Isoperimetric Inequalities for the Nonhomogeneous Clamped Rod and Plate

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Introduction. It was conjectured by Lord RAYLEIGH [5, pp. 339–340], and first proved by FABER [1] and KRAHN [4], that of all (homogeneous) membranes of a given area the circle has the lowest principal frequency. This result may be taken as a model for a class of inequalities occurring in mathematical physics which are called *isoperimetric inequalities* by PÓLYA & SZEGÖ [8]. Although the above result for the vibrating membrane obviously has no analogue in the case of a homogeneous vibrating string, an analogous result can be stated for the nonhomogeneous string. Indeed, B. SCHWARZ and the author [7, Theorem 2] proved that among all strings of a given length and equimeasurable density, the least principal frequency occurs in the case that the density is symmetrically decreasing about the midpoint of the string (and the maximum principal frequency occurs when the density is symmetrically increasing). In a recent paper [6], SCHWARZ proved a similar theorem for the nonhomogeneous membrane.

In §1 of this paper we consider the transverse vibrations of a thin cylindrical rod of variable density and constant flexural rigidity which is clamped at both ends, and prove a result analogous to that quoted above for the nonhomogeneous string: *Among all rods of a given length and equimeasurable density, the least (greatest) principal frequency occurs in the case that the density is symmetrically decreasing (increasing) about the midpoint.*

In §2 we consider the transverse vibrations of a thin plate of variable density and constant flexural rigidity extended over a plane domain D and clamped along the boundary C . In the case of constant density, and under the hypothesis that the first eigenfunction does not change sign in D , SZEGÖ [9] proved that the lowest principal frequency among all plates of a given area occurs when C is a circle. Under the same hypothesis concerning the first eigenfunction we prove a similar result for nonhomogeneous plates of equimeasurable density. In order to state this result precisely we must define the symmetrical rearrangements of a continuous function $p(x, y)$ defined over $D \cup C$. Let $\bar{D}$ denote the open disk $0 \leq x^2 + y^2 = r^2 < A/\pi$, where A is the area of D , and let $\bar{C}$ be the boundary ($r = R_0$) of $\bar{D}$. The rearrangement $p^-(x, y) = p^-(r)$ of $p(x, y)$ in symmetrically decreasing order is defined in the closure of $\bar{D}$ as the (unique)

circularly symmetric, continuous, non-increasing function (of r) which is equimeasurable with $p(x, y)$, and whose maximum and minimum values in the closure of $\bar{D}$ coincide with the respective maximum and minimum values of $p(x, y)$ in the closure of D . By saying that $p^-(x, y)$ and $p(x, y)$ are equimeasurable, we mean that the two sets $\{(x, y): p(x, y) > d\}$, $\{(x, y): p^-(x, y) > d\}$ have the same Lebesgue measure for every d . Similarly, the rearrangement $p^+(x, y) = p^+(r)$ of $p(x, y)$ in symmetrically increasing order is defined in the closure of $\bar{D}$ by replacing the word "non-increasing" in the above definition by "non-decreasing". The symmetrical rearrangements of a function of one real variable are defined in a completely analogous way (cf. [2] and [8]).

Now let λ denote the principal frequency of the plate with density function $p(x, y)$ over $D \cup C$, i.e., λ is the least eigenvalue of the differential problem

$$\nabla^4 u(x, y) - \lambda^4 p(x, y) u(x, y) = 0, \quad u(C) = \frac{\partial u(C)}{\partial n} = 0.$$

Similarly, let λ^+ and λ^- denote the principal frequencies of the circular plate $\bar{D} \cup \bar{C}$ with density functions $p^+(x, y)$ and $p^-(x, y)$ respectively. Then, under the hypothesis that the corresponding first eigenfunctions do not change sign,¹ we prove that $\lambda^- \leq \lambda$ for any domain D , and that $\lambda \leq \lambda^+$ if D is itself a disk.

