A Solvable Exceptional Jordan Algebra

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Communicated by N. JACOBSON

In a recent letter from Paris, NATHAN JACOBSON raised the question of the existence of a solvable exceptional Jordan algebra. A fairly obvious candidate for this place in the family of Jordan algebras can be defined as follows. We let $\mathfrak{S}$ be the simple exceptional Jordan algebra of all three-rowed Hermitian matrices with elements in a Cayley division algebra $\mathfrak{C}$ over a field $\mathfrak{F}$ of characteristic not two, and $\mathfrak{R} = \mathfrak{F}[g]$ be the commutative associative algebra of dimension t over $\mathfrak{F}$ consisting of all polynomials $\alpha_1 g + \alpha_2 g^2 + \cdots + \alpha_t g^t$, with coefficients α_i in $\mathfrak{F}$, such that $g^{t+1} = 0$. We then form the direct product

$$(1) \quad \mathfrak{G} = \mathfrak{S} \times \mathfrak{R}.$$

This is an algebra of dimension $27t$ over $\mathfrak{F}$, and is a Jordan algebra. Indeed, it is trivial to verify that the direct product $\mathfrak{G}$, of any Jordan algebra $\mathfrak{S}$ and any commutative associative algebra $\mathfrak{R}$, is a Jordan algebra. Also $\mathfrak{G}' = \mathfrak{G}^2 = \mathfrak{S} \times \mathfrak{R}^2$, $\mathfrak{G}^{(k)} = [\mathfrak{G}^{(k-1)}]^2 = \mathfrak{S} \times \mathfrak{R}^{2k} = 0$ if $2^k > t$. Hence $\mathfrak{G}$ is a solvable Jordan algebra. We shall show that, *when t is sufficiently large, the algebra $\mathfrak{G}$ is an exceptional Jordan algebra.*

We shall use the method of proof of an earlier note.* The algebra $\mathfrak{S}$ contains three pairwise orthogonal idempotents e_1, e_2, e_3 whose sum, $e = e_1 + e_2 + e_3$, is the unity element of $\mathfrak{S}$. Then $\mathfrak{S}$ is the vector space direct sum

$$(2) \quad \mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2 + \mathfrak{S}_3 + \mathfrak{S}_{12} + \mathfrak{S}_{23} + \mathfrak{S}_{13},$$

where $\mathfrak{S}_i = e_i \mathfrak{S}$ is one-dimensional. The space $\mathfrak{S}_{ij} = \mathfrak{S}_{ij}$ consists of all elements $x_{ij} = x_{ji}$ of $\mathfrak{S}$ such that

$$(3) \quad 2(e_i, x_{ij}) = 2(e_j, x_{ij}) = x_{ij} \quad (i \neq j; i, j = 1, 2, 3),$$

and it follows that

$$(4) \quad 2(e_k, x_{ij}) = 0 \quad (i \neq j; k \neq i, j).$$

Here we are using (a, b) for the product in the Jordan algebra $\mathfrak{S}$. The subspaces $\mathfrak{S}_{ij}$ of $\mathfrak{S}$ then each have dimension 8 over $\mathfrak{F}$, and there are also three linear

* *A note on the exceptional Jordan algebra*, Proc. Nat. Acad. of Sciences, **36** (1950), pp. 374-374.

mappings

$$(5) \quad x \rightarrow x_{ij}$$

of $\mathfrak{C}$ onto these three subspaces $\mathfrak{S}_{ij}$ of $\mathfrak{S}$. Let $x * y$ be the product operation of $\mathfrak{C}$, u be the unity element of $\mathfrak{C}$, and $x \rightarrow \bar{x}$ be the conjugate operation of $\mathfrak{C}$. It is well known that the mappings (5) can be selected so that

$$(6) \quad 2(x_{12}, y_{23}) = (x * y)_{13}, \quad 2(x_{12}, y_{13}) = (\bar{x} * y)_{23}, \quad 2(x_{13}, y_{23}) = (x * \bar{y})_{12}.$$

