On Open Mappings in Banach Algebras¹

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We shall be concerned with a Banach algebra $\mathfrak{A}$ with unit e . The needed information concerning the functional calculus, the Gelfand theory, and Fréchet-analyticity will be found in [2]. Suppose that f is a complex-valued function, holomorphic in an open subset U of the complex plane. If a is an element of $\mathfrak{A}$ with spectrum $\sigma(a) \subset U$, we can form the element $f(a) \in \mathfrak{A}$. Denote by $f(\mathfrak{A})$ the set of all $f(a)$ with $\sigma(a) \subset U$. The problem of determining necessary and sufficient conditions that a neighborhood of $f(a)$ be contained in $f(\mathfrak{A})$ is not yet solved, even in the cases $f(\lambda) = e^\lambda$ or $f(\lambda) = \lambda^n, n \geq 2$, an integer. However, rather natural sufficient conditions are known in these cases [3]. The general approach here is used to make the proofs more transparent, and the specialization to these cases is immediate.³

We use the following additional notations. $\mathfrak{E}(\mathfrak{A})$ is the Banach algebra of all bounded operators on $\mathfrak{A}$. L_a and R_a are those elements of $\mathfrak{E}(\mathfrak{A})$ defined by $L_a(x) = ax, R_a(x) = xa, a, x \in \mathfrak{A}$. $\rho(a)$ is the complement of $\sigma(a)$ in the complex plane, and is the domain of existence of the holomorphic $\mathfrak{A}$ -valued function $R(\lambda, a) = (\lambda e - a)^{-1}$. We will also consider $R(\lambda, L_a)$ and $R(\lambda, R_a)$ in $\mathfrak{E}(\mathfrak{A})$. When λ is a scalar, $(\lambda e - a)$ will be shortened to $(\lambda - a)$.

1. Lemma. $\sigma(L_a) \subset \sigma(a), \sigma(R_a) \subset \sigma(a)$.

Proof. Let $\lambda \in \rho(a)$. Then, in $\mathfrak{E}(\mathfrak{A})$, $\lambda - L_a$ has the two-sided inverse $L_{(\lambda - a)^{-1}}$, so that $\lambda \in \rho(L_a)$. Similarly, $R(\lambda, R_a)$ is $R_{R(\lambda, a)}$ whenever $R(\lambda, a)$ exists.

2. Lemma. Let $a \in \mathfrak{A}$ and $\sigma(a) \subset U$, an open set. Then there is an $\epsilon > 0$ such that if $h \in \mathfrak{A}, \|h\| < \epsilon$, then $\sigma(a + h) \subset U$.

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³ While this manuscript was in preparation, the author discovered another method of attacking this problem which gives less precise results, but has greater scope of application; the present method promises applications to functional calculi, and, in fact, has so been used by J. SCHWARTZ, *Two Perturbation Formulae*, Comm. Pure and Appl. Math., 8, 1955.

Proof. Let Γ be a contour in U surrounding, but disjoint from, $\sigma(a)$. Let M be the maximum of $\|R(\gamma, a)\|$ on Γ . Since $R(\lambda, a)$ is holomorphic at infinity, $\|R(\lambda, a)\| \leq M$ outside of Γ . Then, for $\|h\| < M^{-1}$, $\sum_{n=0}^{\infty} R(\lambda, a) [hR(\lambda, a)]^n$ exists for all λ outside of Γ and is there the inverse of $\lambda - (a + h)$.

3. Theorem. *Let $a \in \mathfrak{A}$. Suppose that f is a complex-valued function, holomorphic and one-to-one on an open set U containing $\sigma(a)$. Then there is a neighborhood of 0 in $\mathfrak{A}$ and a continuous function $y(x)$ defined on this neighborhood with values in $\mathfrak{A}$ such that $\sigma(a + y(x)) \subset U$ and $f(a + y(x)) = f(a) + x$.*

Proof. Set $G(x, y) = f(a + y) - f(a) - x$. We first show that the operator $D(h) = \delta_y G(x, y; h)$ in $\mathfrak{E}(\mathfrak{A})$ is invertible at $x = y = 0$. We have

$$\begin{aligned} D(h) &= \lim_{\eta \rightarrow 0} \frac{1}{\eta} [G(0, \eta h) - G(0, 0)] = \lim_{\eta \rightarrow 0} \frac{1}{\eta} [f(a + \eta h) - f(a)] \\ &= \lim_{\eta \rightarrow 0} \frac{1}{\eta} \cdot \frac{1}{2\pi i} \int_{\Gamma} f(\gamma) [R(\gamma, a + \eta h) - R(\gamma, a)] d\gamma \\ &= \lim_{\eta \rightarrow 0} \frac{1}{\eta} \cdot \frac{1}{2\pi i} \int_{\Gamma} f(\gamma) \sum_{n=1}^{\infty} R(\gamma, a) [hR(\gamma, a)]^n d\gamma \\ &= \frac{1}{2\pi i} \int_{\Gamma} f(\gamma) R(\gamma, a) h R(\gamma, a) d\gamma = \left[\frac{1}{2\pi i} \int_{\Gamma} f(\gamma) R(\gamma, L_a) R(\gamma, R_a) d\gamma \right] (h), \end{aligned}$$

