On the Equation of the Vibrating String

WILLIAM FELLER*

Introduction. The usual derivation of the equation of the vibrating string starts from Newton's laws and certain assumptions concerning tensional forces. The elegance and the usefulness of the theory suffer under somewhat vague and unnatural regularity conditions.

In this note we shall derive a slightly more general and more intuitive form of the equation from a purely analytical hypothesis which may be described very roughly as follows. We view the "vibrating string" as a one parameter family of functions $u(t, \cdot)$ defined on a fixed interval I and speak of $u(t, \cdot)$ as the "position" of the string at time t . It will be assumed that an acceleration d^2u/dt^2 exists (at least for sufficiently many values of t and at sufficiently many points) and is of the form Ωu where Ω is a linear operator of local character such that $\Omega f(s) \leq 0$ at each point s where f attains a positive local maximum. In other words, at a point where u assumes a positive local maximum the acceleration, if it exists, is directed towards I .

Although this hypothesis is milder than the classical ones it suffices to derive a generalized differential equation describing the motion. In this equation there appear two positive measures, of which one is easily interpretable as physical mass, the other as attractive force. The equation itself may then be viewed as expressing Newton's second law under the action of an inner tension. It seems surprising that *these notions and relations of mechanics appear as inevitable logical consequences of a purely geometric hypothesis.*

Our equation appears as a natural generalization of the classical one. It covers the case in which the masses and forces may vary discontinuously along the string to the extreme of being concentrated at individual points.¹ Such cases are frequently treated in the applied literature, using approximations and passages to the limit. By contrast, we obtain a simple unified theory in which one existence and expansion theorem covers all cases, obviating "transition conditions" *etc.* Furthermore, *boundary conditions* appear as an obvious extension of the differential operator to the endpoints of the interval I (if any).

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¹ As a curiosity we can treat *continuous* motions of a string such that its mass is concentrated at the *rational*s; see [3].

It is noteworthy that our string need not be differentiable in the “natural” or physical coordinate system. However, we *prove* the existence of a coordinate system in I in which the position of the string is differentiable, except for corners, at all times.

Finally, we introduce a *Hilbert space* E whose norm may be interpreted as *energy*. We show that our operator generates a *group of unitary transformations* in E . The intuitive content of this theorem is that to a prescribed instantaneous position and velocity at time 0 there corresponds a motion with the prescribed properties and defined for $-\infty < t < \infty$.

It may be mentioned that our operators are the most general generators of one-dimensional diffusion (or heat transfer) processes without creation (but with possible destruction) of mass.

1. The traditional equation. The equation of the vibrating string is usually written in the form

$$(1.1) \quad \frac{d^2 u}{dt^2} = \mu^{-1} \frac{d}{ds} \left(T \frac{du}{ds} \right) - cu,$$

but its meaning becomes clearer in the integrated version

$$(1.2) \quad \int_{\alpha}^{\beta} \frac{d^2 u}{dt^2} \mu \, ds = T \left. \frac{du}{ds} \right|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} uc\mu \, ds.$$

Interpreting $\mu(s)$ as the mass density at the point s we have on the left the acceleration integrated with respect to the mass, that is, the “vertical” component of the force acting on the arc of the string between α and β . The last integral on the right represents the effect of an “elastic force” proportional to the displacement from the s -axis and directed toward it. It follows that if a tensional force acts at each point in the direction of the tangent to the string, then T is the *horizontal component*² of the tension and $T(du/ds)$ the *vertical component*, or effective force.

Let us now replace the space variable s by the *new independent variable* x defined by

$$(1.3) \quad x = \int \frac{ds}{T}$$

(with the constant of integration fixed conveniently). Introduce the *mass distribution*

$$(1.4) \quad m = \int \mu \, ds$$

² Many textbooks start from assumptions which contradict (1.1) and interpret T erroneously as the tension. To reconcile this misconception with (1.1) they introduce the irrelevant restriction that du/ds be small and consider (1.1) not at face value but as an approximation.