We also have

$$(7) \quad 2(x_{ij}, y_{ij}) = (x * \bar{y} + y * \bar{x})(e_i + e_j),$$

where

$$(8) \quad x * \bar{y} + y * \bar{x} = \bar{x} * y + \bar{y} * x$$

is in $u\mathfrak{F}$, that is, is a scalar element of $\mathfrak{S}$. The latter relation is actually a consequence of

$$(9) \quad x * \bar{x} = \bar{x} * x = \Delta(x)u,$$

where $\Delta(x)$ is the norm function of $\mathfrak{C}$. The hypothesis that $\mathfrak{C}$ is a division algebra is equivalent to the hypothesis that $\Delta(x) \neq 0$ if $x \neq 0$.

The algebra $\mathfrak{G}$ is the vector space direct sum

$$(10) \quad \mathfrak{G} = (\mathfrak{S}_1 \times \mathfrak{R}) + (\mathfrak{S}_2 \times \mathfrak{R}) + (\mathfrak{S}_3 \times \mathfrak{R}) \\ + (\mathfrak{S}_{12} \times \mathfrak{R}) + (\mathfrak{S}_{23} \times \mathfrak{R}) + (\mathfrak{S}_{13} \times \mathfrak{R}).$$

The space $\mathfrak{S}_i \times \mathfrak{R}$ is spanned by the t elements which may be represented symbolically by $e_i \cdot g^\alpha$ for $\alpha = 1, 2, \dots, t$ and, in fact, we have

$$(11) \quad \mathfrak{S}_i \times \mathfrak{R} = \mathfrak{F}[g_i], \quad g_i = e_i \cdot g, \quad g_i^\alpha = e_i \cdot g^\alpha.$$

Note that

$$(12) \quad e \cdot g = g_1 + g_2 + g_3.$$

The vector space $\mathfrak{S}_{ij} \times \mathfrak{R}$ is the vector space direct sum

$$(13) \quad \mathfrak{S}_{ij} \cdot g + \mathfrak{S}_{ij} \cdot g^2 + \dots + \mathfrak{S}_{ij} \cdot g^t \quad (i \neq j),$$

where $\mathfrak{S}_{ij} \cdot g^\alpha$ is the set of all elements

$$(14) \quad x_{ij} \cdot g^\alpha \quad (\alpha = 1, 2, \dots, t),$$

for x_{ij} in $\mathfrak{C}$. We now proceed to the assumption that $\mathfrak{G}$ is special. Thus we are assuming that the elements of $\mathfrak{G}$ are actually elements of an associative algebra $\mathfrak{A}$, and that, if a and b are in $\mathfrak{G}$, the product relation

$$(15) \quad 2(a, b) = ab + ba$$

holds, where ab is the product in $\mathfrak{A}$. Let us then reflect the multiplicative properties of $\mathfrak{S}$ as properties of $\mathfrak{S} \times \mathfrak{R}$ in terms of this associative product.

We first observe that (15) implies that powers a^k in $\mathfrak{G}$ coincide with corresponding associative powers in $\mathfrak{A}$. Thus the algebras $\mathfrak{S}_i \times \mathfrak{R} = \mathfrak{R}_i = \mathfrak{F}[g_i]$ are commutative associative subalgebras of $\mathfrak{A}$. These algebras are Jordan orthogonal, that is, we have the associative relation

$$(16) \quad k_i k_j = -k_j k_i \quad (i \neq j; i, j = 1, 2, 3),$$

for every k_i in $\mathfrak{R}_i$ and k_j in $\mathfrak{R}_j$. Then $k_i^2 k_j = k_j k_i^2$ for every k_i in $\mathfrak{R}_i$ and k_j in $\mathfrak{R}_j$, and so we have

$$(17) \quad g_i g_j = -g_j g_i, \quad g_i^2 g_j = g_j g_i^2 = 0 \quad (i \neq j; i, j = 1, 2, 3).$$

