} = b - pu_x.$$

This completes the proof of Lemma 5.

We have proved

Theorem 2. *There is at least one solution of the free boundary problem.*

7. The Uniqueness Proof. The proof of the following theorem is similar to the proof published by DOUGLAS in [1].

Theorem 3. *There is at most one solution of the free boundary problem.*

Proof. Let h and g be two elements of S such that

$$(7.1) \quad h = R(h), \quad g = R(g).$$

We will show that $h = g$.

If we duplicate the proof of Lemma 4, we get the analogue of inequality (6.14), i.e.,

$$(7.2) \quad E(t) \leq (1 + T)Ma^0[j(t) + \tfrac{1}{2}E(t)]E(t),$$

where

$$(7.3) \quad E(t) = \max_s |h(s) - g(s)|, \quad 0 \leq s \leq t, \\ j(t) = \min (h(t), g(t)).$$

Both $j(t)$ and $E(t)$ go to zero with t . We can take t^0 such that $0 < t^0$, and

$$(7.4) \quad (1 + T)Ma^0[j(t) + \tfrac{1}{2}E(t)] < 1 \quad \text{if} \quad 0 \leq t < t^0.$$

This implies that $h = g$ on the interval $[0, t^0]$.

Now let

$$(7.5) \quad T^0 = \sup \{t \text{ such that } h(s) = g(s), 0 \leq s \leq t\}.$$

If T^0 is infinite, then $h = g$. If T^0 is finite, then, for all $r > 0$,

$$(7.6) \quad 0 < A(r) = \max_t |h(t) - g(t)|, \quad T^0 \leq t \leq T^0 + r.$$

We will show that this is impossible, and therefore T^0 must be infinite.

As before, we have the inequality

$$(7.7) \quad 0 < A(r) \leq M(1 + T) [\max_t \int_0^{j(t)} |\Delta u(x, t)| dx + \tfrac{1}{2}a^0 A(r)^2],$$

where

$$(7.8) \quad T^0 \leq t \leq T^0 + r, \quad \Delta u(x, t) = u(x, t; h) - u(x, t; g).$$

The integral in (7.7) can be written

$$(7.9) \quad \int_0^{j(t)} |\Delta u(x, t)| dx = \int_0^m |\Delta u(x, t)| dx + \int_m^{j(t)} |\Delta u(x, t)| dx,$$

where

$$(7.10) \quad m = h(T^0) = g(T^0).$$

Clearly

$$(7.11) \quad \max_t \int_m^{j(t)} |\Delta u(x, t)| dx \leq a^0(j(T^0 + r) - m)A(r), \quad T^0 \leq t \leq T^0 + r.$$

By assumption, $A(r) > 0$, so we can divide the inequality (7.7) by $A(r)$ and obtain by (7.11)

$$(7.12) \quad 1 \leq M(1 + T) \left[\max_t \int_0^m \left| \frac{\Delta u(x, t)}{A(r)} \right| dx + a^0(j(T^0 + r) + \tfrac{1}{2}A(r)) \right].$$

We obtain a contradiction by proving

Lemma 7.

$$(7.13) \quad \lim_{r \rightarrow 0} \max_t \int_0^m \left| \frac{u(x, t; h) - u(x, t; g)}{A(r)a^0} \right| dx = 0, \quad T^0 \leq t \leq T^0 + r.$$

Proof. Let

$$(7.14) \quad v(x, t) = \frac{u(x, t; h) - u(x, t; g)}{A(r)}.$$

Then

$$(7.15) \quad \begin{aligned} & \text{(a) } v_x(0, t) = 0, \quad T^0 \leq t, \\ & \text{(b) } v(x, 0) = 0, \quad 0 \leq x \leq m, \\ & \text{(c) } |v(j(t), t)| \leq 1, \quad T^0 \leq t, \\ & \text{(d) } A(x, t)v_{xx} + B(x, t)v_x + C(x, t)v - D(x, t)v_t = 0. \end{aligned}$$

Equalities (a) and (b) are obvious, (c) follows from the inequality (5.4,b), and (d) is obtained by taking the difference of the two differential equations satisfied by $u(x, t; h)$ and $u(x, t; g)$ and using the law of the mean, e.g.,

$$(7.16) \quad \begin{aligned} & f(x, t, u(x, t; h))u_t(x, t; h) - f(x, t, u(x, t; g))u_t(x, t; g) \\ & = f_u(x, t, u_1(x, t))u_{1t}(x, t)v(x, t) + f(x, t, u_1(x, t))v_t(x, t), \end{aligned}$$

where u_1 is a function between the two given solution functions.