(to be considered as an interval function or measure) and, finally, a *measure*

$$(1.5) \quad \gamma = \int c \, dm$$

representing the effect of the elastic force. Then (1.2) takes on the form

$$(1.6) \quad \int_{\alpha}^{\beta} \frac{d^2 u}{dt^2} dm = \left. \frac{du}{dx} \right|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} u \, d\gamma.$$

This relation remains mathematically and physically meaningful when m represents an arbitrary mass distribution, possibly attaching finite weight to individual points. Similarly, there is no reason why x and γ should be integrals. Indeed, we shall see that the restriction to differentiable functions x , m , γ is neither natural nor convenient.

2. The generalized equation. In an open (possibly unbounded) interval I of the s -axis let x be a *strictly increasing continuous function*; let m be a measure (mass distribution) which is finite and *strictly positive* on closed intervals; and let γ be an arbitrary (non-negative) measure finite on closed intervals. It is convenient to represent m and γ by the corresponding distribution functions which are continuous on the *right* and whose increments in $(\alpha, \beta]$ represent the measures of this half open interval. Then x , m , γ are increasing but may tend to $\pm \infty$ at the boundaries of I .

Given x , m , γ an operator Ω may be defined by

$$(2.1) \quad \Omega f \cdot dm = df' - f \cdot d\gamma.$$

For f to belong to the domain of Ω we require that both f and Ωf be continuous; the derivative $f' = df/dx$ exists at each point where m and γ are continuous, while one-sided derivatives f^+ and f^- exist at all points with the obvious continuity properties; finally, f' is of bounded variation and (2.1) signifies that the integrals of both sides are equal.

This operator Ω may be put into a more familiar form. It is almost evident that there exist *two positive convex functions* ϕ and ψ such that $\Omega\phi = 0$ and $\Omega\psi = 0$. Every solution of $\Omega w = 0$ is then a linear combination of ϕ and ψ .

The equation³

$$(2.2) \quad \Omega f = \frac{1}{\phi} D_m \left\{ \phi^2 D_x \frac{f}{\phi} \right\}$$

³ The symbol D_{ν} is used for differentiation with respect to a strictly increasing function ν ; it is defined in the obvious way taking all increments over *closed* intervals. Thus at a point of discontinuity $D_{\nu} f(s) = \{f(s+) - f(s-)\} \{\nu(s+) - \nu(s-)\}^{-1}$, where the existence of the limits is required for the derivative to exist. At points of discontinuity of m the derivative D_x should be replaced by the right and left derivatives, but for the reader's convenience we indulge in a slight abuse of notation.

is equivalent to (2.1) with

$$(2.3) \quad d\gamma = \frac{d\phi'}{\phi},$$

and may be taken as the definition of Ω . (The same equation holds with ϕ replaced by ψ .) Given an arbitrary convex $\phi > 0$ and a choice of x, m as above, our equations represent an operator Ω such that $\Omega\phi = 0$. The representation is intrinsic, that is, the "canonical scale parameter" x is determined uniquely up to a trivial linear transformation $x \rightarrow px + q$; the measure m is then uniquely determined and ϕ may be replaced only by a solution $\psi > 0$ of $\Omega\psi = 0$. Each such solution is convex and of the form

$$(2.4) \quad \psi = \phi \int \frac{dx}{\phi^2} + K\phi$$

with an arbitrary constant K . (Cf. [2].)

It is now clear that (1.6) may be written in the form

$$(2.5) \quad \frac{d^2u}{dt^2} = \Omega u.$$

This is our *generalized equation of the vibrating string*. Before deriving it analytically and discussing the boundary conditions *etc.*, we give examples to illustrate the role of the three arbitrary entities x, m, ϕ and the analytical advantages of the new form.