The proof of the inequality $\lambda^- \leq \lambda$ relies on SZEGÖ's proof [9] for the corresponding homogeneous problem with $p(x, y) \equiv 1$, and makes use of another circularly symmetric function introduced by SZEGÖ. However, since the proof in [9] contains a minor error (subsequently corrected by Professor SZEGÖ in a private communication to the author²), we shall repeat in §2 some of the details of [9]. Besides the results of SZEGÖ, one of our main tools in both §§1 and 2 is the following theorem of HARDY, LITTLEWOOD & PÓLYA on the rearrangement of functions (cf. [2, Theorem 378] and [8, p. 153]): Let f, f_1, f_2, g, g_1, g_2 be continuous (real-valued) functions such that f_1 and g_1 are similarly ordered, f_2 and g_2 oppositely ordered, f, f_1, f_2 are equimeasurable, and g, g_1, g_2 are equimeasurable on a Lebesgue measurable set E . Then

$$\int_E f_2 g_2 \leq \int_E f g \leq \int_E f_1 g_1.$$

Two functions f and g are called *similarly ordered* on E if

$$[f(p_1) - f(p_2)] \cdot [g(p_1) - g(p_2)] \geq 0$$

holds for every pair p_1, p_2 of points of E ; f and g are called *oppositely ordered* if f and $-g$ are similarly ordered.

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¹ It is not known whether this hypothesis is valid even in the homogeneous case $p(x, y) \equiv 1$. In this connection, see [8, p. 232].

² See also the note in [10].

1. The clamped rod. If we denote by $p(x)$ the quotient of the density of the rod divided by its constant flexural rigidity, and write λ instead of λ^4 , then our result can be stated as

Theorem 1. *Let $p(x)$ be continuous and positive for $-a \leq x \leq a$, and let $p^-(x)$, $p^+(x)$ denote the rearrangements of $p(x)$ in symmetrically decreasing and increasing order respectively. Let λ , λ^- , λ^+ denote the lowest eigenvalues of the differential problems*

$$(1.1) \quad u^{iv}(x) - \lambda p(x)u(x) = 0, \quad u(\pm a) = u'(\pm a) = 0;$$

$$(1.2) \quad v^{iv}(x) - \lambda^- p^-(x)v(x) = 0, \quad v(\pm a) = v'(\pm a) = 0;$$

$$(1.3) \quad w^{iv}(x) - \lambda^+ p^+(x)w(x) = 0, \quad w(\pm a) = w'(\pm a) = 0.$$

Then $\lambda^- \leq \lambda \leq \lambda^+$.

Proof. The systems (1.1)–(1.3) are completely regular eigenvalue problems, and hence [3, Theorem 4] the first eigenfunctions do not vanish on the open interval $(-a, a)$. Let $u(x)$ be a first eigenfunction of (1.1). Since $u(x)$ satisfies (1.1), $u'(x)$ can vanish at most a finite number of times on $-a \leq x \leq a$, and, moreover, $u(x)$ cannot be constant on any subinterval of $-a \leq x \leq a$.

The result $\lambda^- \leq \lambda$ will be proved by constructing two functions $\bar{u}(x)$ and $u^-(x)$ and proving the inequalities

$$(1.4) \quad \lambda = \frac{\int u'^2 dx}{\int p u^2 dx} \geq \frac{\int \bar{u}'^2 dx}{\int p^-(u^-)^2 dx} \geq \frac{\int \bar{u}'^2 dx}{\int p^- \bar{u}^2 dx} \geq \lambda^-.$$

More precisely, we shall prove the inequalities

$$(1.5) \quad \int u'^2 dx = \int \bar{u}'^2 dx,$$

$$(1.6) \quad 0 \leq u^-(x) \leq \bar{u}(x), \quad -a \leq x \leq a.$$

Now the inequality

$$(1.7) \quad \int p^-(u^-)^2 dx \geq \int p u^2 dx$$

follows from the theorem quoted in the introduction since p^- and p are equimeasurable on $[-a, a]$, as are $(u^-)^2$ and u^2 , while p^- and $(u^-)^2$ are similarly ordered, *i.e.*, both symmetrically decreasing. Here we used the fact that $u(x)$, and hence $u^-(x)$, does not change sign on $[-a, a]$. The first inequality of (1.4) follows from (1.5) and (1.7), while the second inequality sign follows from (1.6) since $p^-(x)$ is positive. The last inequality sign follows from the minimum property of the least eigenvalue of the system (1.2), since we shall prove that $\bar{u}(x)$ is an admissible function for this minimum property, *i.e.*, $\bar{u}(x)$ and $\bar{u}'(x)$

are continuous on $[-a, a]$, $\bar{u}''(x)$ is Lebesgue square integrable on $[-a, a]$, and $\bar{u}$ satisfies the boundary conditions.