The relation (3) is reflected in $\mathfrak{G}$ by the property

$$(18) \quad g_i^\alpha (x_{ij} \cdot g^\beta) + (x_{ij} \cdot g^\beta) g_i^\alpha = g_i^\alpha (x_{ij} \cdot g^\beta) + (x_{ij} \cdot g^\beta) g_i^\alpha = x_{ij} \cdot g^{\alpha+\beta} \\ (i \neq j; \alpha, \beta = 1, \dots, t),$$

the relation (4) becomes

$$(19) \quad g_k^\alpha (x_{ij} \cdot g^\beta) + (x_{ij} \cdot g^\beta) g_k^\alpha = 0 \quad (i \neq j; k \neq i, j; \alpha, \beta = 1, \dots, t),$$

and the relations in (6) become

$$(20) \quad (x_{12} \cdot g^\alpha)(y_{23} \cdot g^\beta) + (y_{23} \cdot g^\beta)(x_{12} \cdot g^\alpha) = (x * y)_{13} \cdot g^{\alpha+\beta},$$

$$(21) \quad (x_{12} \cdot g^\alpha)(y_{13} \cdot g^\beta) + (y_{13} \cdot g^\beta)(x_{12} \cdot g^\alpha) = (\bar{x} * y)_{23} \cdot g^{\alpha+\beta},$$

$$(22) \quad (x_{13} \cdot g^\alpha)(y_{23} \cdot g^\beta) + (y_{23} \cdot g^\beta)(x_{13} \cdot g^\alpha) = (x * \bar{y})_{12} \cdot g^{\alpha+\beta}.$$

We note finally that (7) becomes

$$(23) \quad (x_{ij} \cdot g^\alpha)(y_{ij} \cdot g^\beta) + (y_{ij} \cdot g^\beta)(x_{ij} \cdot g^\alpha) = [x * y + y * x](g_i^{\alpha+\beta} + g_j^{\alpha+\beta}) \\ (i \neq j; \alpha, \beta = 1, \dots, t),$$

so that, in particular, we have

$$(24) \quad (x_{ij} \cdot g^\alpha)^2 = [\Delta(x)](g_i^{2\alpha} + g_j^{2\alpha}) \quad (i \neq j; \alpha, \beta = 1, \dots, t).$$

Since $\Delta(u) = 1$ we observe that

$$(25) \quad (u_{ij} \cdot g^\alpha)^2 = g_i^{2\alpha} + g_j^{2\alpha} \quad (i \neq j; \alpha, \beta = 1, \dots, t).$$

Let us now add the relations of (18) to obtain

$$(26) \quad (g_i^\alpha + g_j^\alpha)(x_{ij} \cdot g^\beta) + (x_{ij} \cdot g^\beta)(g_i^\alpha + g_j^\alpha) = 2x_{ij} \cdot g^{\alpha+\beta},$$

and also subtract these same relations to obtain

$$(27) \quad (g_i^\alpha - g_j^\alpha)(x_{ij} \cdot g^\beta) + (x_{ij} \cdot g^\beta)(g_i^\alpha - g_j^\alpha) = 0 \quad (i \neq j; \alpha, \beta = 1, \dots, t).$$

But (17) implies that

$$(28) \quad (g_i^\alpha - g_j^\alpha)^2 = (g_i^\alpha + g_j^\alpha)^2 = g_i^{2\alpha} + g_j^{2\alpha} \quad (i \neq j; \alpha = 2, \dots, t).$$

Since (27) states that $g_i^\alpha - g_j^\alpha$ anticommutes with $x_{ij} \cdot g^\beta$ its square commutes with $x_{ij} \cdot g^\beta$, and (26) implies that

$$(29) \quad x_{ij} \cdot g^{\beta+2\alpha} = (g_i^{2\alpha} + g_j^{2\alpha})(x_{ij} \cdot g^\beta) = (x_{ij} \cdot g^\beta)(g_i^{2\alpha} + g_j^{2\alpha}) \\ (i \neq j; \alpha = 2, \dots, t; \beta = 1, \dots, t).$$