where Γ is a contour in U surrounding $\sigma(a)$ and all the $\sigma(a + \eta h)$ for η sufficiently small. Consider now the commutative subalgebra $\mathfrak{B}$ of $\mathfrak{E}(\mathfrak{A})$ which is the second commutant of the algebra generated by L_a and R_a . The spectrum of an element $T \in \mathfrak{B}$ with respect to $\mathfrak{B}$ is the same as the spectrum of T as an operator on $\mathfrak{A}$. Since D is expressible as the integral of scalar multiples of elements of $\mathfrak{B}$, D is in $\mathfrak{B}$. Let $\mathfrak{M}$ be the maximal ideal space of $\mathfrak{B}$; we calculate $D(m)$ for $m \in \mathfrak{M}$. If m is such that $L_a(m) = R_a(m) = \zeta$, then $\zeta \in \sigma(a)$, and

$$D(m) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\gamma) d\gamma}{(\gamma - \zeta)^2} = f'(\zeta)$$

which is not zero since f is one-to-one on an open set containing $\sigma(a)$. If m is such that $L_a(m) = \zeta_1 \neq R_a(m) = \zeta_2$, then $\zeta_1, \zeta_2 \in \sigma(a)$, and

$$D(m) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\gamma) d\gamma}{(\gamma - \zeta_1)(\gamma - \zeta_2)} = \frac{f(\zeta_1) - f(\zeta_2)}{\zeta_1 - \zeta_2},$$

again not zero since f is one-to-one on $\sigma(a)$. Since $\sigma(D) = \{D(m) | m \in \mathfrak{M}\}$, $0 \notin \sigma(D)$ and D has a bounded inverse.

We may now appeal to the implicit function theorem of GRAVES & HILDEBRANDT [1] to conclude that there exists a Fréchet-analytic function $y(x)$ defined for x in a sufficiently small neighborhood of 0 in $\mathfrak{A}$, with values in $\mathfrak{A}$ and such that $G(x, y(x)) \equiv 0$, which proves the theorem.

An alternative approach is to write $f(a + y)$ as $f(a) + D(y) + r(y)$ where y

is in a suitably small neighborhood of 0 and $\|r(y)\| \leq M|h|^2$ for some M [2, Sec. 3.17]. Set $y(x) = \sum_{n=0}^{\infty} D^{-1}(-rD^{-1})^n x$ for all x such that $\|x\| \leq \rho/MB^2$, where B is the bound of D^{-1} and ρ is a fixed number, $0 < \rho < 1$. For these x , $\|(rD^{-1})^n x\| \leq \rho^n \|x\|$, so that the series defining $y(x)$ converges, and

$$f(a + y(x)) - f(a) = D(y(x)) + r(y(x)) = x.$$

4. Theorem. Let f be holomorphic on an open set U containing $\sigma(a)$ and one-to-one on an open set $V \subset U$ such that $f(\sigma(a)) \subset f(V)$. Then there is a $b \in \mathfrak{A}$ such that $f(b) = f(a)$ and $\sigma(b) \subset V$, so that, by Theorem 3, $f(a)$ is an interior point of $f(\mathfrak{A})$.

Proof. Let $\tilde{f}$ be the restriction of f to V and let $\tilde{f}^{-1}$ be the function inverse to $\tilde{f}$. Let Γ be a contour in U enclosing $\sigma(a)$. Define $\tilde{a} = (2\pi i)^{-1} \int_{\Gamma}^{\tilde{f}^{-1}} f(\gamma) R(\gamma, a) d\gamma$. Then $\sigma(\tilde{a}) \subset \tilde{f}^{-1} f(\sigma(a)) \subset V$, and $f(\tilde{a}) = \tilde{f}(\tilde{a}) = \tilde{f} \cdot \tilde{f}^{-1} \cdot f(a) = f(a)$.

5. Theorem. Suppose that f is holomorphic and one-to-one on an open set U containing $\sigma(a)$ and $\sigma(a_1)$, and $f(a) = f(a_1)$. Then a and a_1 commute.