We now proceed to the construction of the function $\bar{u}(x)$. With no loss of generality we may assume that the first eigenfunction $u(x)$ of (1.1) satisfies the inequalities $0 < u(x) \leq 1$ on $-a < x < a$. Let $u_1, u_2, \dots, u_n$ be the values of $u(x)$, arranged in increasing order of magnitude, at which $u(x)$ has a relative maximum or minimum on $-a < x < a$, and set $u_0 = 0, u_{n+1} = 1$. Then, on each subinterval $u_i < u < u_{i+1}$, the number $2m$ of roots $x_k = x_k^i(u)$ of the equation $u = u(x)$ is constant, i.e., $m = m(i)$. Moreover, $x_k^i(u)$ is an increasing or decreasing function of u according as k is odd or even. In what follows, we may omit the superscript without danger of confusion. Now define

$$L(u) = [x_2(u) - x_1(u)] + [x_4(u) - x_3(u)] + \dots + [x_{2m}(u) - x_{2m-1}(u)],$$

$$L(0) = 2a, \quad L(1) = 0.$$

The function $L(u)$ is a continuous, monotone decreasing function of u , and

$$L'(u) = - \sum_{k=1}^{2m} |x'_k(u)|$$

is integrable on $0 \leq u \leq 1$. However, $L'(u)$ will have infinite discontinuities at the values u_i , as well as at those values of u corresponding to a horizontal inflection point of the curve $u = u(x)$.

We may write the left side of (1.5) as

$$(1.8) \quad \int_{-a}^a u''^2 dx = \sum_{i=0}^n \int_{u_i}^{u_{i+1}} \left(\sum_{k=1}^{2m} \frac{[x'_k(y)]^2}{|x'_k(y)|^5} \right) dy = \sum_{i=0}^n \int_{u_i}^{u_{i+1}} Q_i(y) dy.$$

The functions $Q_i(y)$ are integrable on $u_i \leq y \leq u_{i+1}$, and can have (infinite) discontinuities only for those values of y for which $L'(y)$ has such discontinuities.

We now define, following SZEGÖ, the function $\bar{u}(x)$ by

$$(1.9) \quad \bar{u}(\pm x) = f(y), \quad 0 \leq x \leq a,$$

where x and y are related by $2x = L(y)$, and the function $f(y)$ is to be determined so that (1.5) is satisfied. From (1.9) we have

$$\bar{u}'(x) = f'(y) \frac{dy}{dx} = \frac{2f'(y)}{L'(y)}, \quad \bar{u}''(x) = \frac{2}{L'(y)} \frac{d}{dy} \left\{ \frac{2f'(y)}{L'(y)} \right\}.$$

Hence, using (1.8), we see that (1.5) is satisfied if $f(y)$ is chosen so that

$$(1.10) \quad 2 \int_0^a \bar{u}''^2 dx = \int_0^1 \frac{4}{|L'(y)|} \left[\frac{d}{dy} \left\{ \frac{2f'(y)}{L'(y)} \right\} \right]^2 dy = \sum_{i=0}^n \int_{u_i}^{u_{i+1}} Q_i(y) dy.$$

Let us write

$$Q(y) = Q_i(y), \quad u_i \leq y \leq u_{i+1}, \quad i = 0, 1, \dots, n,$$

and define y_0 ($0 < y_0 < 1$) by the equation

$$\int_0^{y_0} \{Q(y) \cdot |L'(y)|\}^{\frac{1}{2}} dy = \int_{y_0}^1 \{Q(y) \cdot |L'(y)|\}^{\frac{1}{2}} dy.$$

(The integrability of $\{Q(y) \cdot |L'(y)|\}^{\frac{1}{2}}$ on $[0, 1]$ follows from that of $Q(y)$ and $L'(y)$ by Schwarz's inequality.) We now define

$$g(y) = \begin{cases} \frac{|L'(y)|}{4} \cdot \int_0^y \{Q(t) \cdot |L'(t)|\}^{\frac{1}{2}} dt, & 0 < y \leq y_0, \\ \frac{|L'(y)|}{4} \cdot \int_y^1 \{Q(t) \cdot |L'(t)|\}^{\frac{1}{2}} dt, & y_0 \leq y < 1. \end{cases}$$

Finally, we set

$$(1.11) \quad f(y) = \int_0^y g(r) dr, \quad 0 \leq y \leq 1.$$

One easily verifies that $f(y)$, as defined by (1.11), satisfies the condition (1.10). We now show that $\bar{u}(x)$, as defined by (1.9), is admissible. This function is certainly continuous on $-a \leq x \leq a$, and we have

$$\bar{u}'(x) = \frac{2g(y)}{L'(y)}, \quad 2x = L(y), \quad 0 \leq y \leq 1,$$

so that $\bar{u}'$ is continuous for $0 \leq x \leq a$ (and, indeed, for $-a \leq x \leq a$, since $\bar{u}'(0+) = 0$). The boundary conditions $\bar{u}'(\pm a) = 0$ are also satisfied, and $\bar{u}''$ is Lebesgue square integrable by (1.10).

