The Derivation of Stress-Deformation Relations for a Stokesian Fluid

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In continuum mechanics the action of material forces across a surface element is shown to depend upon a certain tensor $\mathbf{T}$, known as the stress tensor. Through its appearance in the equations of motion, this tensor governs the dynamic response of the medium in question; thus by relating $\mathbf{T}$ to other kinematic and thermodynamic variables we effectively define or delimit the type of medium which we study, *e.g.* fluid, elastic, plastic, and so forth. Such a relation between $\mathbf{T}$ and other variables is known as a *constitutive equation*, and its form is obviously of fundamental importance. In the case of fluids, the constitutive equation usually adopted is the classical Newton-Cauchy-Poisson viscosity law, but recently there has been considerable interest in more general stress-deformation relations, [3]–[8], [10]–[12]. The purpose of this paper is to give a simple and complete treatment of the constitutive equation in one of these more general situations, namely when the stress tensor is a function of the deformation tensor (Stokesian fluid). In §1 a clear-cut set of conditions is stated concerning the reaction of the fluid to deformation; from these conditions and these alone the form of the constitutive equation follows unavoidably. The algebraic details are presented in §§ 2 and 4, with the main results being stated in the form of two theorems. In §3 we have included a simple proof of the Newton-Cauchy-Poisson law, which is not so well known as it deserves to be. Finally, in §5 the concept of fluid pressure is discussed and clarified in the light of the preceding work.

Some of the results of the paper are not new. In particular, the main theorems 1 and 2 are essentially due to RIVLIN & ERICKSEN [7], and the discussion of fluid pressure is largely taken from TRUESDELL's memoir [12]. Thus part of the paper is expository; it is hoped, however, that the simplicity of the derivations, the inclusion of some new results, and, above all, the fact that the material is available here for the first time as a unified theory will justify the author's efforts. Finally, we should emphasize that our point of view stems directly from the pioneer work of STOKES, and is greatly influenced by the "phenomenological" ideas of REINER, RIVLIN, and TRUESDELL, who first seriously studied constitutive equations for non-Newtonian fluids.

1. The Stokesian fluid. In a remarkable paper published when he was but 26 years old, STOKES put the concept of fluidity into the following words: "That

the difference between the pressure¹ on a plane in a given direction passing through any point P of a fluid in motion and the pressure would exist in all directions about P if the fluid in its neighborhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P ; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned" ([9], §1). Although this may seem fairly vague, it nevertheless contains the essential idea of fluidity and will serve adequately as the basis for our discussion.

In order to give mathematical form to the above principle, it is first necessary to attach a precise meaning to STOKES' concept of the "pressure . . . in a state of relative equilibrium." This may be done by *introducing a new dynamical variable p , having the physical dimension M/LT^2 , and by formally defining the equilibrium stress to be $-p\mathbf{I}$* .² The "difference in pressures" which STOKES refers to is then represented by the tensor

$$(1) \quad \mathbf{V} = \mathbf{T} + p\mathbf{I}.$$

It is customary to call p the fluid pressure and $\mathbf{V}$ the viscosity tensor. We emphasize that the existence of the primitive variable p is the cornerstone of the mathematical theory of viscous fluids.

Now let $\mathbf{D}$ denote the symmetric part of the velocity gradient tensor. In component form

$$(2) \quad D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}),$$

the notation being obvious. It is well known (see, for example, [1], §30) that local relative fluid motion can be expressed as the superposition of a pure rotation and a pure deformation, the latter depending only on the tensor $\mathbf{D}$. In consequence, we are able to state STOKES' principle in the following precise form:

P1. $\mathbf{V}$ is a function of the deformation tensor $\mathbf{D}$ and the local thermodynamic state, and is independent of all other quantities.

P2. $\mathbf{V} = 0$ when $\mathbf{D} = 0$.

Other sets of postulates are of course possible, and in certain cases even desirable (cf. [3], [11]), but for almost all present day applications STOKES' assumptions are sufficient. *A continuous medium for which the stress tensor obeys postulates P1 and P2 will be called a Stokesian fluid*.³ The analytical statement of P1 is simply

$$(3) \quad \mathbf{V} = f(\mathbf{D}, Z),$$

¹ Modern writers would use the word "stress" instead of "pressure."