Note that all of the differential equations which we consider can be written as linear differential equations in the dependent variable, even though the dependent variable appears in the coefficients. Therefore the maximum principle [9] for linear parabolic equations can be applied.

The maximum principle. *If v is a solution of a linear parabolic equation*

$$(7.17) \quad A(x, t)v_{xx} + B(x, t)v_x + C(x, t)v - D(x, t)v_t + F(x, t) = 0,$$

where the coefficients are bounded and continuous, A and D are strictly positive, and C is non-positive, then

- (a) *F non-positive implies that v attains its positive maximum on the boundary of the domain,*
- (b) *F non-negative implies that v attains its negative minimum on the boundary of the domain.*

Because C , the coefficient of v in equation (7.15,d), is not restricted to be non-positive, we cannot apply the maximum principle directly. However, if

λ is sufficiently large, the function

$$(7.18) \quad y(x, t) = e^{-\lambda t} v(x, t)$$

is a solution of a homogeneous differential equation which is subject to the maximum principle, namely,

$$(7.19) \quad A(x, t)y_{xx} + B(x, t)y_x + (C(x, t) - \lambda D(x, t))y - D(x, t)y_t = 0.$$

Hence y attains its positive maximum and negative minimum on the boundary, say at $(s, j(s))$ and $(s', j(s'))$, respectively. Then, from the fact that $|v|$ is less than unity on the boundary, we conclude that

$$(7.20) \quad -e^{-\lambda(s'-t)} \leq v(x, t) \leq e^{-\lambda(s-t)}.$$

Or, if $r_0 = \min(s, s')$,

$$(7.21) \quad |v(x, t)| \leq 1, \quad 0 \leq x \leq j(t), \quad T^0 \leq t \leq T^0 + r_0.$$

Now let w be the solution of the following problem:

$$(7.22) \quad \begin{aligned} & \text{(a) } w_x(0, t) = 0, \quad T^0 < t, \\ & \text{(b) } w(x, 0) = 0, \quad 0 < x < m, \\ & \text{(c) } w(m, t) = 1, \quad T^0 < t, \\ & \text{(d) } A(x, t)w_{xx} + B(x, t)w_x + C(x, t)w - D(x, t)w_t = 0 \\ & \quad \text{for } 0 < x < m, \quad T^0 < t. \end{aligned}$$

The functions

$$(7.23) \quad z = (w \pm v)e^{-\lambda t}$$

are non-negative on the boundary and, by the maximum principle, in the interior of the domain. This implies that $w \pm v$ are non-negative, i.e.,

$$(7.24) \quad |v(x, t)| \leq w(x, t) \quad 0 < x < m, \quad T^0 < t < T^0 + r_0.$$

Therefore

$$(7.25) \quad \max_t \int_0^m |v(x, t)| dx \leq \max_t \int_0^m w(x, t) dx, \quad T^0 < t < T^0 + r_0.$$

For fixed x , $w(x, t)$ approaches zero as t approaches T^0 . By a theorem due to ARZELÀ [5],

$$(7.26) \quad \lim_{t \rightarrow 0} \int_0^m w(x, t) dx = \int_0^m \lim_{t \rightarrow 0} w(x, t) dx = 0.$$

We conclude that

$$(7.27) \quad \lim_{r \rightarrow 0} \max_t \int_0^m \frac{|u(x, t; h) - u(x, t; g)|}{A(r)} dx = 0, \quad T^0 < t < T^0 + r.$$

Returning now to inequality (7.12), we see that each term inside the brackets goes to zero with r . Hence $A(r)$ must be zero and T^0 must be infinite. This concludes the proof of Theorem 3.

8. The Reduced Problem, II. In this final section, we derive the estimates which were used in the above existence proof and prove that the reduced problem has a unique solution.

Lemma 1. *If $u(x, t; h)$ is a solution of the reduced problem, then there exist numbers a^0 and b^0 , independent of h , such that*

$$(8.1) \quad \begin{aligned} (a) \quad & 0 \leq u(x, t; h) \leq a^0(h(t) - x) \quad \text{for } 0 \leq x \leq h(t), \quad 0 < t < T, \\ (b) \quad & -b^0 \leq p(x, t, u, u_x)u_x < 0. \end{aligned}$$