It remains to prove the inequality (1.6). To prove this, we note that $u^-(x)$ may be defined by the equation

$$u^-(x) = y, \quad 0 \leq x \leq a, \quad 2x = L(y).$$

Hence (1.6) is equivalent to proving that $f(y) \geq y$, $0 \leq y \leq 1$, or

$$(1.12) \quad f'(y) \geq 1, \quad 0 \leq y \leq 1.$$

To prove (1.12) we note first that

$$|L'(y)| \cdot Q_i(y) = \sum_{k=1}^{2m} |x'_k(y)| \cdot \sum_{k=1}^{2m} \frac{|x''_k(y)|^2}{|x'_k(y)|^5} \geq \left\{ \sum_{k=1}^{2m} \frac{|x''_k(y)|}{[x'_k(y)]^2} \right\}^2$$

by the Cauchy-Schwarz inequality. Hence

$$\{Q_i(y) \cdot |L'(y)|\}^{\frac{1}{2}} \geq \sum_{k=1}^{2m} \frac{|x''_k(y)|}{[x'_k(y)]^2}, \quad u_i < y < u_{i+1}.$$

It follows that

$$\int_{u_i}^y \{Q_i(t) \cdot |L'(t)|\}^{\frac{1}{2}} dt \geq \sum_{k=1}^{2m} \int_{u_i}^y \left\{ \frac{d}{dt} |x'_k(t)|^{-1} \right\} dt = \sum_{k=1}^{2m} \left\{ \frac{1}{|x'_k(y)|} - \frac{1}{|x'_k(u_i)|} \right\}.$$

For $i = 0$, this gives (recalling that $u'(\pm a) = 0$)

$$\int_0^y \{Q(t) \cdot |L'(t)|\}^{\frac{1}{2}} dt \geq \sum_{k=1}^{2m(0)} \frac{1}{|x'_k(y)|}, \quad 0 < y < u_1.$$

Now, in successive intervals $u_{i-1} < y < u_i$, $u_i < y < u_{i+1}$, either an $x_k^i(y)$ is a differentiable continuation of an $x_k^{i-1}(y)$ or, for those which are not such a differentiable continuation, we have $(x_k^i(u_i +))' = -\infty$. Thus, for each $i = 0, 1, \dots$, we have

$$(1.13) \quad \int_0^y \{Q(t) \cdot |L'(t)|\}^{\frac{1}{2}} dt \geq \sum_{k=1}^{2m(i)} \frac{1}{|x_k'(y)|}, \quad u_i < y < u_{i+1}.$$

In precisely the same way (beginning with the interval $u_n < y < 1$) we also obtain

$$(1.14) \quad \int_y^1 \{Q(t) \cdot |L'(t)|\}^{\frac{1}{2}} dt \geq \sum_{k=1}^{2m(i)} \frac{1}{|x_k'(y)|}, \quad u_i < y < u_{i+1}.$$

It now follows from (1.11) and the definition of $g(y)$ that

$$f'(y) \geq \frac{1}{4} \sum_{k=1}^{2m} |x_k'(y)| \cdot \sum_{k=1}^{2m} \frac{1}{|x_k'(y)|}, \quad u_i < y < u_{i+1}.$$

Since each sum contains at least two terms, the right side of this inequality is greater than or equal to

$$\frac{1}{4} \cdot \min_{\substack{x > 0 \\ y > 0}} (x + y) \left(\frac{1}{x} + \frac{1}{y} \right) = \frac{1}{4} \min_{z > 0} \left(2 + z + \frac{1}{z} \right) = 1.$$

Thus $f'(y) \geq 1$, completing the proof of the inequality $\lambda^- \leq \lambda$.