² It is convenient to use matrix notation in expressing the various results of the paper; we assume, therefore, that the reader is familiar with the matrix representation of stress and with a few of the elementary facts of matrix theory. The letter $\mathbf{I}$ denotes, as usual, the identity matrix.

³ TRUESDELL [10]-[12] uses the terminology "Stokesian fluid" in a more restricted sense than that proposed here.

where Z denotes the thermodynamical state. Equation (3) implies that each component of $\mathbf{V}$ is a definite function of the six independent components of $\mathbf{D}$ and of the thermodynamical state.

In addition to postulates P1 and P2 relating to the specific form of the constitutive equation, we shall adopt the following general isotropy postulate concerning physical space itself.

P3. *There are no distinguished or preferred directions in space. That is, a given motion, regardless of its orientation in space, will produce the same intrinsic stress field.*⁴

In order that the constitutive equation (3) should be compatible with this postulate, it is necessary and sufficient that the matrix identity

$$(4) \quad \mathbf{S}\mathbf{V}\mathbf{S}^{-1} = f(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}, Z)$$

should hold for all orthogonal transformations $\mathbf{S}$. Equation (4) implies that the function f is invariant with respect to orthogonal coordinate transformations.

Remarks. It should be observed that postulate P1 already defines an *isotropic fluid*, since it explicitly rules out the possibility of the fluid particles themselves carrying a preferred orientation. Nevertheless, P1 certainly does not imply equation (4); the realization that two distinct forms of isotropy (material and spatial, here evidenced by P1 and P3, respectively) play a part in determining the nature of constitutive equations is due to W. NOLL, *cf.* the paper already cited.

The next sections of the paper are devoted to finding the most general constitutive equation which is compatible with P1–P3. In this regard it is not enough for us merely to exhibit *some* formula satisfying (3) and (4), for there is always the question whether such a formula might be more restrictive than the postulates themselves would allow.⁵

2. Two preliminary lemmas. Conditions (3) and (4) have several simple consequences which will be of use in the sequel. Though these results are well known, it is nevertheless valuable to include their complete proofs.

Lemma 1. *The principal directions of the matrices $\mathbf{V}$, $\mathbf{T}$, and $\mathbf{D}$ coincide.*

Proof. It is sufficient to show that any orthogonal transformation reducing $\mathbf{D}$ to diagonal form likewise reduces $\mathbf{V}$ to diagonal form. Indeed, suppose that $\mathbf{D}$ has been transformed to the form

$$\bar{\mathbf{D}} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix},$$

⁴ This postulate is a weakened form of a general principle of the isotropy of space, first stated by W. NOLL ([3], pp. 17–19).

⁵ LAMB's treatment of viscosity (pp. 573–574) is somewhat obscure on this point.

and let $\bar{\mathbf{V}}$ be the corresponding transformation of $\mathbf{V}$. Then by (4) we have $\bar{\mathbf{V}} = f(\bar{\mathbf{D}})$ (suppressing for the moment the dependence of $\mathbf{V}$ on the thermodynamic state). Now the orthogonal transformation

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

leaves $\bar{\mathbf{D}}$ unaltered, whence by (4)

$$(5) \quad \mathbf{S}\bar{\mathbf{V}}\mathbf{S}^{-1} = f(\mathbf{S}\bar{\mathbf{D}}\mathbf{S}^{-1}) = f(\bar{\mathbf{D}}) = \bar{\mathbf{V}}.$$

Thus $\bar{\mathbf{V}}$ is also unaltered by $\mathbf{S}$. An easy computation shows that this can be the case if and only if

$$V_{12} = V_{13} = V_{21} = V_{31} = 0.$$

In a similar way we find $V_{23} = V_{32} = 0$, so $\bar{\mathbf{V}}$ is indeed in diagonal form. *Q.E.D.*

The above proof becomes geometrically obvious if we observe that $\mathbf{S}$ represents a rotation of 180° about the x_1 axis, while (5) shows that $\bar{\mathbf{V}}$ is symmetric about this axis.

Having shown that $\bar{\mathbf{V}}$ is diagonal, let us set

$$\bar{\mathbf{V}} = \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix}.$$

The condition $\bar{\mathbf{V}} = f(\bar{\mathbf{D}})$, which we have already noted, can now be written in the form

$$(6) \quad \begin{aligned} v_1 &= \phi(d_1, d_2, d_3), \\ v_2 &= \psi(d_1, d_2, d_3), \\ v_3 &= \omega(d_1, d_2, d_3), \end{aligned}$$

where ϕ, ψ, ω are certain definite functions of the variables d_i . The next lemma concerns the behavior of these functions when the values d_i are subjected to permutation.

