On Some Transformation Semigroups Invariant Under Euclidean or Non-Euclidean Isometries*

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It is well known that the only non-trivial invariant subgroups of the group $\mathcal{G}$ of isometries of the Euclidean geometry (proper and improper) which are generated by infinitesimal transformations of $\mathcal{G}$ are the group of translations and the group consisting of the proper transformations of $\mathcal{G}$. If $\mathcal{G}$ is the group of isometries of a non-Euclidean geometry (spherical or hyperbolic) the only invariant subgroup, generated by infinitesimal transformations of $\mathcal{G}$, is the group of proper isometries. On the other hand, the author has shown** that certain *semigroups* associated with the Klein and Poincaré models of the hyperbolic geometry may, in this geometry, be considered as analogues of the translation group of the Euclidean geometry. The transformations of these semigroups do not belong to $\mathcal{G}$. In this paper we shall consider the problem of finding *all* transformation semigroups which share certain properties with the semigroups mentioned. In order to be able to formulate them we need a short review of some definitions and facts regarding transformation semigroups.

(A) A transformation semigroup $\mathcal{S}$ on a manifold V is a system of transformations of V into (not necessarily onto) itself which is closed under composition. Beyond that we shall further assume:

- (1) $\mathcal{S}$ contains the identity transformation.
- (2) All transformations of $\mathcal{S}$ are continuous.
- (3) $\mathcal{S}$ is closed in the topology based on uniform convergence in each compact subset of V .

(B) If V is of class C^1 the concept of an *infinitesimal transformation* (abbreviated as i.t.) can be introduced. It is synonymous with a *continuous contravariant vector field* on V .

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** C. LOEWNER, On some Transformation Semigroups, *Journal of Rational Mechanics and Analysis*, 5 (1956), pp. 791-804.

(C) With each semigroup $\mathfrak{S}$ defined on a manifold of class C^1 , certain i.t.'s can be associated. They are called the i.t.'s of $\mathfrak{S}$. The set $\mathfrak{s}$ of all i.t.'s of $\mathfrak{S}$ is called the *infinitesimal semigroup* of $\mathfrak{S}$. An i.t. of $\mathfrak{S}$ is defined in the following way: Let σ_n ($n = 1, 2, \dots$) be a sequence of transformations of $\mathfrak{S}$ which converge in each compact set in V uniformly to the identity transformation and a_n ($n = 1, 2, \dots$) a sequence of non-negative numbers. Let N be an open set of V in which coordinates $x^1, x^2, \dots, x^m$ can be introduced* and C a compact subset of N . If n is sufficiently large the image $\sigma_n(C)$ will lie in N and the mapping can be represented in C by m functions $\bar{x}_n^i = f_n^i(x^1, x^2, \dots, x^m)$ with $x^1, x^2, \dots, x^m$ the coordinates of a point p in C and $\bar{x}_n^1, \bar{x}_n^2, \dots, \bar{x}_n^m$ the coordinates of its image $\bar{p}_n = \sigma_n(p)$. Suppose now that the limits

$$\lim_{n \rightarrow \infty} a_n(\bar{x}_n^i - x^i) = \xi^i(x^1, x^2, \dots, x^m) \quad (i = 1, 2, \dots, m)$$

exist and the convergence is uniform on C for every choice of N and C . It is then easily shown that if one changes coordinates** the ξ^i transform like components of a contravariant vector ξ . Any vector field $\xi(p)$ ($p \in V$) obtained in this way is called an i.t. of $\mathfrak{S}$.

One proves easily that the sum $\xi_1 + \xi_2$ of two i.t.'s of $\mathfrak{S}$ and a scalar multiple $a\xi$ of an i.t. of $\mathfrak{S}$ with a non-negative scalar $a \geq 0$ are again i.t.'s of $\mathfrak{S}$. This can be expressed by saying that the i.t.'s of $\mathfrak{S}$ form a *convex cone* in the vector space formed by all i.t.'s on V with the usual definition of addition and multiplication by scalars. One can further easily prove that $\mathfrak{s}$ is a *closed set* in this space if its topology is based on uniform convergence in each compact set of V .

(D) If $\xi(p)$ is an i.t. on a manifold V we may consider the differential equation $dp/dt = \xi(p)$. By this we mean, if we operate in a coordinate system $x^1, x^2, \dots, x^m$, the system of differential equations $dx^i/dt = \xi^i(x^1, x^2, \dots, x^m)$ ($i = 1, 2, \dots, m$). If we assume that V is of class C^2 and $\xi(p)$ is of class C^1 we know that there exists a unique solution for any initial point $p_0(x_0^1, \dots, x_0^m)$ at $t = 0$ in some interval $< 0, t_1 >$ ($t_1 > 0$) with a t_1 which might depend on p_0 . If t_1 can be chosen independent of p_0 the transition from p_0 to the value of the corresponding solution for $t = t_1$ represents a mapping $p = \sigma_{t_1}(p_0)$. One can then show that $\sigma_t(p_0)$ exists for any choice of $t \geq 0$. We call the transformations $\sigma_t(p_0) = p_t$ ($t \geq 0$) the transformations *generated by the i.t. $\xi(p)$* .

Now let V be of class C^2 and $\mathfrak{s}$ a closed convex cone of i.t.'s in V , all of class C^1 . We assume further that all $\xi \in \mathfrak{s}$ are such that we can speak of transformations generated by ξ . We form all transformations obtained as composites of a finite number of transformations generated by i.t.'s of $\mathfrak{s}$ and limits of such composites in the sense of uniform convergence on any compact set in V . They evidently form a semigroup $\mathfrak{S}$. We call it the *semigroup generated by $\mathfrak{s}$* .

It is clear that all i.t.'s of $\mathfrak{s}$ belong to the infinitesimal semigroup of $\mathfrak{S}$, but it

* m is the dimension of V .

** Since V is of class C^1 the transition from one coordinate system to the other is of class C^1 .

can happen that the latter is bigger than $\mathfrak{s}$. On the other hand, if all i.t.'s of a given semigroup $\mathfrak{S}$, on a manifold of class C^2 , are of class C^1 they generate transformations of $\mathfrak{S}$. This follows from the fact that the solution of the equation $dp/dt = \xi(p)$ gives transformations which, by the nature of the equation, are limits of composites of transformations of $\mathfrak{S}$.