3. Examples.

(a) *Dirac function for the elastic force.* In an interval containing the origin put $dm = dx$ and

$$(3.1) \quad \phi = 1 + \frac{1}{2}|x|.$$

Taking x as the natural physical scale we have a string with uniform mass distribution, constant "horizontal" component of tension, and a possible *position of equilibrium* given by (3.1). Intuitively this implies an *elastic force concentrated at the origin*, and, in fact, the measure γ of (2.3) attaches unit weight to the origin and zero to each interval not containing the origin.

Equation (2.2) now reduces to

$$(3.2) \quad \Omega f = \frac{1}{1 + \frac{1}{2}|x|} \frac{d}{dx} \{(1 + \frac{1}{2}|x|)f' - \frac{1}{2}f \cdot \text{sgn } x\}$$

and thus, as should be expected, $\Omega f = f''$ in each interval not containing the origin. Furthermore, the continuity of the expression within the braces on the right requires that

$$(3.3) \quad f'(0+) - f'(0-) = f(0).$$

Therefore

$$(3.4) \quad \int_{\alpha}^{\beta} f'' dx = f'(\beta) - f'(\alpha) - \delta_0 f(0)$$

where δ_0 equals 1, or 0, according as (α, β) does, or does not, contain the origin. This is equivalent to (2.1).

It is clear intuitively, and may be verified from (2.4), that the possible positions of equilibrium are represented by functions ψ which are linear for $x > 0$ and for $x < 0$ with the slope jumping at the origin by $\psi(0)$.

Needless to say, these results could be obtained "classically" by considering an elastic force of density $(2\epsilon)^{-1}$ in an interval $|x| < \epsilon$ and zero outside and letting $\epsilon \rightarrow 0$. The direct method appears more natural; it is certainly more convenient inasmuch as eigenfunction expansions, energy integrals, etc. may be applied directly without further justification.

(b) *Positive mass concentrated at the origin.* Put $\gamma = 0$ and define m by $m(x) = x$ for $x < 0$ and $m(x) = 1 + x$ for $x \geq 0$. We have a uniform mass-distribution except that the origin carries unit weight. Clearly $\Omega f = f''$ in every interval not containing the origin, while at the origin

$$(3.5) \quad \lim_{x \rightarrow 0} f''(x) = f^+(0) - f^-(0) = \lim_{x \rightarrow 0+} f'(x) - \lim_{x \rightarrow 0-} f'(x).$$

In principle, this case could be treated as a system of *two* hyperbolic equations (one for $x > 0$ and one for $x < 0$) with (3.5) serving as a "transition condition." However, this indirect treatment needlessly repeats a general existence theorem and does not lead to eigenfunction expansions; besides, the transition condition (3.5) would require an elaborate justification.

(c) *Two versions of*

$$(3.6) \quad \Omega f(s) = \begin{cases} \frac{d^2}{ds^2} f(s) & s < 0 \\ \frac{1}{2} \frac{d^2}{ds^2} f(s) & s > 0. \end{cases}$$

Here s may be interpreted as the natural (or physical) scale. If we let the canonical scale $x = s$, we have to put $dm = ds$ for $s < 0$ and $dm = 2ds$ for $s > 0$. We are here dealing with two homogeneous strings of different densities joining at the origin so that the "horizontal" component of the tension remains constant. Putting Ω into the form $D_m D_x$ we find as "transition condition" that df/ds is *continuous* at the origin, while the second derivative jumps so as to render Ωf continuous.

On the other hand, (3.6) is compatible with the notion of a *homogeneous* string with $dm = ds$. Then we have to put $x = s$ for $s < 0$ and $x = 2s$ for $s > 0$. In this case the continuity of $D_x f$ requires the *transition condition*

$$(3.7) \quad \lim_{s \rightarrow 0-} \frac{df(s)}{ds} = \frac{1}{2} \lim_{s \rightarrow 0+} \frac{df(s)}{ds}.$$

Here the horizontal component of the tension jumps at the origin; this could be due to a constant horizontal force acting at a single point. (In principle, any canonical scale x may be interpreted as the result of horizontal forces, and their arbitrariness explains why the position of the string need not be differentiable with respect to s .)