Lemma 2. *Any permutation of the d_i leads to a corresponding permutation of the v_i .*

Proof. A permutation of the d_i can be effected by an orthogonal transformation of $\mathbf{D}$. But, according to (4), this transformation has the same effect on $\bar{\mathbf{V}}$ as on $\bar{\mathbf{D}}$, thus proving the lemma.

3. The linear case. If to postulate P1—P3 we add the condition that $\mathbf{T}$ be

linear in the components of $\mathbf{D}$, then $\mathbf{T}$ must have the form

$$(7) \quad \mathbf{T} = (-p + \lambda\Theta)\mathbf{I} + 2\mu\mathbf{D},$$

where λ and μ are functions of the thermodynamic state and $\Theta = v_{i,i}$. This is the famous Newton-Cauchy-Poisson viscosity law ([11], p. 125). For completeness we include a simple derivation, due to W. MÜLLER [2].

To begin with, in virtue of P2 and the assumed linearity, formula (6) must take the form

$$(8) \quad \begin{aligned} v_1 &= a_1 d_1 + a_2 d_2 + a_3 d_3, \\ v_2 &= b_1 d_1 + b_2 d_2 + b_3 d_3, \\ v_3 &= c_1 d_1 + c_2 d_2 + c_3 d_3, \end{aligned}$$

the coefficients being independent of d_i . Since permutation of the d_i causes a corresponding permutation of the v_i , the cyclic change $\langle 231 \rangle$ leads to the relations

$$a_1 = b_2 = c_3, \quad a_2 = b_3 = c_1, \quad a_3 = b_1 = c_2,$$

while the acyclic change $\langle 213 \rangle$ gives, in addition, $a_2 = b_1$. These equalities yield the formulas

$$(9) \quad a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = \lambda, \quad a_1 = b_2 = c_3 = \lambda + 2\mu,$$

where λ and μ are appropriate parameters, depending at most on the thermodynamic state. With the help of (9), we can write (8) in the matrix form

$$\tilde{\mathbf{V}} = \lambda(d_1 + d_2 + d_3)\mathbf{I} + 2\mu\tilde{\mathbf{D}}.$$

Since $d_1 + d_2 + d_3 = \text{Trace } \mathbf{D} = v_{i,i}$, the required formula (7) follows at once by reverting to the original coordinate system.

This elegant proof should be compared with the ungainly treatment based on the equation $T_{ij} = a_{ijk} D_{kl}$, which is occasionally found in books on Cartesian tensors.

4. The general constitutive equation. The matrix $\mathbf{D}$ possesses three *principal invariants* Θ , II , III which are unaltered by orthogonal transformations. These invariants are defined as the coefficients of λ in the determinantal expansion

$$(10) \quad D(\lambda) = \det(\lambda\mathbf{I} - \mathbf{D}) = \lambda^3 - \Theta\lambda^2 + II\lambda - III;$$

the first invariant, in particular, is given by the formula $\Theta = \text{Trace } \mathbf{D} = v_{i,i}$. The numbers d_1, d_2, d_3 which appear on the main diagonal when $\mathbf{D}$ is written in diagonal form are called the *principal (or characteristic) values of $\mathbf{D}$* . These principal values obviously are roots of the equation $D(\lambda) = 0$, whence the unordered number triple (d_1, d_2, d_3) is a continuous function of Θ, II, III . With the above elementary facts of matrix theory understood, we can now state the central result of the paper.

Theorem 1. *If postulates P1-P3 hold, then $\mathbf{T}$ can be written in the form*

$$(11) \quad \mathbf{T} = (-p + \alpha)\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2,$$

where α, β, γ are functions of the principal invariants of $\mathbf{D}$ and the thermodynamical state $\mathbf{Z}$, and $\alpha = 0$ when $\mathbf{D} = \mathbf{0}$.⁶

Remarks. The converse of this theorem, namely that every relation of the form (11) satisfies postulates P1-P3, is obvious. Theorem 1 may therefore be summarized in the statement that (11) is the most general constitutive equation compatible with P1-P3.