We will now formulate the problem treated in this paper. Let V_m be the space of the m -dimensional Euclidean or a non-Euclidean geometry and $\mathfrak{G}$ the group of its isometries. We want to find all systems $\mathfrak{s}$ of i.t.'s on V_m which satisfy the following conditions:

- (a) $\mathfrak{s}$ is a *closed convex cone of i.t.'s* not consisting only of $\xi = 0$.
- (b) $\mathfrak{s}$ is *invariant* under $\mathfrak{G}$.
- (c) $\mathfrak{s}$ is *finite dimensional*.
- (d) $\mathfrak{s}$ is *minimal*. By this we mean that there is no proper subcone of $\mathfrak{s}$ satisfying (a) and (b).

We will find all solutions of this problem for arbitrary dimensions in the Euclidean case and in non-Euclidean geometries for $m = 1, 2$. It will be shown that in the Euclidean geometry there is only one solution for $m > 1$, namely the translation group, and a continuum of solutions for $m = 1$. In the spherical geometry ($m = 1$ or 2) there is a denumerable number of solutions. This is also the case for the hyperbolic geometry of two dimensions. The one-dimensional case can be disregarded since its isometry group is similar to that of the Euclidian line.

Of special interest are those solutions which generate *semigroups proper*. There is a one-parameter family of such solutions in the one-dimensional Euclidean and hyperbolic geometry, but they are all transformable into each other by simple transformations. Further, there exists a *denumerable* number of such solutions in the two-dimensional hyperbolic geometry. All other solutions are linear spaces and generate, therefore, *groups*.

The last paragraph of this paper contains some sketchy remarks about the generated semigroups. A detailed discussion will be presented in another paper.

1. The Euclidean Case. We operate in a Euclidean m -space E_m in which cartesian coordinates $x = (x^1, x^2, \dots, x^m)$ have been introduced. An i.t. ξ is given by its m components $\xi^i(x)$. We start with the case $m = 1$ and can then drop the index. The group consists of the transformations $x' = \pm x + a$ where a is an arbitrary real number. An i.t. $\xi(x)$ transforms like a scalar under the translations $x' = +x + a$ and changes only its sign if it is subjected to an improper transformation $x' = -x + a$. Let now $\mathfrak{s}$ be a solution of our problem and $\xi(x)$ an element of $\mathfrak{s}$. Then $\xi(x + a)$ and all linear combinations of functions $\xi(x + a)$ with non-negative coefficients belong to $\mathfrak{s}$. But $\mathfrak{s}$ is supposed to be *finite dimensional*. This is only possible if also the linear combinations of the $\xi(x + a)$ with *arbitrary real* coefficients form a finite dimensional space. We need now the following

Lemma 1'. *If a function $f(x)$, continuous on $-\infty < x < \infty$, is such that its transforms $f(x + a)$ by all translations $x^1 = x + a$ span a finite dimensional space then it is of the form*

$$(1) \quad f(x) = \sum_{\nu=1}^k P_{\nu}(x) e^{-\gamma_{\nu} x}$$

where the $P_{\nu}(x)$ are polynomials and the γ_{ν} constants.*

Although the lemma may be considered as known we give a proof of it here. Let p be the dimension of the space L spanned by the $f(x + a)$ and let us first assume that $f(x)$ is of class C^p . Then the difference quotient $g_a(x) = [f(x + a) - f(x)]/a$ must also belong to L and therefore also its limit $f'(x)$. If $p > 1$ we operate in the same way with $f'(x)$ and obtain the result that $f''(x)$ belongs to L . By p such steps we arrive at the result that $f(x), f'(x), \dots, f^{(p)}(x)$ all belong to L . But L is p -dimensional and therefore these derivatives are linearly dependent, in other words, $f(x)$ satisfies a linear homogeneous differential equation with constant coefficients. As a consequence $f(x)$ has form (1).

Now let us only assume that $f(x)$ is continuous. We form with an arbitrary $\epsilon > 0$ the functions

$$(2) \quad f_{\epsilon}(x) = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(x + a) da.$$

They evidently also belong to L , but are of class C^1 . We apply the averaging process (2) p times, choosing ϵ arbitrarily at each step. Call the chosen ϵ 's $\epsilon_1, \epsilon_2, \dots, \epsilon_p$ and the resulting function $f_{\epsilon_1, \epsilon_2, \dots, \epsilon_p}(x)$. It again belongs to L and is of class C^p . Applying the previous result we see that $f_{\epsilon_1, \epsilon_2, \dots, \epsilon_p}(x)$ is of the form (1), the P_{ν} , γ_{ν} , and k depending on the choice of the ϵ 's. But it is clear that only a finite number of constants γ_{ν} can occur and the degrees of the P_{ν} must lie below a fixed limit. Otherwise L would not be of finite dimension. We now perform successively the limiting processes $\epsilon_p \rightarrow 0, \epsilon_{p-1} \rightarrow 0, \dots, \epsilon_1 \rightarrow 0$ and obtain the result that $f_{\epsilon_1, \epsilon_2, \dots, \epsilon_p}(x), f_{\epsilon_1, \epsilon_2, \dots, \epsilon_{p-1}}(x), \dots$, and finally $f(x)$ itself must have the form (1).

By application of the lemma to the elements of $\mathfrak{S}$ we see they all must be of the form (1). We conclude further, from the finite dimensionality of L , that the γ_{ν} appearing in their representations belong to a fixed finite set of numbers and that the degrees of the polynomials P_{ν} are uniformly bounded. We write now

$$\gamma_{\nu} = \alpha_{\nu} + i\beta_{\nu} \quad (\alpha_{\nu}, \beta_{\nu} \text{ real})$$

and designate with α the maximal α_{ν} . Further, let ξ be an element of $\mathfrak{S}$ such that not all P_{ν} with $\alpha_{\nu} = \alpha$ vanish identically and designate the maximal degree of these distinguished P_{ν} with l . We form

*The γ_{ν} and the coefficients of the P_{ν} might be complex. If f is real, then with each complex term the complex conjugate term must also occur.

$$\xi_a = \frac{\xi(x+a)}{e^{\alpha a} a^i} \quad (a > 0)$$

and let $a \rightarrow \infty$. Each $\lim_{a \rightarrow \infty} \xi_a(x) = \xi^*(x)$ is then an element of $\mathfrak{s}$ of the form $\xi^*(x) = \sum c_\mu e^{\gamma_\mu x}$ in which all c_μ are non-zero constants and all γ_μ have real part α . Since $\xi^*(x)$ is real, with each term $c_\mu e^{\gamma_\mu x}$, $\bar{c}_\mu e^{\bar{\gamma}_\mu x}$ will also occur. Let β be the maximum of all β_μ . If $\beta > 0$ we form

$$\xi^{**}(x) = \int_0^{2\pi/\beta} e^{-\alpha a} \xi^*(x+a) da$$

which is again an element of $\mathfrak{s}$. By the integration process the terms belonging to $\gamma = \alpha + i\beta$ are eliminated, but none of the other ones. By possible continuation of this process we obtain finally the result that $\mathfrak{s}$ contains either one of the elements $\pm e^{\alpha x}$ or an element of the form $\operatorname{Re}(e^{(\alpha+i\beta)x})$ ($\beta > 0$). If one applies to these elements again all transformations of $\mathfrak{G}$ and forms their convex hull one obtains easily

