An Analogue of the Heinz-Hopf Inequality

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Of the many proofs known for BERNSTEIN's theorem on minimal surfaces, the first one which was based on geometric properties of a finite piece of the surface was due to E. HEINZ [1]. Later, E. HOPF [2] improved this result. It gives a bound on the Gauss curvature at a point of a minimal surface if one can represent the surface in the form $z = f(x, y)$ over a circle $x^2 + y^2 < R^2$ about the given point.

Recently the author [3] generalized BERNSTEIN's theorem to surfaces which need not be of the form $z = f(x, y)$ but which are restricted by the condition that the normals to the surface omit a neighborhood of some direction. In the present paper we obtain a result analogous to that of HEINZ and HOPF for this wider class of surfaces.

Theorem. *Let S be a simply-connected portion of a minimal surface, and assume that there is some fixed direction in space and a number $\alpha > 0$ such that the normals to S all make an angle at least α with the fixed direction. Let p be any point of S and let s be the geodesic distance from p to the boundary of S . Then the Gauss curvature K at p satisfies*

$$(1) \quad |K| \leq h(s, \alpha)$$

where h does not depend on the particular surface, but only on the numbers s and α . In particular, we may use for the function $h(s, \alpha)$

$$(2) \quad h(s, \alpha) = \left(\frac{2}{s} \cot \frac{\alpha}{2} \csc^2 \frac{\alpha}{2} \right)^2.$$

Furthermore, this bound can be improved by using the direction of the normal at p . Namely, if this normal makes an angle ω with the fixed direction, then

$$(3) \quad |K| \leq \left[\frac{R^2 - r^2}{s} \cdot \frac{2(1 + R^2)}{R(1 + r^2)^2} \right]^2$$

where $R = \cot \frac{1}{2}\alpha$ and $r = \cot \frac{1}{2}\omega$.

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Remark. Inequality (3) shows, in particular, that for a given value of α $K \rightarrow 0$ as either $s \rightarrow \infty$ or $\omega \rightarrow \alpha$. The classical case of surfaces of the form $z = f(x, y)$ corresponds to $\alpha = \frac{1}{2}\pi$.

Proof. Choose coordinates ξ, η, ζ in space so that the fixed direction corresponds to the positive ζ -axis. Then the Gauss normal map of S into the unit sphere Σ followed by the stereographic projection of Σ onto the $w = u + iv$ plane gives us a map of S into the region $|w| < R$, where $R = \cot \frac{1}{2}\alpha$. The line element $d\sigma$ on Σ is given by $d\sigma = \lambda |dw|$ where $\lambda = 2/(1 + |w|^2)$. It was proved in [3] that we can find a locally conformal map of S into the complex z -plane satisfying $|dz| = f ds$ where the value of f at each point of S is chosen to be the value of λ at the corresponding point of the w -plane. Thus, in particular, $f \geq 2/(1 + R^2)$.

The image of S in the z -plane will be an unbranched Riemann surface W . Let z_0 be the image of the point p . We wish to show that W contains an unbranched disk $D: |z - z_0| < d$, where $d = 2s/(1 + R^2)$. But suppose z_1 is the nearest singularity to z_0 on W . Since there are no algebraic branch points, the line segment L from z_0 to z_1 must correspond to an arc γ from p to the boundary of S . The length of this arc is given by

$$\int_{\gamma} ds = \int_L \frac{|dz|}{f} \leq \frac{|z_1 - z_0|}{2} (1 + R^2).$$

Since the length of γ is at least s , we have $|z_1 - z_0| \geq 2s/(1 + R^2)$.

Consider now the composed map of D into S into $|w| < R$. Since S is a minimal surface this will be an analytic map, and by the Schwarz-Pick lemma we have

$$\left| \frac{dw}{dz} \right|_{z_0} \leq \frac{R^2 - |w_0|^2}{Rd}$$

where w_0 is the point corresponding to p . But

$$\left| \frac{dw}{dz} \right| = \frac{|dw|}{d\sigma} \frac{d\sigma}{ds} \frac{ds}{|dz|},$$

and at z_0 we have

$$\frac{|dw|}{d\sigma} = \frac{1 + |w_0|^2}{2}, \quad \frac{d\sigma}{ds} = \sqrt{|K|}, \quad \frac{ds}{|dz|} = \frac{1}{f} = \frac{1 + |w_0|^2}{2},$$

so that

$$(4) \quad \sqrt{|K|} \leq \frac{4(R^2 - |w_0|^2)}{Rd(1 + |w_0|^2)^2}.$$

Substituting $d = 2s/(1 + R^2)$ and letting $r = |w_0| = \cot \frac{1}{2}\omega$ we arrive at (3).

Note that, with R and d fixed, the right-hand side of (4) is greatest when $|w_0| = 0$. Hence we have, at any rate,

$$\sqrt{|K|} \leq \frac{4R}{d} = \frac{2R(1 + R^2)}{s} = \frac{2}{s} \cot \frac{\alpha}{2} \csc^2 \frac{\alpha}{2}$$

giving (1) and (2).

REFERENCES

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