The proof of the inequality $\lambda \leq \lambda^+$ is much easier since in this case there is no need to introduce a function corresponding to $\bar{u}(x)$. Indeed, let $w(x)$ be a first eigenfunction of the system (1.3); according to a previous remark we may assume that $w(x) > 0$ on $-a < x < a$. Since $p^+(-x) = p^+(x)$, $w(-x)$ also satisfies (1.3), as does the function $y(x) = w(x) + w(-x)$. We shall prove that $y'(x) < 0$ for $0 < x < a$, so that $y(x)$ is a symmetrically decreasing function.

To this end we note that $y'(0) = y'''(0) = 0$. Hence

$$y'(x) = xy''(0) + \int_0^x \int_0^s y'''(t) dt ds = xy''(0) + \lambda^+ \int_0^x \int_0^s \int_0^t p^+(r)y(r) dr dt ds,$$

since $y(x)$ satisfies (1.3). Since p^+ and y are positive, we must have $y''(0) < 0$. Now $y''(x)$ has only one zero on $0 < x < a$, since $y'''(x)$ is positive on this interval. It follows that $y'(x)$ does not vanish on $0 < x < a$, and hence that $y'(x) < 0$ for $0 < x < a$. The functions $p^+(x)$ and $y^2(x)$ are thus oppositely ordered, and an application of the theorem of HARDY, LITTLEWOOD & PÓLYA now gives

$$\lambda^+ = \frac{\int y''^2 dx}{\int p^+ y^2 dx} \geq \frac{\int y''^2 dx}{\int p y^2 dx} \geq \lambda,$$

the last inequality again following from the minimum property of λ . This completes the proof of Theorem 1.

2. The clamped plate. As with the rod, we shall let $p(x, y)$ denote the quotient of the density by the constant flexural rigidity of the plate. As we indicated in the introduction, our Theorem 2 is proved by merely extending the argument given by SZEGÖ for the case $p(x, y) \equiv 1$. For this reason our notation will follow closely that of SZEGÖ as given in [8, pp. 230–237], and we shall repeat only some of the details given there.

Theorem 2. *Let D be a plane domain bounded by a simple, smooth curve C , and suppose that $p(x, y)$ is positive and continuous in $D \cup C$. Denote by $\bar{D}$ the disk $0 \leq x^2 + y^2 = r^2 < A/\pi$, where A is the area of D , and let $\bar{C}$ be the boundary ($r = R_0$) of this disk. Let $p^-(x, y) = p^-(r)$ be the rearrangement of $p(x, y)$ in symmetrically decreasing order, defined in $\bar{D} \cup \bar{C}$. Let λ denote the least eigenvalue of the system*

$$(2.1) \quad \nabla^4 u(x, y) - \lambda p(x, y) u(x, y) = 0, \quad u(C) = \frac{\partial u(C)}{\partial n} = 0,$$

which we assume has a corresponding first eigenfunction $u(x, y)$ that does not change sign in D . Then $\lambda^- \leq \lambda$, where

$$(2.2) \quad \lambda^- = \inf \left\{ \frac{\iint_{\bar{D}} (\nabla^2 v)^2 d\sigma}{\iint_{\bar{D}} p^- v^2 d\sigma} \right\}$$

and the infimum is taken over all functions $v(x, y)$ which, together with their first derivatives, are continuous in $\bar{D} \cup \bar{C}$, for which $\nabla^2 v = v_{xx} + v_{yy}$ is Lebesgue square integrable over $\bar{D}$, and which satisfy the boundary conditions

$$(2.3) \quad v(\bar{C}) = \frac{\partial v(\bar{C})}{\partial n} = 0.$$

Proof. We may assume that $0 \leq u(x, y) \leq 1$ in D . As in the case of the first part of Theorem 1, the theorem will be proved by constructing two functions $\bar{u}(x, y) = \bar{u}(r)$ and $u^-(x, y) = u^-(r)$, and establishing the inequalities

$$(2.4) \quad \lambda = \frac{\iint_D (\nabla^2 u)^2 d\sigma}{\iint_D p u^2 d\sigma} \geq \frac{\iint_{\bar{D}} (\nabla^2 \bar{u})^2 d\sigma}{\iint_{\bar{D}} p^-(u^-)^2 d\sigma} \geq \frac{\iint_{\bar{D}} (\nabla^2 \bar{u})^2 d\sigma}{\iint_{\bar{D}} p^-(\bar{u})^2 d\sigma} \geq \lambda^-.$$