Theorem 1'. *A solution $\mathfrak{s}$ of our problem consists of elements ξ of one of the following types:*

$$(a) \quad \xi = \operatorname{Re} (c_1 e^{(\alpha+i\beta)x} + c_2 e^{-(\alpha+i\beta)x})$$

(α, β fixed positive constants, c_1 and c_2 arbitrary complex constants),

$$(b) \quad \xi = \operatorname{Re} (ce^{i\beta x})$$

(β a fixed positive constant, c an arbitrary complex constant),

$$(c) \quad \xi = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$$

(α a fixed positive constant, $c_1 \geq 0, c_2 \leq 0$ or $c_1 \leq 0, c_2 \geq 0$),

$$(d) \quad \xi = c$$

(c arbitrary real) (infinitesimal translations).

Remark. (a), (b) and (d) are linear sets, but not (c). By the transformation $x' = e^{\alpha x}$ it goes into an infinitesimal semigroup, mentioned in the paper quoted in the introduction, or its opposite.

We will now assume that the dimension m is > 1 . First we need a generalization of Lemma 1' to arbitrary dimensions.

Lemma 1. *Lemma 1' remains true if we replace x, a and γ , by m -dimensional vectors and read γ, x as the scalar product of γ , and x .*

The proof is almost immediate by application of Lemma 1' to translations in coordinate directions.

We need a second lemma in which the group of translations $x' = x + a$ in V_m is replaced by the group $\mathfrak{G}$ of all isometries of V_m .

Lemma 2. *If a continuous function in V_m is such that its transforms by all elements of $\mathfrak{G}$ span a finite dimensional space then $f(x)$ is a polynomial in the components x^i of x .*

Proof. Since $\mathfrak{G}$ contains the group of translations $f(x)$ must be of the form (1). If a rotation is applied to the space, $f(x)$ keeps its form (1) with vectors γ , obtained from the original ones by application of the rotation. From this one concludes that all $\gamma_v = 0$. Otherwise the transforms of $f(x)$ would span a space of infinite dimension.

From Lemma 2 we derive

Lemma 3. *Let ξ be an i.t. in V_m whose transforms span a finite dimensional space. Then its components ξ^i are polynomials in the x^i ($i = 1, \dots, m$).*

Proof. Form the scalar products $\alpha\xi$ with all constant vectors α . Their transforms span a finite dimensional space. Apply now Lemma 2.

From Lemma 3 follows immediately

Lemma 4. *Let $\mathfrak{s}$ be a solution of our problem. Then all $\xi \in \mathfrak{s}$ are vector polynomials on the x^i with uniformly bounded degrees.*

That the degrees are uniformly bounded is a consequence of the finite dimensionality of $\mathfrak{s}$.

Now let ξ be a non-zero element of $\mathfrak{s}$ of degree k . This should mean that its components are of degree $\leq k$ and at least one of them is of degree k . By subjecting ξ to a suitable rotation we can achieve that ξ^1 is of degree k , even in the single variable x^1 . Then apply a translation to ξ in the x^1 -direction and form

$$\xi(x) = \frac{\xi(x^1 + a, x^2, \dots, x^m)}{a^k} \quad (a > 0)$$

which again is an element of $\mathfrak{s}$. The limiting process $a \rightarrow \infty$ then leads to an element of $\mathfrak{s}$ with constant components, the first component not being zero, i.e. a *non-zero infinitesimal translation*. By subjecting this translation to all rotations and multiplications by non-negative scalars, we obtain the fact that $\mathfrak{s}$ contains *all* infinitesimal translations. Since $\mathfrak{s}$ is *minimal* we obtain the fact that $\mathfrak{s}$ is the infinitesimal translation group. We summarize the result in

Theorem 1". *For dimension $m > 1$ the infinitesimal translation group is the only solution of our problem in the Euclidian geometry of dimension m .*

2. The Case of Spherical Geometry for $m = 1, 2$. If $m = 1$ we may operate on a circle: $x^1 = \cos \theta$, $x^2 = \sin \theta$, and an infinitesimal transformation is given by a continuous function $\xi(\theta)$ with period 2π . The group $\mathfrak{G}$ of isometries consists of the transformations $\bar{\theta} = \pm \theta + a$. Applying Theorem 1' and the periodicity of all $\xi(\theta)$ we obtain almost immediately

Theorem 2. *We have a denumerable number of solutions*

$$(3) \quad \mathfrak{s}_n : \xi = a \cos n\theta + b \sin n\theta \quad (n = 0, 1, 2, \dots)$$

(a, b arbitrary real constants).

We go over to $m = 2$. We may operate on the 2-dimensional sphere $V_2 : (x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ imbedded in the 3-dimensional Euclidian space E_3 . $\mathfrak{G}$ is the group of orthogonal transformations in E_3 applied to V_2 . An i.t. on V is described by a continuous field of vectors ξ of the imbedding space attached to points of V_2 and tangential to it.

For solving our problem we need again a lemma analogous to Lemma 1 in the Euclidean geometry.

Lemma 5. *If the transforms of a continuous function f on V_2 by $\mathfrak{G}$ span a finite dimensional space than f coincides on V_2 with a polynomial in x^1, x^2, x^3 .*

Proof. We expand f formally in spherical harmonics $f \sim \sum_{k=0}^{\infty} \varphi_k$. If f is subjected to a transformation of $\mathfrak{G}$ the same happens with each φ_k . As is well known, the transforms of a spherical harmonic $\varphi_k \neq 0$ span the whole space $\mathfrak{S}_k$ of spherical harmonics of degree k , which is of dimension $2k + 1$. Hence $2k + 1$ cannot exceed the dimension of the space spanned by the transforms of f . This shows that only a finite number of the φ_k occur in the expansion. This proves the lemma.

From Lemma 5 we derive

Lemma 6. *Let ξ be an i.t. on V_2 whose transforms span a finite dimensional space. Then its components ξ^i ($i = 1, 2, 3$) in E_3 are representable on V_2 as polynomials in x^1, x^2, x^3 .*

Proof. Form the scalar products $\alpha\xi$ with all constant vectors α . Their transforms again span a finite dimensional space. Now apply Lemma 5.