This result is clearly equivalent to its special case

$$(30) \quad x_{ij} \cdot g^{\beta+4} = (g_i^4 + g_j^4)(x_{ij} \cdot g^\beta) = (x_{ij} \cdot g^\beta)(g_i^4 + g_j^4) \quad (i \neq j; \beta = 1, \dots, t).$$

We also use (26) to see that

$$(g_i^{2\alpha} + g_j^{2\alpha})(x_{ij} \cdot g^\beta) + 2(g_i^\alpha + g_j^\alpha)(x_{ij} \cdot g^\beta)(g_i^\alpha + g_j^\alpha) + (x_{ij} \cdot g^\beta)(g_i^{2\alpha} + g_j^{2\alpha}) \\ = 2(g_i^\alpha + g_j^\alpha)(x_{ij} \cdot g^{\alpha+\beta}) + 2(x_{ij} \cdot g^{\alpha+2})(g_i^\alpha + g_j^\alpha) = 4x_{ij} \cdot g^{2\alpha+\beta}.$$

Thus $(g_i^\alpha + g_j^\alpha)(x_{ij} \cdot g^\beta)(g_i^\alpha + g_j^\alpha) = x_{ij} \cdot g^{2\alpha+\beta}$, a result equivalent to its special case

$$(31) \quad (g_i^2 + g_j^2)(x_{ij} \cdot g^\beta)(g_i^2 + g_j^2) = x_{ij} \cdot g^{\beta+4}.$$

We now use (27) to see that

$$g_i^\alpha(g_i^\alpha - g_j^\alpha)(x_{ij} \cdot g^\beta)g_i^\alpha = g_i^{2\alpha}(x_{ij} \cdot g^\beta)g_i^\alpha \\ = g_i^\alpha(x_{ij} \cdot g^\beta)(g_i^\alpha - g_j^\alpha)g_i^\alpha = -g_i^\alpha(x_{ij} \cdot g^\beta)g_i^{2\alpha} = -g_i^\alpha[(x_{ij} \cdot g^\beta)g_i^\alpha]g_i^\alpha \\ = -g_i^\alpha[(x_{ij} \cdot g^{\alpha+\beta}) - g_i^\alpha(x_{ij} \cdot g^\beta)]g_i^\alpha = g_i^{2\alpha}(x_{ij} \cdot g^\beta)g_i^\alpha - g_i^\alpha(x_{ij} \cdot g^{\alpha+\beta})g_i^\alpha.$$

Thus

$$(32) \quad g_i^\alpha(x_{ij} \cdot g^{\alpha+\beta})g_i^\alpha = 0 \quad (i \neq j; \alpha = 2, \dots, t; \beta = 1, \dots, t),$$

and this result is equivalent to

$$(33) \quad g_i^2(x_{ij} \cdot g^\beta)g_i^2 = 0 \quad (i \neq j; \beta = 3, \dots, t).$$

However, we know then that

$$0 = g_i^2(x_{ij} \cdot g^{\beta+4})g_i^2 = g_i^2(g_i^2 + g_j^2)(x_{ij} \cdot g^\beta)(g_i^2 + g_j^2)g_i^2$$

by (31), and we have shown that

$$(34) \quad g_i^4(x_{ij} \cdot g^\beta)g_i^4 = 0 \quad (i \neq j; \beta = 1, \dots, t).$$

Formula (19) states that g_k^α anticommutes with $x_{ij} \cdot g^\beta$, and so $g_k^{2\alpha}$ both commutes and anticommutes with $x_{ij} \cdot g^\beta$, that is,

$$(35) \quad g_k^{2\alpha}(x_{ij} \cdot g^\beta) = (x_{ij} \cdot g^\beta)g_k^{2\alpha} = g_k^\alpha(x_{ij} \cdot g^\beta)g_k^\alpha = 0 \\ (i \neq j; k \neq i, j; \alpha, \beta = 1, \dots, t).$$

This relation is, in fact, equivalent to

$$(36) \quad g_k^2(x_{ij} \cdot g^\beta) = (x_{ij} \cdot g^\beta)g_k^2 = g_k(x_{ij} \cdot g^\beta)g_k = 0 \\ (i \neq j; k \neq i, j; \beta = 1, \dots, t).$$