In (2.4), $u^-(r)$ is the rearrangement of $u(x, y)$ in symmetrically decreasing order, while $\bar{u}(r)$ is the (corrected) function introduced by SZEGÖ in [8]. In order to establish (2.4) it suffices to prove

$$(2.5) \quad \iint_D (\nabla^2 u)^2 d\sigma = \iint_{\bar{D}} (\nabla^2 \bar{u})^2 d\sigma,$$

$$(2.6) \quad 0 \leq u^-(x, y) \leq \bar{u}(x, y), \quad (x, y) \in \bar{D},$$

$$(2.7) \quad \iint_D p^-(u^-)^2 d\sigma \geq \iint_D pu^2 d\sigma.$$

The inequality (2.7) is a special case of the theorem quoted in the introduction, since one can obtain a common domain of definition by setting $p^- = u^- = 0$ outside $\bar{D}$ and $p = u = 0$ outside D . Both (2.5) and (2.6) were established by SZEGÖ, the proofs being subsequently modified by him.³ The final inequality of (2.4) depends on the admissibility of $\bar{u}$ for (2.2).

Now let C_ρ denote the level curve (or curves) $u = \rho$, $0 < \rho < 1$, and let $A(\rho)$ be the area of the subset D_ρ of D characterized by $u \geq \rho$. Thus $A(0) = A$, $A(1) = 0$, while $A(\rho)$ is continuous and monotone decreasing for $0 \leq \rho \leq 1$. With SZEGÖ we write

$$G = |\text{grad } u|, \quad P(\rho) = \int_{C_\rho} G ds,$$

and note that [8, pp. 233–4]

$$\iint_D G^2 d\sigma = \int_0^1 P(\rho) d\rho, \quad -A'(\rho) = |A'(\rho)| = \int_{C_\rho} \frac{ds}{G}.$$

The circularly symmetric function $\bar{u}(x, y) = \bar{u}(r)$ is defined by

$$(2.8) \quad \bar{u}(r) = f(\rho), \quad 0 \leq r \leq R_0,$$

where r and ρ are related by $\pi r^2 = A(\rho)$, and $f(\rho)$ is to be determined in such a way that (2.5) and (2.6) are valid. We have

$$\nabla^2 \bar{u}(r) = \frac{1}{r} \frac{d}{dr} [r \bar{u}'(r)] = \frac{4\pi}{A'(\rho)} \frac{d}{d\rho} \left[\frac{A(\rho)f'(\rho)}{A'(\rho)} \right],$$

hence the right side of (2.5) becomes

$$(2.9) \quad 2\pi \int_0^{R_0} r(\nabla^2 \bar{u})^2 dr = 16\pi^2 \int_0^1 \frac{1}{|A'(\rho)|} \left\{ \frac{d}{d\rho} \left[\frac{A(\rho)f'(\rho)}{A'(\rho)} \right] \right\}^2 d\rho.$$

On the other hand, setting

$$(2.10) \quad Q(\rho) = \int_{C_\rho} \frac{(\nabla^2 u)^2}{G} ds,$$

the left side of (2.5) becomes $\int_0^1 Q(\rho) d\rho$. Hence (2.5) is satisfied if we choose $f(\rho)$ so that

$$(2.11) \quad \frac{16\pi^2}{|A'(\rho)|} \left\{ \frac{d}{d\rho} \left[\frac{A(\rho)f'(\rho)}{A'(\rho)} \right] \right\}^2 = Q(\rho).$$

Since $\int_0^1 Q(t) dt$ and $\int_0^1 |A'(t)| dt = A(0)$ are finite, it follows from Schwarz's inequality that $\{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}}$ is integrable. Moreover [8, p. 236],

³ See [10] and the introduction of the present paper.

$$(2.12) \quad \{Q(\rho) \cdot |A'(\rho)|\}^{\frac{1}{2}} \geq \int_{c_\rho} \frac{\nabla^2 u}{G} ds,$$

$$(2.13) \quad \int_0^\rho \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt \geq P(\rho).$$

In precisely the same way as (2.13) follows from (2.12), we also obtain

$$(2.14) \quad \int_\rho^1 \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt \geq P(\rho).$$