From Lemma 6 follows immediately

Lemma 7. *Let $\mathfrak{s}$ be a finite dimensional invariant convex cone of i.t.'s in V_2 . Then all $\xi \in \mathfrak{s}$ can be represented on V_2 as vector polynomials in x^1, x^2, x^3 with uniformly bounded degrees.*

That the degrees are uniformly bounded is a consequence of the finite dimensionality of $\mathfrak{s}$.

We will now construct infinitely many solutions of our problem utilizing well known properties of spherical harmonics. As was mentioned already, the spherical harmonics of degree n form a space $\mathfrak{S}_n$ of dimension $2n + 1$. If $\varphi_n \in \mathfrak{S}_n$ we form the vector field

$$(4) \quad \xi = r^{n+2} \nabla \frac{\varphi_n}{r^n} \quad \text{on} \quad r = ||\vec{r}|| = 1,$$

$\vec{r}$ being the radius vector of the point (x^1, x^2, x^3) . Since φ_n/r^n is homogeneous

of order zero its radial component vanishes and (4) represents a tangential vector field on $r = 1$, i.e. on V_2 .* The fields (4) form a linear space $\mathfrak{s}_n$ of dimension $2n + 1$. We prove

Lemma 8. $\mathfrak{s}_n$ is a solution of our problem.

Proof. One knows that if φ_n is any non-zero spherical harmonic the linear combinations of its transforms by $\mathfrak{G}$ even with non-negative coefficients span the total $\mathfrak{G}_n$. Hence no proper invariant subcone of $\mathfrak{s}_n$ can exist.

We now construct solutions in the following way: Take a $\xi \in \mathfrak{s}_n$ and subject it at each point of V_2 to a rotation by $\frac{1}{2}\pi$ in its tangent plane. From (4) we obtain in this way the i.t.

$$(5) \quad \xi = \nabla \varphi_n \times \vec{r} \quad \text{on } V_2$$

where the usual symbol $\alpha \times \beta$ is used for the cross product of the vectors α and β .

We have further

Lemma 9. The i.t.'s (5) form a solution $\mathfrak{t}_n$ of our problem.

Indeed $\mathfrak{t}_n$ is evidently invariant under proper transformations of $\mathfrak{G}$. But this is also the case with improper transformations which change the angle $\frac{1}{2}\pi$ into $-\frac{1}{2}\pi$, since with ξ also $-\xi$ belongs to $\mathfrak{t}_n$. Evidently $\mathfrak{t}_n$ is also minimal.

As preparation for the proof that the $\mathfrak{s}_n$ and $\mathfrak{t}_n$ ($n = 1, 2, \dots$) exhaust all possibilities we need some further lemmas. First we are interested in whether the spaces represented by the $\mathfrak{s}_n$ and $\mathfrak{t}_n$ are linearly independent. In order to verify this we introduce a metric in the space of the i.t.'s on V_2 based on the following definition of a scalar product of two i.t.'s ξ and η :

$$(6) \quad (\xi, \eta) = \int_{V_2} \xi(x) \cdot \eta(x) \, d\omega$$

where the integrand is the ordinary scalar product of ξ and η at the point $x \in V_2$ and $d\omega$ designates the ordinary area element of V_2 . The norm of ξ in the sense of definition (6) is $\sqrt{(\xi, \xi)}$ and is zero only if $\xi = 0$. The linear independence follows in a well known way from

Lemma 10. Any two of the systems of spaces $\mathfrak{s}_n, \mathfrak{t}_n$ ($n = 1, 2, \dots$) are orthogonal to each other, the orthogonality concept being based on definition (6).

Proof. Let us first consider two spaces $\mathfrak{s}_p, \mathfrak{s}_q$ ($p \neq q$). We have to show that for any two harmonics $\varphi_p \in \mathfrak{S}_p, \varphi_q \in \mathfrak{S}_q$

$$(7') \quad \int_{V_2} \nabla \frac{\varphi_p}{r^p} \cdot \nabla \frac{\varphi_q}{r^q} \, d\omega = 0.$$

* It represents the intrinsic gradient field of φ_n on V_2 .

This means

$$(7'') \quad \int_{V_2} (\nabla \varphi_p - p \varphi_p \vec{r}) \cdot (\nabla \varphi_q - q \varphi_q \vec{r}) d\omega = 0$$

or, by use of the homogeneity of φ_p and φ_q of orders p and q respectively,

$$(7) \quad \int_{V_2} \nabla \varphi_p \cdot \nabla \varphi_q d\omega - pq \int_{V_2} \varphi_p \varphi_q d\omega = 0.$$

But it is known that the second integral in (7) vanishes since $p \neq q$ and, on the same account, the first integral too since the derivative of a spherical harmonic is also a spherical harmonic, with an order decreased by one.

The orthogonality of t_p and t_q ($p \neq q$) follows immediately from the orthogonality of $\mathfrak{s}_p$ and $\mathfrak{s}_q$ by use of the fact that rotation by the same angle of two vectors ξ and η in the tangent plane of V_2 does not change the scalar product $\xi \cdot \eta$.

We have finally to show that any two spaces $\mathfrak{s}_p$ and t_q (p and q arbitrary) are orthogonal to each other. This means that for any $\varphi_p \in \mathfrak{S}_p$, $\varphi_q \in \mathfrak{S}_q$

$$(8') \quad \int_{V_2} \nabla \frac{\varphi_p}{r^p} \cdot \nabla \varphi_q \times \vec{r} d\omega = \int_{V_2} \left(\nabla \varphi_p, \nabla \varphi_q, \nabla \frac{r^2}{2} \right) d\omega = 0$$

where the symbol (α, β, γ) for the determinant of the 3 times 3 components of 3 spatial vectors α, β, γ is used. The homogeneity of φ_p and φ_q allows us to write (8') in the form

$$(8) \quad \int_{|\vec{r}| \leq 1} \left(\nabla \varphi_p, \nabla \varphi_q, \nabla \frac{r^2}{2} \right) dv = 0,$$

the integral to be extended over the solid unit sphere with the ordinary volume element dv . Indeed, the integrand is homogeneous of order $p + q - 1$ and therefore integral (8') differs from integral (8) by the positive factor $p + q$. But (8) is true because the left side represents the algebraic volume of the map of the unit sphere by the transformation

$$(9) \quad \bar{x}^1 = \varphi_p, \quad \bar{x}^2 = \varphi_q, \quad \bar{x}^3 = \frac{r^2}{2}$$

and $\bar{x}^3$ is *constant* on the boundary.