Proof. The function u satisfies

$$(8.2) \quad (p(x, t, u, u_x)u_x)_x - f(x, t, u)u_t = 0.$$

By the maximum principle, if u is less than zero anywhere, it must have a negative minimum on $x = 0$. But, since u_x is negative on $x = 0$, the function u must decrease as x increases, and therefore cannot have a negative minimum, *i.e.*

$$(8.3) \quad 0 \leq u(x, t).$$

Since u equals zero on $x = h(t)$, we can use a similar argument to conclude that

$$(8.4) \quad u_x(h(t), t) \leq 0, \quad 0 < t < T.$$

The derivative u_x satisfies a homogeneous parabolic differential equation which is obtained from (8.2) by dividing by f and differentiating with respect to x . Using the maximum principle and the non-positivity of u_x on the boundary, we conclude that

$$(8.5) \quad u_x(x, t) \leq 0, \quad 0 \leq x \leq h(t), \quad 0 < t < T.$$

At the left boundary

$$(8.6) \quad u_x(0, t) = -a(t)p_0(t) \leq -a^0,$$

where a^0 is a positive constant.

We now show that the function $u^0(x, t)$, which satisfies the differential equation (8.2) subject to the boundary conditions

$$(8.7) \quad \begin{aligned} u^0(h(t), t) &= 0, \\ u_x^0(0, t) &= -a^0, \end{aligned} \quad 0 < t < T,$$

dominates $u(x, t)$, *i.e.*,

$$(8.8) \quad \begin{aligned} 0 &\leq u^0(x, t) - u(x, t), \quad 0 \leq x \leq h(t), \quad 0 < t, \\ (u_x^0(h(t), t) - u_x(h(t), t)) &\leq 0, \quad 0 < t. \end{aligned}$$

To prove (8.8), we duplicate the derivation of (8.3) and (8.4), after checking that the maximum principle can be applied to the differential equation satisfied by $(u^0 - u)e^{-\lambda t}$.

We will now obtain a bound for u_x^0 which, by (8.8), will serve as a bound for u_x . Let $v = u^0 + a^0 x$. Then

$$(8.9) \quad (p - a^0 p_x)v_{xx} + (p_x + p_u v_x - 2a^0 p_u + p_x v_{xx})v_x - f v_t = a^0(p_x - p_u a^0).$$

Since $p_x \geq 0$, $p_u \leq 0$, and $p_x \leq 0$, the coefficients in (8.9) have the proper sign, and the maximum principle applies.

Since v_x is zero on $x = 0$, v must, by the maximum principle, attain its maximum on $x = h(t)$. Furthermore, since v is a monotonic increasing function there, we have

$$(8.10) \quad 0 \leq v(x, t) \leq a^0 h(t), \quad 0 \leq x \leq h(t), \quad 0 < t < T,$$

and

$$(8.11) \quad 0 \leq v_x(h(t), t) = u_x^0 + a^0, \quad 0 < t < T.$$

By (8.8), u_x^0 is not greater than u_x on $x = h(t)$, hence

$$(8.12) \quad -a^0 \leq u_x \leq 0$$

on $x = 0$ and on $x = h(t)$. Once again we take $y = e^{-\lambda t} u_x$ and choose λ so that the differential equation satisfied by y is subject to the maximum principle. From (8.12), we get

$$(8.13) \quad -a^0 e^T \leq u_x(x, t) \leq 0, \quad 0 < x < h(t), \quad 0 < t < T.$$

Inequality (8.13) gives us now a bound on pu_x , since p is positive and bounded.

Finally, we use our bound on u^0 to obtain the desired bound on u . From (8.10), we have

$$(8.14) \quad u^0(x, t) \leq a^0(h(t) - x).$$

But u^0 dominates u , and u is non-negative, therefore

$$(8.15) \quad 0 \leq u(x, t) \leq a^0(h(t) - x).$$

This completes the proof of Lemma 1.

Lemma 2. *If h and g are two boundary functions, if $u(x, t; h)$ and $u(x, t; g)$ are the corresponding solutions of (5.1), and if*

$$(8.16) \quad j(t) = \min(h(t), g(t)),$$

then

$$(8.17) \quad \begin{aligned} & \text{(a) } |u(j(t), t; h) - u(j(t), t; g)| \leq a^0 |h(t) - g(t)|, \quad 0 < t, \\ & \text{(b) } |u(x, t; h) - u(x, t; g)| \leq a^0 \max_s |h(s) - g(s)| \\ & \quad \text{for } 0 \leq x \leq j(t), \quad 0 \leq s \leq t. \end{aligned}$$

Proof. We have

$$(8.18) \quad 0 \leq u(x, t; h) \leq a^0(h(t) - x), \quad 0 \leq u(x, t; g) \leq a^0(g(t) - x).$$

At $x = j(t)$, we get (8.17,a).