It will be apparent from the following proof that the coefficients α, β, γ inherit all the differentiability properties of the original relation $f(\mathbf{D})$, provided that the principal values d_1, d_2, d_3 are different. On the other hand, the example

$$\mathbf{V} = (\Theta\mathbf{I} - 3\mathbf{D})^2 \sin \Lambda^{-\frac{1}{2}} = (\Theta^2\mathbf{I} - 6\Theta\mathbf{D} + 9\mathbf{D}^2) \sin \Lambda^{-\frac{1}{2}}$$

where $\Lambda = 2(\Theta^2 - 2II) = (d_1 - d_2)^2 + (d_2 - d_3)^2 + (d_3 - d_1)^2$, shows that α, β, γ need not be everywhere continuous even for a differentiable relation $f(\mathbf{D})$. This fact is fortunately not the general rule, for, as we shall show in §6, discontinuities in α, β, γ cannot occur if $f(\mathbf{D})$ is three times differentiable. In other words, it is only for quite pathological examples that discontinuities in α, β, γ can appear.

Proof of Theorem 1. Let us assume, to begin with, that the principal values of $\mathbf{D}$ are all different. Consider the following system of equations for the unknown multipliers α, β, γ ;

$$(12) \quad v_i = \alpha + \beta d_i + \gamma d_i^2, \quad i = 1, 2, 3,$$

where v_i is given by (6). Equations (12) are a linear system with coefficient determinant

$$\Delta = \begin{vmatrix} 1 & d_1 & d_1^2 \\ 1 & d_2 & d_2^2 \\ 1 & d_3 & d_3^2 \end{vmatrix} = (d_1 - d_2)(d_2 - d_3)(d_3 - d_1) \neq 0.$$

Thus, by Cramer's rule,

$$(13) \quad \alpha = \frac{1}{\Delta} \begin{vmatrix} v_1 & d_1 & d_1^2 \\ v_2 & d_2 & d_2^2 \\ v_3 & d_3 & d_3^2 \end{vmatrix},$$

with similar expressions for β and γ . In virtue of Lemma 2 it is seen that α is unchanged by a permutation of the d_i , and naturally the same is true of β and γ . In other words, α, β and γ are symmetric functions of the variables d_i .

⁶ Equation (11) was derived by both REINER [4] and RIVLIN [6] under the assumption that $f(\mathbf{D})$ is analytic.

It follows from the preceding that α, β, γ are single-valued functions of the principal invariants of $\mathbf{D}$. Moreover, according to (12),

$$(12)' \quad \bar{\mathbf{V}} = \alpha \mathbf{I} + \beta \bar{\mathbf{D}} + \gamma \bar{\mathbf{D}}^2,$$

and this, when transformed to the original matrices $\mathbf{T}$ and $\mathbf{D}$, is precisely (11).

When two principal values of $\mathbf{D}$ coincide, we can establish in the same way as above that $\mathbf{V} = \alpha \mathbf{I} + \beta \mathbf{D}$, while if all the principal values are equal, $\mathbf{V} = \alpha \mathbf{I}$. Since these formulas are of the type (11), the proof is completed.

According to Theorem 1, in order to specify the response of a Stokesian fluid to deformation one must give definite form to three scalar functions. (There are several problems which can be solved even when α, β, γ are left arbitrary, cf. [6], [8], but these must be considered quite special cases.) Most frequently, $\mathbf{T}$ is assumed to be linear in the components of $\mathbf{D}$, as in §3. Of next importance is quadratic dependence of $\mathbf{T}$ on $\mathbf{D}$. Since the invariants Θ, II, III are respectively of degrees 1, 2, 3 in the components of $\mathbf{D}$, this can be achieved by specializing (11) to the form

$$(14) \quad \mathbf{T} = (-p + \lambda\Theta + \lambda'\Theta^2 + \lambda''II)\mathbf{I} + (2\mu + 2\mu'\Theta)\mathbf{D} + 4\nu\mathbf{D}^2,$$

and higher order stress-deformation relations are likewise easily written down. It is of interest to know whether (14) and its higher order analogues are the most general relations possible when quadratic or higher order dependence is assumed. The crucial result is this:

Theorem 2. *The most general constitutive equation satisfying postulates P1–P3, and such that the stress components are polynomials of degree N in the deformation components, is*

$$(15) \quad \mathbf{T} = (-p + P_0)\mathbf{I} + P_1\mathbf{D} + P_2\mathbf{D}^2$$

where the P_k , $k = 0, 1, 2$, are polynomials in the principal invariants of $\mathbf{D}$, with weight $P_k = N - k$.⁷ Moreover, the constant term of P_0 must vanish.