Examples of the type described occur frequently in the applied literature and are treated by piecing together two solutions. We obtain the transition conditions as part of the differentiable equation itself and require no special treatment for each case.

4. The derivation. Let M be a linear manifold of "admissible positions" of the string, *i.e.* of continuous functions defined on the same open interval I . For each fixed $t > 0$ the position $u(t, \cdot)$ of the string is supposed to be an element of M (and, conversely, M may be taken as the set of all linear combinations of *one* such family of functions). It would be tempting to postulate that an acceleration u_{tt} exists for each $t > 0$ and at each point, but this is impossible.⁴ What we want to express rigorously are the following two intuitive properties of the acceleration *wherever it exists*:

(i) For fixed t , s the acceleration $u_{tt}(t, s)$ depends only on the behavior of $u(t, \cdot)$ in a neighborhood of s .

(ii) If $u(t, \cdot)$ attains a positive local maximum at s then at this point $u_{tt}(t, s) \leq 0$ (the acceleration is directed towards I).

If one wishes to eliminate the possibility of elastic forces, the second condition should be replaced by the more restrictive

(ii') At any place s of a local maximum $u_{tt}(t, s) \leq 0$.

To formulate these conditions rigorously we introduce the following *definitions*: A function f belongs to the *local domain* of the linear operator Ω at the point s if both f and Ωf are defined and continuous in a neighborhood of s . The operator Ω is of *local character* if each function f vanishing identically near s belongs to the local domain of Ω at s and $\Omega f(s) = 0$. (In other words, $\Omega f(s)$ depends only on the behavior of f near s .) The operator Ω has the *positive maximum property* if at each point s where f reaches a positive maximum $\Omega f(s) \leq 0$ (assuming that f belongs to the local domain of Ω). Finally, Ω has the *strict maximum property* if $\Omega f(s) \leq 0$ at each point s of a local maximum of f .

We can now express the intuitive properties (i) and (ii) analytically by the

Basic assumption. If for fixed t the acceleration $u_{tt}(t, \cdot)$ exists and is continuous in some interval then in that interval $u_{tt} = \Omega u$ where Ω is a linear operator (independent of t) of local character with the positive maximum property.

⁴For example, consider the simple equation $u_{tt} = u_{xx}$ in the interval $0 < x < 1$ with the boundary conditions $u(t, 0) = u(t, 1) = 0$ (homogeneous string with fixed ends). All solutions have period 2π , and if the initial position has a corner, the same will be true of the position at times $2\pi n$. See the precise result of section 6.

(For (ii') the positive maximum property is to be replaced by the strict maximum property.)

It will be noted that so far nothing has been assumed concerning the existence of u_{tt} . Now the operator Ω defined in (2.1) or (2.2) clearly is of local character and has the positive maximum property. Conversely, it has been shown in [2] that, within its domain of definition, *every operator of local character with the positive maximum property coincides with an operator of the form (2.2)*, except if it degenerates into a first order differential operator. In the case of the strict maximum property one has to take $\phi = 1$.

In other words, *our basic assumption implies that the motion of the vibrating string is governed by an equation of the form $u_{tt} = \Omega u$ with Ω defined by (2.2)* except that theoretically Ω could (globally or locally) degenerate into a first order operator. This cannot happen if the domain of Ω is sufficiently large, *e.g.*, if in the neighborhood of each point there exist both increasing and decreasing functions f such that $\Omega f > 0$.⁵ In particular, Ω is necessarily of the form (2.2) if it is possible to prescribe arbitrarily an initial position of the string. We shall discuss the question of the existence of the acceleration together with the integration of the equation $u_{tt} = \Omega u$.

5. Boundary conditions. Consider the operator Ω of (2.2) in an open interval I parametrized by the canonical scale x . For simplicity we shall assume that I thus parametrized is bounded, say (β_1, β_2) . (Note that this implies a limitation on the tension rather than on the length of the vibrating string.)