We know that a solution $\mathfrak{s}$ consists of i.t.'s representable by vector polynomials of uniformly bounded degrees. Therefore we introduce the spaces $\mathfrak{P}_n$ consisting of all i.t.'s described by vector polynomials of degree $\leq n$. We will prove

Lemma 11. $\mathfrak{P}_n$ is the direct sum

$$(10) \quad \mathfrak{P}_n = \sum_{i=1}^{n-1} \mathfrak{s}_i + \sum_{j=1}^n t_j.$$

Proof. It is clear that all the terms on the right-hand side of (10) are included

in $\mathfrak{P}_n$. Since they are linearly independent it is sufficient to show that the dimension of $\mathfrak{P}_n$ is equal to the sum of the dimensions of the right-hand members, which is

$$(11) \quad \sum_{i=1}^{n-1} (2i+1) + \sum_{j=1}^n (2j+1) = 2n^2 + 2n - 1.$$

In the computation of the dimension of $\mathfrak{P}_n$ we have to pay attention to the fact that the representation of $\xi \in \mathfrak{P}_n$ on V_2 by a polynomial is not unique. By adding to each term of ξ whose degree is $< n-1$ a suitable power of $||\vec{r}||^2$ we may raise its degree to n or $n-1$ without changing its value on V_2 . Hence *one may assume* that ξ contains only terms of degrees n and $n-1$. But then the representation is unique. We further have to take into account that $\xi \cdot \vec{r} \equiv 0$ on V_2 and is therefore identically zero in space. Hence the number of independent coefficients in ξ is

$$(12) \quad 3 \left[\frac{(n+1)(n+2)}{2} + \frac{n(n+1)}{2} \right] - \frac{(n+2)(n+3)}{2} - \frac{(n+1)(n+2)}{2} = 2n^2 + 2n - 1$$

which coincides with (11).

We are now able to determine all finite dimensional invariant closed convex cones of i.t.'s, dropping first the condition of minimality. We will prove

Lemma 12. *Every finite dimensional invariant closed convex cone $\mathfrak{s}$ is the direct sum of a finite number of the $\mathfrak{s}_n$ and $\mathfrak{t}_n$ and therefore a linear space.*

This can be seen in the following way: We know that $\mathfrak{s}$ lies completely in a $\mathfrak{P}_n$ with sufficiently high index. Call the right-hand members of (10) in some order $\mathfrak{v}_i$ ($i = 1, 2, \dots, 2n-1$). Every element $\xi \in \mathfrak{s}$ can be resolved uniquely according to

$$(13') \quad \xi = \sum_{i=1}^{2n-1} \xi_i \quad (\xi_i \in \mathfrak{v}_i, i = 1, 2, \dots, 2n-1).$$

The components ξ_i for a fixed i taken for all $\xi \in \mathfrak{s}$ again form an invariant cone included in $\mathfrak{v}_i$. Because of the minimality of $\mathfrak{v}_i$ they are all zero or fill out the complete $\mathfrak{v}_i$. Call those $\mathfrak{v}_i$ which are completely filled out

$$(14) \quad \mathfrak{w}_1, \mathfrak{w}_2, \dots, \mathfrak{w}_m \quad m \leq 2n-1.$$

We order them according to dimension

$$(15) \quad \dim \mathfrak{w}_1 \leq \dim \mathfrak{w}_2 \leq \dots \leq \dim \mathfrak{w}_m.$$

Equation (13') can be replaced by

$$(13) \quad \xi = \sum_{i=1}^m \omega_i \quad (\omega_i \in \mathfrak{w}_i).$$

We apply complete induction with respect to m . If $m = 1$ the lemma coincides with Lemmas 8 and 9. If $m > 1$ call, for the sake of brevity,

$$(16) \quad \sum_{i=1}^m \mathfrak{w}_i = \mathfrak{B}, \quad \sum_{i \neq j} \mathfrak{w}_i = \mathfrak{B}_j$$

and write (13) in the form

$$(17) \quad \xi = \omega_1 + \eta_1 \quad (\omega_1 \in \mathfrak{w}_1, \eta_1 \in \mathfrak{B}_1).$$

The components η_1 of all $\xi \in \mathfrak{s}$ again form an invariant cone for which $\mathfrak{B}_1$ plays the same role as $\mathfrak{B}$ for $\mathfrak{s}$, but m is replaced by $m - 1$. Applying complete induction we can assert that the η_1 's fill out the complete space $\mathfrak{B}_1$.

For any element $\xi \in \mathfrak{s}$ we can find another element $\bar{\xi} \in \mathfrak{s}$ such that

$$(18) \quad \bar{\xi} = -\omega_1 + \bar{\eta}_1.$$

There are now two possibilities.

(a) For some $\xi \in \mathfrak{s}$ the sum $\eta_1 + \bar{\eta}_1 \neq 0$. We find then in $\xi + \bar{\xi} = \eta_1 + \bar{\eta}_1$ an element of the intersection $\mathfrak{s} \cap \mathfrak{B}_1$ which is *not zero*. Using complete induction we can assert that $\mathfrak{s} \cap \mathfrak{B}_1$ is the direct sum of some $\mathfrak{w}_j$ ($j > 1$). Take one of them, say $\mathfrak{w}_k$ ($k > 1$) and write any element of $\mathfrak{s}$ in the form

$$(17') \quad \xi = \omega_k + \eta_k \quad (\omega_k \in \mathfrak{w}_k, \eta_k \in \mathfrak{B}_k).$$

Since ω_k and also $-\omega_k$ belong to $\mathfrak{s}$, we see that $\xi + (-\omega_k) = \eta_k$ must also fall into $\mathfrak{s}$. Again using complete induction we can assert that the η_k fill out the complete $\mathfrak{B}_k$ which proves that $\mathfrak{s} = \sum_{i=1}^m \mathfrak{w}_i$.

(b) For all ξ the sum $\eta_1 + \bar{\eta}_1 = 0$ or $\bar{\eta}_1 = -\eta_1$. From this two consequences can be drawn:

(b₁) With ξ also $-\xi$ belongs to $\mathfrak{s}$, so that $\mathfrak{s}$ is a linear space.

