On the Conformal Capacity in Space

CHARLES LOEWNER*

Let D be a domain in Euclidean 3-space with two boundary components C_1 and C_2 . The Newtonian capacity of D is defined as the infimum of the Dirichlet integral

$$(1) \quad \int_D |\nabla u|^2 d\omega$$

for the set of functions u continuously differentiable in D which have boundary values 0 and 1 on C_1 and C_2 respectively. ∇u represents the gradient of u and $d\omega$ the spatial volume element. The capacity is invariant under the group of all (proper and improper) motions. It has the dimension of a length.

For some geometrical considerations it is convenient to replace the above Dirichlet integral by

$$(2) \quad \int_D |\nabla u|^3 d\omega$$

and define the capacity as the infimum of (2) for the same class of functions. One sees easily that the new integral is invariant under the groups of all conformal (Möbius) transformations of space. We may then operate in the Möbius space (Euclidean space plus point at infinity) and admit also domains D which have ∞ in its interior. The new capacity is dimensionless. Because of its conformal invariance we will call it the *conformal capacity* of D .

We will now prove the following

Theorem 1. *The conformal capacity of a domain D with two boundary components C_1 and C_2 is positive if and only if neither component degenerates into a point.*

Proof. Suppose that neither of the boundary components consists only of one point. Then we can evidently fix a finite point O in space such that two different spheres with center O , say with radii r_1 and r_2 ($0 < r_1 < r_2$), intersect with both C_1 and C_2 . Because C_1 and C_2 are continua every sphere S_r with a radius $r \in \langle r_1, r_2 \rangle$ will intersect both boundary components. Let D_r be the

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intersection of S_r with D . Then

$$(3) \quad \int_D |\nabla u|^3 d\omega \geq \int_{r_1}^{r_2} \left(\int_{D_r} |\nabla u|^3 d\sigma \right) dr,$$

$d\sigma$ being the area element of S_r . We wish to estimate the inner integral of (3). Let us first assume that an S_r meets C_1 and C_2 in a pair of diametrically opposed points, which we make the north and south poles of a spherical coordinate system on S_r . Each half meridian contains an arc in D_r in whose end points u has the values 0 and 1 respectively. Calling the polar distance ϑ and ϑ_1 and ϑ_2 the values of ϑ at the end points of the arc and, finally, $\nabla' u$ the gradient of u on S_r , we have

$$(4') \quad \int_{\vartheta_1}^{\vartheta_2} |\nabla' u|^3 \sin \vartheta d\vartheta \geq \frac{1}{r^3} \int_{\vartheta_1}^{\vartheta_2} \left| \frac{\partial u}{\partial \vartheta} \right|^3 \sin \vartheta d\vartheta.$$

Use of Hölder's inequality gives further

$$(4'') \quad 1 \leq \int_{\vartheta_1}^{\vartheta_2} \left| \frac{\partial u}{\partial \vartheta} \right| d\vartheta \leq \left(\int_{\vartheta_1}^{\vartheta_2} \left| \frac{\partial u}{\partial \vartheta} \right|^3 \sin \vartheta d\vartheta \right)^{\frac{1}{3}} \left(\int_{\vartheta_1}^{\vartheta_2} \frac{d\vartheta}{\sqrt{\sin \vartheta}} \right)^{\frac{2}{3}}$$

and therefore

$$(4''') \quad \int_{\vartheta_1}^{\vartheta_2} \left| \frac{\partial u}{\partial \vartheta} \right|^3 \sin \vartheta d\vartheta \geq k = \frac{1}{\left(\int_0^\pi \frac{d\vartheta}{\sqrt{\sin \vartheta}} \right)^2}.$$