It is intuitively clear that when not only x but also the mass m and the elastic form γ remain bounded, the definition of Ω must be supplemented by boundary conditions expressing the behavior of the string at the endpoints. The mathematical content of this vague statement will be seen in the next section; here we wish only to give an interpretation for the boundary conditions derived abstractly in [1] and generalizing the classical boundary conditions found in textbooks.

In [1] it has been shown that if Ω is to be a self-adjoint operator in an appropriate Hilbert space the domain of Ω must be restricted by boundary conditions of the form⁶

$$(5.1) \quad m_1 \Omega f(\beta_1) + \gamma_1 f(\beta_1) = f^+(\beta_1), \quad m_1 \geq 0, \quad \gamma_1 > 0,$$

⁵ If the manifold M of admissible positions reduces to the multiples of e^{-s} one could consider the equation $u_{tt} = u_s$ with the solutions $u = (a \cos t + b \sin t)e^{-s}$. Similarly, if one defines M in terms of the single particular solution $u(t, s) = \sin t \sin s$ of $u_{tt} = u_{ss}$, the acceleration is given either by u_{ss} or by $\frac{1}{2}(u_{ss} - u)$. To obtain the precise form of Ω we require at least three independent functions in its domain.

⁶ In the most general form a term involving $f(\beta_2)$ is added to (5.1), but the requirement that Ω be of local character eliminates this possibility (unless the two boundaries are identified and we have a circular ring with the requirement of periodicity taking the place of boundary conditions). There is no difficulty in showing that *no other boundary conditions are compatible with a conservation of energy*, but we shall not dwell on this point.

or

$$(5.2) \quad f(\beta_1) = 0$$

and similarly at the limit β_2 . Obviously (5.2) is a limiting form of (5.1) when $\gamma_1 \rightarrow \infty$. In (5.1) it is supposed that $\Omega f(x)$ tends to the limit $\Omega f(\beta_1)$ as $x \rightarrow \beta_1$. The special case $m_1 = 0$ represents the classical boundary condition of textbooks.

The point of interest here is that (5.1) is really no boundary condition but a natural (and physically obvious) extension of the operator Ω to the boundary point β_1 . In fact, consider β_1 as a point of the string carrying weight $m_1 \geq 0$ and acted upon by an elastic force of intensity $\gamma_1 > 0$. (In other words, extend the measures m and γ to the closed interval.) Then the "boundary condition" (5.1) simply coincides with the original definition (2.1) of Ω at the point β_1 if one puts formally $f^-(\beta_1) = 0$. In physical terms $m_0 \Omega f$ represents mass times acceleration, and (5.1) states that this equals the sum of the elastic force plus the vertical component of the tension to the right. Thus (5.1) expresses the law of motion stating that the external world acts on the string horizontally, and therefore does not influence the acceleration.

Note that this simple interpretation is made possible only by permitting discontinuities in the coefficients of the differential operator.

6. Energy. Groups of transformations generated by Ω . For physical reasons one expects that the "initial position" and "initial velocity" of a string can be prescribed arbitrarily and that they uniquely determine the subsequent motion. We shall give a precise meaning to this statement and show that it applies to the *past* motion as well.

For simplicity we consider only the case treated in the preceding section where x , m , and γ remain bounded and where the operator Ω has been extended to the boundary points β_1 , β_2 by two boundary conditions of the form (5.1) or (5.2).

Using the usual conventions we introduce two Hilbert spaces V and U of functions on the interval $I = [\beta_1, \beta_2]$ with respective norms

$$(6.1) \quad V : \quad ||g|| = \int_{\beta_1-}^{\beta_2+} g^2 dm$$

and

$$(6.2) \quad U : \quad N(f) = \int_{\beta_1-}^{\beta_2+} f^2 d\gamma + \int_{\beta_1}^{\beta_2} f'^2 dx.$$

Here $f' = D_x f$, and the integrations over dm and $d\gamma$ extend over the closed interval, so that they include terms $f^2(\beta_1)m_1$ etc.