(b₂) η_1 is uniquely determined by ω_1 and the mapping $\omega_1 \rightarrow \eta_1$ is a linear mapping of $\mathfrak{w}_1$ onto the whole $\mathfrak{B}_1$. From (b₁) and (b₂) we conclude the inequality

$$(19) \quad \dim \mathfrak{w}_1 \geq \dim \mathfrak{B}_1 = \dim \mathfrak{w}_2 + \dim \mathfrak{w}_3 + \cdots + \dim \mathfrak{w}_m.$$

This is compatible with (15) only if $m = 2$ and $\dim \mathfrak{w}_1 = \dim \mathfrak{w}_2$. Further, the pair $\mathfrak{w}_1, \mathfrak{w}_2$ must be identical with a pair $\mathfrak{s}_j, \mathfrak{t}_j$ since $\dim \mathfrak{s}_j = \dim \mathfrak{t}_j = 2j + 1$. We have to show that both $\mathfrak{s}_j$ and $\mathfrak{t}_j$ are contained in $\mathfrak{s}$. We write an element of $\mathfrak{s}$ in the form

$$(20) \quad \xi = \omega_1 + \omega_2$$

where ω_1 and ω_2 are of the form

$$(21) \quad \omega_1 = r^{j+2} \nabla \frac{\varphi_j^1}{r^j}, \quad \omega_2 = \nabla \varphi_j^2 \times \vec{r}$$

with suitable harmonics φ_j^1, φ_j^2 of order j . We perform now the *improper transformation* $\bar{x}^i = -x^i$. If j is even, ω_1 does not change, but the cross-product ω_2 changes its sign. If j is odd, ω_1 changes its sign and ω_2 remains unchanged.

In each case the arithmetic mean of ξ and the transformed element is either ω_1 or ω_2 . This shows that either w_1 or w_2 belongs to $\mathfrak{s}$, and hence both belong to $\mathfrak{s}$ since we assumed that $m = 2$. Lemma 12 is now completely proved.

If one adds the minimality condition one obtains the final

Theorem 2. *The $\mathfrak{s}_n$ and $\mathfrak{t}_n$ ($n = 1, 2, \dots$) are the only solutions of our problem for the two-dimensional spherical geometry.*

We conclude this section with the following

Remark. One may pose the same problem for the *elliptic* geometry. The latter is obtained from the spherical one by identifying two opposite points on V_2 . It is easy to see that the $\mathfrak{s}_n$ with an *even* n and the $\mathfrak{t}_n$ with an *odd* n are the only solutions.

3. The Case of Hyperbolic Geometry in Two Dimensions. We will use the following model of the hyperbolic geometry, which is essentially KLEIN's model in homogeneous coordinates: We operate in the 3-space E_3 with coordinates x^1, x^2, x^3 and introduce in it the *Lorentzian* metric, in which the scalar product of two vectors $\xi = (\xi^1, \xi^2, \xi^3)$ and $\eta = (\eta^1, \eta^2, \eta^3)$ is given by

$$(22) \quad \xi \cdot \eta = \xi^3 \eta^3 - \xi^1 \eta^1 - \xi^2 \eta^2.$$

The Lorentzian norm $||\xi||$ of a vector ξ is then given by

$$(23) \quad ||\xi||^2 = (\xi^3)^2 - (\xi^1)^2 - (\xi^2)^2.$$

We will operate only with vectors for which (23) is *non-negative* and call the positive square root of (23) the norm of ξ . If further $\xi^3 \geq 0$, ξ is called *timelike*, if the terminology of relativity theory is adopted. Only timelike vectors will occur in this paper. The *unit sphere* of this geometry consists of those points whose radius vectors are timelike vectors of unit norm. We obtain a model of the hyperbolic geometry if we take the unit sphere V_2 as the plane of the geometry and introduce as group $\mathfrak{G}$ the group of Lorentz transformations, *i.e.* the group of linear homogeneous transformations in E_3 which keep V_2 invariant. A vector field in our geometry is represented by a vector field in E_3 distributed over V_2 and tangential to it.

We start with a few lemmas which are analogous to those discussed in the spherical case.

Lemma 13. *If a continuous function $f(x)$ on V_2 is subjected to all transformations of $\mathfrak{G}$ and the transforms span a finite dimensional space then $f(x)$ is representable on V_2 as a polynomial in x^1, x^2, x^3 .*

Lemma 14. *Let ξ be an i.t. on V_2 whose transforms by $\mathfrak{G}$ span a finite dimensional subspace of the space of all i.t.'s on V_2 . Then its components ξ^i ($i = 1, 2, 3$) in E_3 are representable on V_2 by polynomials in x^1, x^2, x^3 .*

Lemma 15. *If a finite dimensional closed convex cone $\mathfrak{s}$ of i.t.'s in V_2 is invariant*

under $\mathfrak{G}$ then for all $\xi \in \mathfrak{s}$ the conclusion of Lemma 14 holds with uniformly bounded degrees.

Since Lemmas 14 and 15 are almost immediate consequences of Lemmas 13 only the latter will be proved.

Proof. The proof seems to appear in the simplest form if one uses the Poincaré model of the hyperbolic geometry in the upper half of the complex $z = x + iy$ -plane: $y > 0$. The group $\mathfrak{G}$ consists then of the transformations $z' = (\alpha z + \beta)/(\gamma z + \delta)$ with real coefficients $\alpha, \beta, \gamma, \delta$ and positive determinant

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix},$$

combined also with $z' = -\bar{z}$. Let $f(x, y)$ be a function in the upper half-plane satisfying the conditions of Lemma 13 and subject it first only to the translations $z' = z + \beta$ (β real). Since each line $y = \text{constant}$ is translated into itself, Lemma 1' can be applied and we see that f must have the form

$$(24) \quad f = \sum P_\nu(x, y) e^{\gamma_\nu x/y}$$

where the P_ν are polynomials in x for each y . We wrote the exponents in the form $\gamma_\nu x/y$ in order to show the γ_ν may be considered as a finite system of mutually different fixed constants. Indeed, the transformations $z = \alpha z$ ($\alpha > 0$) from $\mathfrak{G}$ transform the lines $y = \text{constant}$ among themselves transitively and do not change the quotient x/y . If variation of y would lead to infinitely many γ_ν , the transform of f would evidently not span a space of finite dimension. We see further that the P_ν have uniformly bounded degrees, since the transforms of a polynomial of degree k in x by all translations already span a space of dimension k . Now we will even show that only $\gamma_\nu = 0$ appears. We apply again all transformations $z' = z + \beta$. The highest coefficient of the polynomial $P_\nu(x, y)$ in x is then multiplied by $e^{\gamma_\nu \beta/y}$. If a γ_ν were not zero infinitely many linearly independent transforms of f would evidently result, since one of the P_ν is not identically zero.

We now see that f can be written in the form

$$(25) \quad f = \sum_{\rho=0}^m \varphi_\rho(y) x^\rho$$

and we have to determine the character of the coefficients. We start with two remarks.