We now proceed to the (corrected) definition of $f(\rho)$. To this end, let the constant ρ_0 , $0 < \rho_0 < 1$, be defined by the equation

$$\int_0^{\rho_0} \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt = \int_{\rho_0}^1 \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt,$$

and set

$$g(\rho) = \begin{cases} \frac{|A'(\rho)|}{4\pi A(\rho)} \int_0^\rho \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt, & 0 < \rho \leq \rho_0, \\ \frac{|A'(\rho)|}{4\pi A(\rho)} \int_\rho^1 \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt, & \rho_0 \leq \rho < 1. \end{cases}$$

Then $f(\rho)$ is defined by

$$(2.15) \quad f(\rho) = \int_0^\rho g(t) dt, \quad 0 \leq \rho \leq 1.$$

The convergence of (2.15) at $\rho = 0$ follows from the fact that $g(t) \leq k|A'(t)|/A(t)$, which is integrable on $(0, \rho)$ for $\rho < 1$. To establish convergence at $\rho = 1$, we note that

$$(2.16) \quad \int_\rho^1 \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt \leq A^{\frac{1}{2}}(\rho) \cdot \left\{ \int_\rho^1 Q(t) dt \right\}^{\frac{1}{2}}$$

by Schwarz's inequality, and hence $g(t) \leq kA^{-\frac{1}{2}}(t) \cdot |A'(t)|$ which is integrable on $(\rho, 1)$.

It is now clear from the definition of $g(\rho)$ that the condition (2.11) is satisfied. Moreover, the function $\bar{u}(r)$ defined by (2.8) is admissible as we now show. We have

$$\bar{u}'(r) = f'(\rho) \frac{d\rho}{dr} = g(\rho) \frac{2\{\pi A(\rho)\}^{\frac{1}{2}}}{A'(\rho)},$$

so that $\bar{u}(r)$ has continuous first derivatives in $\bar{D}$ (except, perhaps, at $r = 0$, corresponding to $\rho = 1$), and the boundary condition

$$\left(\frac{\partial \bar{u}}{\partial n} \right)_{r=R_0} = \lim_{\rho \rightarrow 0} f'(\rho) \frac{d\rho}{dr} = 0$$

is satisfied. The remaining boundary condition (2.3) is clearly satisfied. Finally,

$$\bar{u}'(0) = -\frac{2\pi^{\frac{1}{2}}}{4\pi} \lim_{\rho \rightarrow 1} \frac{1}{A^{\frac{1}{2}}(\rho)} \int_{\rho}^1 \{Q(t) \cdot |A'(t)|\}^{\frac{1}{2}} dt = 0,$$

by (2.16), since $\int_{\rho}^1 Q(t) dt \rightarrow 0$ as $\rho \rightarrow 1$. This completes the proof of the admissibility of $\bar{u}$ and establishes (2.5).

In order to prove the inequality (2.6), we note that $u^{-}(x, y) = u^{-}(r)$ is defined by

$$u^{-}(r) = \rho, \quad 0 \leq r \leq R_0, \quad \pi r^2 = A(\rho).$$

Hence, according to (2.8), (2.6) is equivalent to proving that $f(\rho) \geq \rho$. This follows, precisely as in [8, pp. 236–7], from (2.13) and (2.14) together with the inequality

$$(2.17) \quad P(\rho) \geq \frac{4\pi A(\rho)}{|A'(\rho)|},$$

which is a consequence of Schwarz's inequality, and the general isoperimetric inequality [8, p. 234, (9)]. Indeed, (2.13), (2.14), and (2.17) imply $g(\rho) \geq 1$, hence $f(\rho) \geq \rho$. This completes the proof of (2.6) and hence of (2.4).

Theorem 3. *Let D be the disk $0 \leq x^2 + y^2 = r^2 < R_0^2$ with boundary C , and suppose that $p(x, y)$ is positive and continuous in $D \cup C$. Let $p^{+}(x, y) = p^{+}(r)$ be the rearrangement of $p(x, y)$ in symmetrically increasing order defined in $D \cup C$, and let λ^{+} denote the least eigenvalue of the system*

$$(3.1) \quad \nabla^4 u(x, y) - \lambda^{+} p^{+}(x, y) u(x, y) = 0, \quad u(C) = \frac{\partial u(C)}{\partial r} = 0.$$

If this system has a corresponding first eigenfunction $u(x, y)$ which does not change sign in D , then $\lambda \leq \lambda^{+}$, where

$$(3.2) \quad \lambda = \inf \left\{ \frac{\iint_D (\nabla^2 v)^2 d\sigma}{\iint_D p v^2 d\sigma} \right\}.$$

The infimum in (3.2) is taken over the class of functions admissible in the disk D (cf. (2.2)); in particular, the boundary conditions of (3.1) are to be satisfied.