It is known⁷ that in the space V the operator Ω possesses a complete ortho-

⁷ The spectral resolution of the operators Ω has been given by McKean [4]; cf. the footnote 7 in [1] about the validity of McKean's expansion when the operator is extended to the closed interval.

normal system of eigenfunctions $\{z_n\}$ such that

$$(6.3) \quad \lambda_n^2 z_n + \Omega z_n = 0, \quad \lambda_n > 0, \lambda_n \uparrow \infty,$$

all eigenvalues being simple. A simple integration by parts shows that $\|z_n\| = 1$ is equivalent to $N(z_n) = \lambda_n^2$ and that $\{\lambda_n^{-1} z_n\}$ is a complete orthonormal system in U . Any pair $f \in U, g \in V$ is representable in the form

$$(6.4) \quad f = \sum f_n \lambda_n^{-1} z_n, \quad g = \sum g_n z_n$$

with $\sum f_n^2 < \infty, \sum g_n^2 < \infty$, and any two series of this form converge in the quadratic mean and represent elements of U and V respectively.

In order to define the admissible initial conditions we now introduce the Hilbert space⁸ E of pairs $\{f, g\}$, where $f \in U, g \in V$, as the direct sum of U and V ; the norm of $\{f, g\}$ equals $N(f) + \|g\|$ and is called the *energy* of $\{f, g\}$.

Consider the transformation T_t mapping $\{f, g\} \in E$ into $\{u, v\} \in E$ where

$$(6.5) \quad \begin{aligned} u &= \sum \{f_n \cos \lambda_n t + g_n \sin \lambda_n t\} \lambda_n^{-1} z_n, \\ v &= \sum \{g_n \cos \lambda_n t - f_n \sin \lambda_n t\} z_n. \end{aligned}$$

An obvious calculation shows that $\{T_t\}$ is a unitary group of transformations of E .⁹ Moreover, v is obtained from u by formal differentiation, and the difference ratios of u tend in the quadratic mean to v . If the coefficients f_n and g_n tend sufficiently fast to zero, then u and v will be continuous and $v = du/dt$ pointwise. Finally, if

$$(6.6) \quad \sum (f_n^2 + g_n^2) \lambda_n^2 < \infty$$

we shall have

$$(6.7) \quad \Omega u = \frac{dv}{dt} = - \sum \{f_n \cos \lambda_n t + g_n \sin \lambda_n t\} \lambda_n z_n,$$

and Ωu will be continuous if $f_n \lambda_n$ and $g_n \lambda_n$ tend to zero sufficiently fast. In summary:

To an arbitrary "initial position" $f \in U$ and "initial velocity" $g \in V$ there corresponds by (6.5) a motion defined for $-\infty < t < \infty$ where $u \in U$ is the position and $v \in V$ the velocity; for $t \rightarrow 0$ we have $u \rightarrow f, v \rightarrow g$, and the energy $N(u) + \|v\| = N(f) + \|g\|$ is constant. If f and g are sufficiently smooth, the acceleration $d^2 u/dt^2$ will exist at all points and equal Ωu .

Note. A great deal of the foregoing theory applies also when the elastic force is not necessarily attractive, that is, if $d\gamma$ is a signed measure or, equivalently,

⁸ The structure of E can be derived from the requirement that there exists a group of unitary transformations of the type derived below. Similarly, the possible boundary conditions follow from the same requirement. In this way a purely analytical consideration leads to the concepts of mass, tension, attractive force and energy.

⁹ A similar expansion is given in RIESZ-SZ. NAGY [5], under more restrictive conditions and without use of the generalized differential equation.

if ϕ is not convex. However, in this case the “energy integral” will not be positive definite. It should be observed that the most general differential operator of the form (2.2) cannot be reduced to the form (2.1).

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Princeton University
Princeton, New Jersey