(α) All the terms $\varphi_\rho(y) x^\rho$ of (25) belong to the space spanned by the transforms of f .

(β) The derivatives of f with respect to x belong to the space spanned by the transforms of f .

To prove (α) subject f to $m+1$ different translations $z' = z + \beta_\mu$ ($\mu = 1, \dots, m+1$). Suitable combinations of the resulting functions give the separate $\varphi_\rho x^\rho$.

To prove (β) form $[f(x + h, y) - f(x, y)]/h$ which belongs to the space spanned by the transforms of f . The limiting process $h \rightarrow 0$ shows that $\partial f/\partial x$ is contained in this space.

Combinations of the remarks (α) and (β) show that the coefficients $\varphi_\rho(y)$ also belong to the space. Now apply to $\varphi_\rho(y)$ the transformation $z' = 1/\bar{z}$. One obtains $\varphi_\rho(y/(x^2 + y^2))$ which *must be a polynomial* in x for each $y > 0$. Setting $y/(x^2 + y^2) = u$ one sees that

$$(26) \quad \varphi_\rho(u) = P_\nu \left(\sqrt{\frac{y}{u} - y^2} \right)$$

P_ν being a polynomial in its argument depending on y . From this it follows that $\varphi_\rho(u)$ is an *analytic* function. Therefore, in computing the right side of (26), the square root must disappear. Thus $\varphi_\rho(u)$ is a *rational* function in u whose denominator is a power of u with a non-negative integral exponent.

Returning now to (25) we see that f can be written in the form

$$(27) \quad f(x, y) = \frac{P(x, y)}{y^k}$$

where P is a polynomial in x and y and k is a non-negative integer. We may naturally assume that P is not divisible by y . Again apply the transformation $z' = 1/\bar{z}$ to f . The result is, if l is the degree of $P(x, y)$, the function

$$\frac{P(x, y)}{y^k(x^2 + y^2)^{l-k}}.$$

This is a function of the type (27) *only if* $k \geq l$. On the other hand, it is clear that if $k \geq l$ the transforms of f span a finite dimensional space. Thus we have found in (27), with the condition $k \geq l$, the totality of functions satisfying the conditions of Lemma 13. Transforming the expression of (27) into the previously used model of the hyperbolic geometry we obtain a complete proof of Lemma 13.

We return to our main problem. As in the spherical case we shall construct solutions and later show that they exhaust all possibilities. We first introduce the concept of a *Lorentz (L) harmonic*. Let

$$L(u) \equiv \frac{\partial^2 u}{\partial (x^3)^2} - \frac{\partial^2 u}{\partial (x^1)^2} - \frac{\partial^2 u}{\partial (x^2)^2} = 0$$

be the wave equation in E_3 . By an L -harmonic we mean a solution of $L(u) = 0$ which is representable by a form in x^1, x^2, x^3 . Formally one obtains the L -harmonics from the spherical harmonics by replacing x^3 by ix^3 , but we are interested only in real functions. There are, of course, as in the spherical case, $2n + 1$ linearly independent L -harmonics of degree n , or they form a $(2n + 1)$ -dimensional space $\mathfrak{S}_n$. The spaces $\mathfrak{S}_n$ are linearly independent. This again follows from the mentioned transformation through the imaginary domain.

By a Lorentz gradient (L -gradient) $\nabla_L u$ of a function $u(x^1, x^2, x^3)$ in E_3 we mean the contravariant vector field obtained from the ordinary gradient ∇u ,

which is a covariant vector field, by pulling up the indices according to the Lorentzian metric. Its components are, therefore, successively $-\partial u/\partial x^1$, $-\partial u/\partial x^2$, $\partial u/\partial x^3$. We form now the analogues of the spaces $\mathfrak{s}_n$ of the spherical geometry by introducing the i.t.'s

$$(28) \quad \xi = r^{n+2} \nabla_L \frac{\varphi_n}{r^n} \quad (\varphi_n \in \mathfrak{S}_n, r = \|\vec{r}\| = 1)$$

where $\|\vec{r}\|$ is the Lorentz norm of the radius vector of (x^1, x^2, x^3) . They are tangential to V_2 and represent, therefore, i.t.'s on V_2 . *By definition, the totality of these i.t.'s forms $\mathfrak{s}_n$.*

We introduce further the vector field on V_2 given by

$$(29) \quad \xi = \nabla_L \varphi_n \times \vec{r} \quad (\varphi_n \in \mathfrak{S}_n, r = 1)$$

where the crossproduct is to be understood in the Lorentz sense. (29) is obtained from (28) by rotation in the tangent plane of V_2 through the angle $\frac{1}{2}\pi$, rotation to be understood with respect to the Lorentz metric. *The linear space formed by the i.t.'s (29) will be called $\mathfrak{t}_n$.* Both $\mathfrak{s}_n$ and $\mathfrak{t}_n$ have dimension $2n + 1$ and are invariant under $\mathfrak{G}$. From the formal relation of these spaces to the corresponding ones in spherical geometry one concludes immediately that $\mathfrak{s}_1, \mathfrak{t}_1, \mathfrak{s}_2, \mathfrak{t}_2, \dots, \mathfrak{s}_n, \mathfrak{t}_n, \dots$ are *linearly independent*. Further, if we introduce the space $\mathfrak{P}_n$ of all i.t.'s on V_2 whose components in E_3 can be represented on V_2 by polynomials of no higher degree than n , then *the formula (10) also holds here as in the spherical case.*

From now on the analogy with the spherical case is not complete. We will first consider a closed convex and finite dimensional cone $\mathfrak{f}$ of continuous *functions* on V_2 invariant under $\mathfrak{G}$. Call $\mathfrak{f}_k$ the space of functions on V_2 representable there by polynomials whose degrees are not greater than k . We know that $\mathfrak{f}$ must be contained in an $\mathfrak{f}_k$ of sufficiently high index, and let n be the least possible k . Then $\mathfrak{f}$ contains an element f whose n^{th} order terms compose a form not divisible by $\|\vec{r}\|^2 = (x^3)^2 - (x^1)^2 - (x^2)^2$ or, which means the same thing, which does not vanish identically on the "light cone" $\|\vec{r}\|^2 = 0$. Otherwise all $f \in \mathfrak{f}$ could be represented on V_2 by polynomials of lower degree than n . We will now show that under our assumption the following lemma holds:

Lemma 16. *$\mathfrak{f}$ contains all functions of the forms $(\alpha \cdot \vec{r})^n$ or $-(\alpha \cdot \vec{r})^n$, where α represents all possible null vectors (i.e. $\alpha \cdot \alpha = 0$) with a non-negative third component $\alpha^3 \geq 0$.*