Proof. We show first that the coefficient α in (11) is a polynomial of weight N in Θ, II, III . According to hypothesis, the functions ϕ, ψ, ω in (6) must be polynomials of degree N at most in the variables d_i . Therefore the numerator determinant in (13) is a polynomial of degree $N + 3$ at most. It vanishes when $d_1 = d_2$, hence contains $(d_1 - d_2)$ as a factor. Similarly it has $(d_2 - d_3)$ and $(d_3 - d_1)$ as factors. Consequently the denominator in (13) can be cancelled out, leaving for α a polynomial of degree at most N in the variables d_i .

We have shown, however, that α is a symmetric function of d_i ; thus by a well known theorem of algebra⁸ α is a polynomial of weight at most N in the principal invariants of $\mathbf{D}$. To satisfy P2, the constant term in this polynomial

⁷ The weight of the polynomial P_k is the maximum weight of any of its terms, a term $\Theta^l II^m III^n$ being counted of weight $k + 2l + 3m$.

⁸ Cf. B. L. VAN DER WAERDEN, *Modern Algebra*, Vol. I, pp. 78–81.

must vanish. In the same way, β and γ can be proved to be polynomials of respective weights $N - 1$ and $N - 2$, and the theorem is proved.⁹

5. Fluid pressure. For a compressible fluid there is a well defined thermodynamic pressure $\pi = \pi(Z)$. It is customary in fluid mechanics to identify the thermodynamic pressure with the dynamic pressure p introduced in §1. The reasons for this identification seem to be, firstly, its intuitive appeal and, secondly, the fact that without some postulated relation between p and π the general equations of motion would constitute an undetermined system. Finally, since one naturally expects $\mathbf{T} = -\pi\mathbf{I}$ when $\mathbf{D} = 0$, the identification is compatible with postulate P2. *For a compressible fluid, therefore, it is assumed that the fluid pressure is a thermodynamic state variable, $p = \pi(Z)$, and*

$$(16) \quad \mathbf{T} = (-p + \alpha)\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2.$$

For an incompressible fluid, on the other hand, there is no such thing as thermodynamic pressure. In this case p remains one of the primitive unknowns of the flow. We observe, moreover, that p appears in the equations of motion always in the combination $(-p + \alpha)$. Because of this fact there is no loss of generality in assuming $\alpha = 0$, and this particular specification of pressure is the one we adopt. If the reader feels that a disturbing element of arbitrariness has thus been introduced into the theory, he should recall that whatever the definition of pressure may be, the quantity called "pressure" in the hydrodynamical equations is not what is recorded by a measuring device; what one measures is some component of the stress tensor. In other words, the somewhat arbitrary nature of fluid pressure is inherent in the concept itself. With this understanding, then *the final formula for stress in an incompressible Stokesian fluid becomes*

$$(17) \quad \mathbf{T} = -p\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2,$$

where β and γ are functions of the principal invariants II, III and the thermodynamical state Z .

In some works (notably LAMB's *Hydrodynamics*), fluid pressure is treated as a dynamic variable defined by $\bar{p} = -\frac{1}{3} \text{Trace } \mathbf{T}$. Although this is entirely permissible, it is certainly less convenient than the present approach. In particular, for incompressible fluids it leads to a less simple formula than (17), while for compressible fluids one is left with the problem of relating $\bar{p}$ to the thermodynamic pressure π .

The difference between the pressure p and the mean pressure $\bar{p}$ is given by the formula

$$3(p - \bar{p}) = \begin{cases} 3\alpha + \beta\Theta + \gamma\mathbf{D}:\mathbf{D} & (\text{compressible}) \\ \gamma\mathbf{D}:\mathbf{D} & (\text{incompressible}). \end{cases}$$

⁹ Formula (15) appeared first in the work of REINER [4] and RIVLIN [6]. The theorem itself is due to RIVLIN & ERICKSEN ([7], §§39–41), though their proof is far from simple.