Proof. Let D be the operator of Theorem 3, and set $T_a = L_a - R_a$; we then have

$$\begin{aligned} DT_a(a_1) &= T_a D(a_1) = T_a \left[\frac{1}{2\pi i} \int_{\Gamma} f(\gamma) R(\gamma, L_a) R(\gamma, R_a) d\gamma \right] (a_1) \\ &= \left[\frac{1}{2\pi i} \int_{\Gamma} f(\gamma) [(\gamma - R_a) - (\gamma - L_a)] R(\gamma, L_a) R(\gamma, R_a) d\gamma \right] (a_1) \\ &= \left[\frac{1}{2\pi i} \int_{\Gamma} f(\gamma) R(\gamma, L_a) d\gamma - \frac{1}{2\pi i} \int_{\Gamma} f(\gamma) R(\gamma, R_a) d\gamma \right] (a_1) \\ &= [L_{f(a)} - R_{f(a)}](a_1) = f(a)a_1 - a_1 f(a) = f(a_1)a_1 - a_1 f(a_1) = 0. \end{aligned}$$

Since D is an invertible operator, we must then have $T_a(a_1) = 0$; that is, a and a_1 commute.

We now apply these results to the case $f(\lambda) = e^\lambda$. The case $f(\lambda) = \lambda^n$ is much the same.

Following HILLE [3], we say that $\sigma(a)$ is stratified modulo $2\pi i$ if there is a connected curve C in the complex plane such that

(1) the map from C into the real numbers given by $z \rightarrow \Re(z)$ is one-to-one and onto,

(2) none of the curves $C_n = \{z + 2\pi ni \mid z \in C\}$ intersect $\sigma(a)$.

We call the domain between C_n and C_{n-1} U_n , and assume that C has been chosen so that $0 \in U_0$. Theorem 4 then tells us that $\exp(a)$ has a neighborhood contained in $\exp(\mathfrak{A})$.

6. Lemma. If $\sigma(a)$ is stratified modulo $2\pi i$ and $\exp(a) = \exp(b)$, then $\sigma(b)$ is also stratified modulo $2\pi i$; the C_n 's which stratify $\sigma(a)$ also stratify $\sigma(b)$.

Proof. It is sufficient to show that $\sigma(b) \cap C_n$ is empty for all n . Since the C_n

stratify $\sigma(a)$, $\exp(\sigma(a)) \cap \exp(C_n)$ is empty, but $\exp(\sigma(a)) = \sigma(\exp(a)) = \sigma(\exp(b)) = \exp(\sigma(b))$, so that $\exp(\sigma(b)) \cap \exp(C_n)$ is empty for all n . If there were a $\lambda \in \sigma(b) \cap C_n$, then e^λ would be in $\exp(\sigma(b)) \cap \exp(C_n)$, an impossibility.

We denote by Log the mapping of the complement of $\exp(C)$ onto U_0 given by $\text{Log } \zeta = \lambda$ where $e^\lambda = \zeta$. Log is one-to-one. If $a \in A$ and $\sigma(a) \subset U_0$, $\text{Log } \exp(a) = a$.

7. Theorem. *Suppose that $\sigma(a)$ is stratified modulo $2\pi i$ and $\exp(a) = \exp(b)$. Then there exists a finite set of idempotents $\{j_\alpha\}$, $j_\alpha \in \mathfrak{A}$, and integers m_α such that $b = a + \sum_\alpha 2\pi i m_\alpha j_\alpha$.*

Proof. By Lemma 6, we know that we can find contours Γ_n in U_n surrounding both $\sigma_n = \sigma(a) \cap U_n$ and $\sigma'_n = \sigma(b) \cap U_n$. Define

$$j_n = \frac{1}{2\pi i} \int_{\Gamma_n} R(\gamma, a) d\gamma, \quad j'_n = \frac{1}{2\pi i} \int_{\Gamma_n} R(\gamma, b) d\gamma.$$

We first assume that $\sigma(b) \subset U_0$, so that, by Theorem 5, a and b commute, and hence the j_n and b commute. Now

$$\begin{aligned} \exp(a j_n - 2\pi n i j_n) &= \exp(a j_n) = j_n \exp(a) + (e - j_n) \\ &= j_n \exp(b) + (e - j_n) = \exp(b j_n). \end{aligned}$$

Both the elements $(a - 2\pi n i)j_n$ and $b j_n$ have their spectra in U_0 , for

$$\sigma((a - 2\pi n i)j_n) \subset (\sigma_n - 2\pi n i) \cup \{0\} \subset U_0 \quad \text{and} \quad \sigma(b j_n) \subset \sigma(b) \cup \{0\};$$

it is here that the technical requirement $0 \notin U_0$ is needed. Since the exponentials of these two elements are equal, their Logs are equal, so that we have $a j_n - 2\pi n i j_n = b j_n$. Summing on n , and noting that the sum of the j_n 's is e and only finitely many are non-zero, we have the theorem in this special case. In the general case, we set $\tilde{a} = \text{Log } \exp(a)$ and apply what we have just proved to a and $\tilde{a}$, then to b and $\tilde{a}$, and obtain

$$a - \sum_n 2\pi n i j_n = \tilde{a} = b - \sum_n 2\pi n i j'_n.$$

We cannot expect that the j_n will always commute with the j'_n .

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