Proof. Let f be an element of $\mathfrak{f}$ whose n^{th} order terms represent a form not vanishing on the light cone. Since $\mathfrak{G}$ acts transitively on the null vectors we may assume that the form does not vanish at $(1, 0, 1)$. We subject f to the special Lorentz transformations

$$(30) \quad L_a : \bar{x}^1 = \frac{x^1 + ax^2}{\sqrt{1 - a^2}}, \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = \frac{ax^1 + x^3}{\sqrt{1 - a^2}} \quad (|a| < 1)$$

and obtain the transformed functions

$$(31) \quad f_a = f\left(\frac{x^1 + ax^3}{\sqrt{1-a^2}}, x^2, \frac{ax^1 + x^3}{\sqrt{1-a^2}}\right) \quad (|a| < 1).$$

We perform the limiting process $a \rightarrow 1$. Then

$$(32) \quad \lim_{a \rightarrow 1} (\sqrt{1-a^2})^n f_a = C(x^1 + x^3)^n$$

where C is the value of the homogeneous part of f of order n at $(1, 0, 1)$ and therefore is *not zero*. Thus we see that either $(x^1 + x^3)^n$ or $-(x^1 + x^3)^n$ belongs to $\mathfrak{f}$. Since $x^1 + x^3$ can be transformed into any $\alpha \cdot \vec{r}$ with $||\alpha|| = 0$, $\alpha^3 \geq 0$ by suitable Lorentz transformations the lemma is proved.

We introduce the notation $\mathfrak{h}_n$ for the least convex closure of the functions $(\alpha \cdot \vec{r})^n$.

From Lemma 16 immediately follows

Lemma 17. *The only solutions of our problem contained in $\mathfrak{s}_n$ are obtained by substituting in (28) for φ_n all elements of $\mathfrak{h}_n$ or all elements of $-\mathfrak{h}_n$.*

We will call these solutions $\bar{\mathfrak{s}}_n$ and $-\bar{\mathfrak{s}}_n$.

We further prove

Lemma 18. *All $\mathfrak{t}_n$ are solutions of our problem.*

For the proof we have only to remark that if an improper Lorentz transformation is applied to an i.t. of the form (29) φ_n has to be changed in sign. But the least convex closure of the union of $\mathfrak{h}_n$ and $-\mathfrak{h}_n$ is the space $\mathfrak{G}_n$ of all L -harmonics of n^{th} degree.

We will finally prove that the $\bar{\mathfrak{s}}_n$, $-\bar{\mathfrak{s}}_n$ and $\mathfrak{t}_n$ exhaust all solutions of our problem. Let $\mathfrak{s}$ be a solution and ξ an element of $\mathfrak{s}$, not zero. We may write ξ in the form

$$\xi = \sum_{i=1}^n (\xi_i + \eta_i) \quad (\xi_i \in \mathfrak{s}_i, \eta_i \in \mathfrak{t}_i)$$

and designate the harmonics leading, according to (28) and (29), to ξ_n and η_n with φ'_n and φ''_n . Without loss of generality we may assume that either φ'_n or φ''_n does not vanish identically on $||\vec{r}|| = 0$. Otherwise n could be replaced by a smaller index. We may also again assume that either $\varphi'_n(1, 0, 1)$ or $\varphi''_n(1, 0, 1)$ does not vanish. Again apply transformations (30) to ξ , then multiply the resulting i.t. by $(\sqrt{1-a^2})^n$ and let a converge to 1. The limiting i.t. is a non-zero linear combination of two i.t.'s ξ' and ξ'' from $\mathfrak{s}_n$ and $\mathfrak{t}_n$ respectively, derived, according to (28) and (29), from the harmonic $(x^1 + x^3)^n$. Apply to the limit the improper Lorentz transformation $\bar{x}^1 = x^1$, $\bar{x}^2 = -x^2$, $\bar{x}^3 = x^3$. The element ξ' does not change, whereas ξ'' changes its sign. By addition of $\xi' + \xi''$ and its transform $\xi' - \xi''$ one obtains again an element of $\mathfrak{s}$. If $\xi' \neq 0$ then $\bar{\mathfrak{s}}_n$ or $-\bar{\mathfrak{s}}_n$ must be completely included in $\mathfrak{s}$. If $\xi' = 0$ and ξ'' is not zero, the whole $\mathfrak{t}_n$ forms a part of $\mathfrak{s}$.

We finally arrive at

Theorem 3. *The $\bar{\mathfrak{s}}_n$, $-\bar{\mathfrak{s}}_n$, $\mathfrak{t}_n$ ($n = 1, 2, \dots$) are the only solutions of our problem.*

4. Some Results on the Generated Semigroups. After all solutions of our problem have been found, the study of the generated semigroups or groups is of primary interest. A detailed discussion will be given in another paper. Here I would like only to state some results obtained regarding those solutions which generate semigroups *proper*. We have then to exclude *a priori* those solutions $\mathfrak{s}$ which are *linear* spaces and which, therefore, generate groups. Then there remain only the cases (c) of Theorem 1' and the corresponding cases for $n = 1$ in the hyperbolic geometry which are essentially the same, and also the solution $\bar{\mathfrak{s}}_n$ and $-\bar{\mathfrak{s}}_n$ ($n = 1, 2, 3, \dots$). In the first case we have to exclude the possibility $c_1 \geq 0$, $c_2 \leq 0$, since the generated transformations do not map the whole line into itself, which was required in the concept of "generated semigroup". On the other hand, the cases $c_1 \leq 0$, $c_2 \geq 0$ lead to semigroups which are all *similar to the semigroup mentioned on page 792* of the previously quoted paper. In the hyperbolic geometry of dimension 2 the solutions $-\bar{\mathfrak{s}}_n$ have to be excluded, since again the generated transformations cannot be defined on the whole V_2 . However, *all $\bar{\mathfrak{s}}_n$ generate semigroups proper*. The semigroups generated by $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$ turn out to be identical with those discussed in the quoted paper and called there $\mathfrak{M}_D$ and $\mathfrak{P}_D$, D being in each case the unit circle. $\mathfrak{M}_D$ is 6-dimensional and $\mathfrak{P}_D$ 8-dimensional. For all $n > 2$ the semigroups generated by $\bar{\mathfrak{s}}_n$ turn out to be of *infinite dimension*. The solutions $\bar{\mathfrak{s}}_1$ and $\bar{\mathfrak{s}}_2$ can therefore be characterized as *the only solutions generating finite dimensional semigroups*.

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