Thus for a compressible fluid there is no particularly simple relation between $\bar{p}$ and the (thermodynamic) pressure $p = \pi(Z)$, unless special hypotheses are made concerning the coefficients α, β, γ . Similarly, for an incompressible fluid $p = \bar{p}$ if and only if $\gamma = 0$; that is, general equality of pressure and mean pressure in an incompressible fluid is equivalent to a quasi-linear stress-deformation relation. The distinction between p and $\bar{p}$ therefore vanishes in the classical viscosity theory; whether the pressure is defined as here, or is defined as $-\frac{1}{3} \text{Trace } \mathbf{T}$, the final result is the same. We emphasize this point because in many places the erroneous impression is given that the notion of mean pressure is the only logical basis for the study of viscous fluids.

6. Appendix. We are interested here in the continuity properties of the coefficients α, β, γ . In §4 it was seen that α, β, γ might be discontinuous at a coalescence of the principal values of $\mathbf{D}$, even though the original relation $f(\mathbf{D})$ was differentiable. The question therefore arises, are these discontinuities a phenomenon to be expected in general (except of course for polynomial dependence), or do they arise only in pathological examples? As remarked in §4, the answer is that discontinuities are in fact somewhat pathological. More precisely, *if $\mathbf{V}$ is a three times continuously differentiable function of $\mathbf{D}$, then $\mathbf{T}$ can be written in the form*

$$(18) \quad \mathbf{T} = (-p + \alpha)\mathbf{I} + \beta\mathbf{D} + \gamma\mathbf{D}^2,$$

where the coefficients α, β, γ are continuous functions of the principal invariants of $\mathbf{D}$.

Proof. Since (18) is already known to hold with continuous α, β, γ whenever the principal values of $\mathbf{D}$ are different, it will be enough to show that these coefficients can be extended to be continuous functions for *all values* of the d_i . Consider first the coefficient α , which by (13) has the form $\alpha = A/\Delta$. We observe that A is three times differentiable and is *anti-symmetric* in the d_i . Corresponding to a double coalescence, we may apply l'Hopital's rule to determine the limit of the indeterminate form A/Δ as $d_1, d_2 \rightarrow a$ and $d_3 \rightarrow b$, provided d_1, d_2, d_3 tend to their limits according to some parametric relation $d_i = d_i(t)$, $i = 1, 2, 3$. In this way we find

$$(19) \quad \lim \frac{A}{\Delta} = \frac{A_1\dot{d}_1 + A_2\dot{d}_2 + A_3\dot{d}_3}{(a-b)^2(\dot{d}_2 - \dot{d}_1)}$$

where the dots denote differentiation with respect to t , and

$$A_1 = \frac{\partial A}{\partial d_1}, \quad A_2 = \frac{\partial A}{\partial d_2}, \quad A_3 = \frac{\partial A}{\partial d_3}.$$

In writing formula (19) it is of course assumed that the functions $d_i(t)$ are continuously differentiable. Now the function A is anti-symmetric, whence

$$A_1(a, a, b) = -A_2(a, a, b), \quad A_3(a, a, b) = 0.$$

It follows that

$$(20) \quad \lim \alpha = \frac{A_2(a, a, b)}{(a - b)^2}.$$

This limit relation is, so far, established only for differentiable approaches to the coalescence. This restriction is easily set aside, however, by observing that the limit (19) is attained *uniformly* on straight paths normal to the surface $d_1 = d_2$. It follows, therefore, that α can be extended to be continuous at a double coalescence,

$$\alpha(a, a, b) = \frac{A_2(a, a, b)}{(a - b)^2}.$$

To show that α is continuous at a triple coalescence we adopt the same method. For paths ending at a triple coalescence (d, d, d) we find, in particular,

$$\lim \frac{A}{\Delta} = \lim \frac{A_2(a, a, b)}{(a - b)^2} = \frac{1}{2} A_{233}(d, d, d).$$

This computation is made more apparent if one observes that, because of the anti-symmetry of A , the third order terms in A must be of the form $\text{Const. } \Delta$, while the second order terms in $A_2(a, a, b)$ are of the form $\text{Const. } (a - b)^2$. Thus, with the definition

$$\alpha(d, d, d) = \frac{1}{2} A_{233}(d, d, d),$$

we see that α is continuous at a triple coalescence. A similar argument holds for β and γ , and our assertion is thereby proved.

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