The function $w(x, t) = u(x, t; h) - u(x, t; g)$ satisfies a homogeneous partial differential equation which is subject to the maximum principle. Since $w_x = 0$ on $x = 0$, we can extend it as an even function defined on the domain

$$(8.19) \quad -j(t) \leq x \leq j(t), \quad 0 < t.$$

The function $we^{-\lambda t}$ is bounded on the boundary (see (8.17,a), and therefore in the interior. We redefine a^0 to be the larger of $(a^0, e^{\lambda T} a^0)$. Hence we obtain (8.17,b).

Theorem 1. *There exists a unique solution of the reduced problem.*

Proof. We have

$$(8.20) \quad pu_{xx} + p_x u_x + p_u u_x^2 + p_x u_x u_{xx} - fu_t = 0.$$

Let

$$(8.21) \quad v(x, t) = u(x, t) + b(t)(x - h(t)) + e(T - t), \quad 0 < t \leq T.$$

Then

$$(8.21) \quad (p - bp_x)v_{xx} + (p_x + p_u v_x - 2bp_u + p_x v_{xx})v_x - fv_t = bp_x - p_u b^2 + f(e + b\dot{h} + b(h - x)),$$

where $\dot{h} = dh/dt$. Recalling that p_x is non-positive, and that p , f , and b are strictly positive, we have that, for sufficiently large positive e , the right side of (8.22) is positive, so the maximum principle obtains.

It is sufficient to prove the existence of a function satisfying (8.22) and the following boundary conditions:

$$(8.23) \quad v(h(t), t) = e(T - t), \quad v_x(0, t) = 0.$$

We now extend the domain by reflection so that v is an even function of x . The domain of v is given by

$$(8.24) \quad -h(t) \leq x \leq h(t), \quad 0 < t \leq T.$$

The classical existence theorems [8] do not insure existence of the solution of this problem, since the domain has a sharp corner at $t = 0$. However, we can conclude that, for each positive s , there is a unique function $v(x, t; s)$ satisfying the differential equation (8.21) and assuming the boundary values

$$(8.25) \quad \begin{aligned} v &= e(T - t) \quad \text{on } x = \pm h(t), \quad 0 < s \leq t \leq T, \\ v(x, s; s) &= e(T - s), \quad -h(s) \leq x \leq h(s). \end{aligned}$$

We will show that as s approaches zero, the functions $v(x, t; s)$ converge uniformly to a continuous function $v(x, t)$ which is defined on the domain (8.24). Clearly

$$(8.26) \quad v(\pm h(t), t) = e(T - t).$$

In order to prove convergence, we apply the maximum principle to the equation satisfied by the function $v(x, t; s) - v(x, t; s')$. Suppose that $s' < s$; then, by Lemma 1,

$$(8.27) \quad |v(x, t; s) - v(x, t; s')| = |u(x, t; s) - u(x, t; s')| \leq 2a^0 h(s).$$

The right side of (8.26) goes to zero with s . This implies convergence.

We must now show that $v(x, t)$ is a solution of the differential equation (8.21). This is done by proving that the uniform limit of solutions of a parabolic differential equation, subject to the maximum principle, is also a solution of the differential equation.

Let (z, k) be an interior point of the domain of $v(x, t)$. We take $r > 0$, so that $z \pm r$ and $k \pm r$ are also interior points. $v(x, t; s)$ is the solution of the differential equation which assumes the boundary values

$$(8.27) \quad \begin{aligned} v(z \pm r, t; s) & \text{ on } x = z \pm r, & k - r \leq t \leq k + r, \\ v(x, k - r; s) & \text{ on } t = k - r, & z - r \leq x \leq z + r. \end{aligned}$$

Let $v^0(x, t)$ be the solution of the differential equation which agrees with $v(x, t)$ on the boundary of the r -domain, *i.e.*

$$(8.29) \quad \begin{aligned} v^0(z \pm r, t) &= v(z \pm r, t) \text{ on } x = z \pm r, & k - r \leq t \leq k + r, \\ v^0(x, k - r) &= v(x, k - r) \text{ on } t = k - r, & z - r \leq x \leq z + r. \end{aligned}$$

As s approaches zero, $v(x, t; s)$ approaches $v^0(x, t)$ on the boundary of the r -domain and, by the maximum principle, in the interior. Since $v(x, t)$ is also the limit of $v(x, t; s)$ we have $v(x, t) = v^0(x, t)$, where $v^0(x, t)$ is a solution of the differential equation.

We have proved that $v(x, t)$ is a solution of the differential equation at every interior point of its domain and that it equals $e(T - t)$ on the boundary. It is therefore a solution of the reduced problem, and, by the maximum principle, it is the only one.

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