Proof. For $0 \leq r \leq R_0$, set

$$v(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r \cos \theta, r \sin \theta) d\theta.$$

Then $v(x, y) = v(r)$ is also a first eigenfunction of (3.1), and

$$(3.3) \quad \lambda^{+} = \frac{\iint_D (\nabla^2 v)^2 d\sigma}{\iint_D p^{+} v^2 d\sigma}.$$

We may assume that $v(r)$ is non-negative, and shall prove that $v'(r) < 0$, $0 < r < R_0$, so that p^+ and v^2 are oppositely ordered. It will then follow that

$$\iint_D p^+ v^2 d\sigma \leq \iint_D p v^2 d\sigma,$$

whence $\lambda^+ \geq \lambda$ will follow from (3.2) and (3.3), since v is clearly an admissible function.

In order to prove $v'(r) < 0$, we note that since $v(r)$ satisfies (3.1), it satisfies the ordinary differential system

$$(3.4) \quad \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{dv}{dr} \right) \right] \right\} = \lambda^+ p^+ v, \quad v(R_0) = v'(R_0) = 0.$$

From the definition of $v(r)$, we have $v'(0) = 0$. Multiplying (3.4) by r and integrating, we obtain

$$r \left(v'' + \frac{1}{r} v' \right)' = \lambda^+ \int_0^r s p^+(s) v(s) ds,$$

since

$$\lim_{r \rightarrow 0} r \left(v'' + \frac{1}{r} v' \right)' = \lim_{r \rightarrow 0} \left(r v''' + v'' - \frac{v'}{r} \right) = 0.$$

A further integration gives

$$(rv')' = 2rv''(0) + \lambda^+ r \int_0^r \int_0^t \frac{1}{t} s p^+(s) v(s) ds dt.$$

Finally, using $v'(0) = 0$, we obtain

$$\frac{v'(r)}{r} = v''(0) + \frac{\lambda^+}{r^2} \int_0^r \int_0^x \int_0^t \frac{x}{t} s p^+(s) v(s) ds dt dx.$$

Now set $f(r) = v'(r)/r$. Since $f(R_0) = 0$, it suffices to prove that $f'(r) > 0$, $0 < r < R_0$. This is equivalent to proving that $g(r) > 0$, $0 < r < R_0$, where

$$g(r) = r \int_0^r \int_0^t r \frac{s}{t} p^+ v ds dt - 2 \cdot \int_0^r \int_0^x \int_0^t x \frac{s}{t} p^+ v ds dt dx.$$

This result follows easily since $g(0) = 0$, while

$$g'(r) = r \int_0^r s p^+(s) v(s) ds > 0.$$

BIBLIOGRAPHY

- [1] G. FABER, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitzungsberichte der Bayrischen Akademie der Wissenschaften, 1923, pp. 169–172.
- [2] G. H. HARDY, J. E. LITTLEWOOD & G. PÓLYA, *Inequalities*, 2nd ed., Cambridge, 1952.

- [3] S. A. JANCZEWSKY, *Oscillation theorems for the differential boundary value problems of the fourth order*, Ann. of Math. (2), **29**, 1928, pp. 521–542.
- [4] E. KRAHN, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann., **94**, 1924, pp. 97–100.
- [5] Lord RAYLEIGH, *The theory of sound*, 2nd ed., London, 1894.
- [6] B. SCHWARZ, *Bounds for the principal frequency of the nonhomogeneous membrane and for the generalized Dirichlet integral*, Pac. J. of Math., **7**, 1957, pp. 1653–1676.
- [7] B. SCHWARZ & P. R. BEESACK, *On the zeros of solutions of second order linear differential equations*, Can. J. of Math., **8**, 1956, pp. 504–515.
- [8] G. SZEGÖ & G. PÓLYA, *Isoperimetric inequalities in mathematical physics*, Princeton, 1951.
- [9] G. SZEGÖ, *On membranes and plates*, Proc. Nat. Ac. Sc. U.S.A., **36**, 1950, pp. 210–216.
- [10] ———, *Note to my paper "On membranes and plates,"* Proc. Nat. Ac. Sc. U.S.A., **44**, 1958, pp. 314–316.

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