Combination of (4') and (4''') and subsequent integration with respect to the geographical longitude φ from 0 to 2π gives finally

$$(5') \quad \int_{D_r} |\nabla' u|^3 d\sigma \geq \frac{2\pi k}{r}.$$

Suppose now S_r meets C_1 and C_2 in two points P_1 and P_2 with the spherical distance ω ($0 < \omega \leq \pi$). We perform a *conformal* mapping of S_r onto itself so that these points are mapped into two diametrically opposed points and call the function into which u is transformed u_1 . Because of conformality we have

$$|\nabla' u|^2 d\sigma = |\nabla' u_1|^2 d\sigma_1,$$

$d\sigma_1$ being the transformed area element, and from this

$$(6') \quad |\nabla u|^3 d\sigma \geq |\nabla' u_1|^3 d\sigma_1 \cdot \left(\frac{d\sigma_1}{d\sigma} \right)^{\frac{1}{3}}.$$

We now try to arrange the desired conformal mapping in such a way that the minimum of $d\sigma_1/d\sigma$ becomes as large as possible. A simple consideration shows that this minimax problem is solved by a mapping which transforms the great circle bisecting the geodesic arc connecting P_1 and P_2 again into a great circle and that then

$$\left(\frac{d\sigma_1}{d\sigma}\right)_{\min} = \tan^2 \frac{\omega}{4},$$

and therefore in this case

$$(6'') \quad |\nabla u|^3 d\sigma \geq |\nabla' u_1|^3 d\sigma_1 \cdot \tan \frac{\omega}{4}.$$

Using the inequality (5') for u_1 we obtain finally

$$(5'') \quad \int_{D_r} |\nabla u|^3 d\sigma \geq \frac{2\pi k}{r} \tan \frac{\omega}{4}.$$

We remark now that ω can never sink below a fixed positive constant since C_1 and C_2 have a positive distance. There exists, therefore, a positive constant κ such that

$$(5) \quad \int_{D_r} |\nabla u|^3 d\sigma \geq \frac{\kappa}{r} \quad (r_1 \leq r \leq r_2),$$

and the inequality (3) gives finally

$$(7) \quad \int_D |\nabla u|^3 d\omega \geq \kappa \log \frac{r_2}{r_1}.$$

The right hand side of (7) is a constant which depends only on the geometrical structure of D and not on u . Hence it represents also a lower bound for the conformal capacity of D . This proves the essential part of Theorem 1. That the conformal capacity is zero if C_1 or C_2 degenerates into a point is evident.

We will make an application of Theorem 1 which answers a question of A. PFLUGER on quasiconformal mappings in space. We prove

Theorem 2. *A univalent quasiconformal mapping of class C^1 of the Euclidean space E_3 into itself gives necessarily the whole E_3 as image.*

A continuously differentiable mapping of a domain of E_3 is K -quasiconformal there if at each point the linear approximation mapping transforms a sphere into an ellipsoid whose greater axis has a ratio to the lesser axis not greater than a fixed positive constant $K \geq 1$ for all points of the domain.

Proof. Let u be a continuously differentiable function and let u be subjected to a univalent K -quasiconformal mapping. Call u' the transformed function. An elementary consideration shows then that

$$(8') \quad |\nabla u|^3 d\omega \geq \frac{1}{K^2} |\nabla' u'|^3 d\omega',$$

$d\omega$ and $d\omega'$ being corresponding volume elements. From (8') it follows that also the conformal capacities Γ and Γ' of corresponding domains satisfy the inequality

$$(8'') \quad \Gamma \geq \frac{1}{K^2} \Gamma'.$$

Let us apply this inequality to a mapping satisfying the conditions of the theorem by choosing for the domain the exterior of a sphere. Then one boundary component is the point ∞ and hence $\Gamma = 0$. But then we conclude from (8'') that also $\Gamma' = 0$ and the image domain must, according to Theorem 1, also have a degenerate component which again must be the point ∞ . This proves Theorem 2.

Final remark. Both theorems can be generalized to n dimensions ($n \geq 3$). In the definition of the conformal capacity the integral (2) is to be replaced by

$$\int_D |\nabla u|^n d\omega.$$

The wording and proofs of the theorems remain unchanged.

Stanford University
Stanford, California