We are now in a position to derive an *impossible consequence* of our assumption that $\mathfrak{G}$ is special, in the case *where t is sufficiently large*. We define a mapping $x \rightarrow \phi(x; \beta)$ of $\mathfrak{G}$ into $\mathfrak{A}$ by

$$(37) \quad \phi(x; \beta) = g_1^4(x_{12} \cdot g^\beta)(u_{12} \cdot g^2).$$

Then we have

$$[\phi(x; \beta)](u_{12} \cdot g^2) = g_1^4(x_{12} \cdot g^\beta)(g_1^4 + g_2^4) = g_1^4(x_{12} \cdot g^{\beta+4})$$

by (30). But $(x_{12} \cdot g)g_1^{\beta+3} + g_1^{\beta+3}(x_{12} \cdot g) = x_{12} \cdot g^{\beta+4}$ and (34) implies that $g_1^4(x_{12} \cdot g)g_1^4 = 0$ from which $[\phi(x; \beta)](u_{12} \cdot g^2) = g_1^{\beta+7}(x_{12} \cdot g)$. Hence

$$[\phi(x; \beta)](u_{12} \cdot g^2)(x_{12} \cdot g) = g_1^{\beta+7}(g_1^2 + g_2^2)\Delta(x) = g_1^{\beta+9}\Delta(x).$$

If $t \geq \beta + 9$ the power $g_1^{\beta+9} \neq 0$ so that $\phi(x; \beta) \neq 0$ if $x \neq 0$. We state this result as follows.

Lemma 1. *The mapping $x \rightarrow \phi(x; \beta)$, of $\mathfrak{G}$ onto the subspace of $\mathfrak{A}$ consisting of all $\phi(x; \beta)$, is a nonsingular mapping if $t > \beta + 8$.*

We now proceed to use relations (20), (21), (22). We see first that if $\alpha > 4$ and $\beta > 4$, then

$$\begin{aligned} (x_{12} \cdot g^\alpha)(y_{13} \cdot g^\beta)(z_{12} \cdot g^\gamma) &= (x_{12} \cdot g^{\alpha-4})(g_1^4 + g_2^4)(y_{13} \cdot g^{\beta-4})(g_1^4 + g_3^4)(z_{12} \cdot g^\gamma) \\ &= (x_{12} \cdot g^{\alpha-4})[g_1^4(y_{13} \cdot g^{\beta-4})g_1^4](z_{12} \cdot g^\gamma). \end{aligned}$$

By (34) we have proved that

$$(38) \quad (x_{12} \cdot g^\alpha)(y_{13} \cdot g^\beta)(z_{12} \cdot g^\gamma) = 0 \quad (\alpha > 4, \beta > 4, \gamma > 0).$$

We also compute

$$\begin{aligned} (u_{23} \cdot g^\alpha)(u_{13} \cdot g^\beta)(y_{12} \cdot g^\gamma) &= (u_{23} \cdot g^\alpha)(u_{13} \cdot g^\beta)(g_1^4 + g_2^4)(y_{12} \cdot g^{\gamma-4}) \\ &= (u_{23} \cdot g^\alpha)(u_{13} \cdot g^\beta)g_1^4(y_{12} \cdot g^{\gamma-4}) = [u_{12} \cdot g^{\alpha+\beta} - (u_{13}g^\beta)(u_{23} \cdot g^\alpha)]g_1^4(y_{12} \cdot g^{\gamma-4}) \end{aligned}$$

and have derived the formula

$$(39) \quad (u_{23} \cdot g^\alpha)(u_{13} \cdot g^\beta)(y_{12} \cdot g^\gamma) = (u_{12} \cdot g^{\alpha+\beta})g_1^4(y_{12} \cdot g^{\gamma-4}) \quad (\alpha > 0, \beta > 0, \gamma > 4).$$

We are now able to use our relations to write

$$\begin{aligned} \phi(x * y; \lambda) &= g_1^4[(x * y)_{12} \cdot g^\lambda](u_{12} \cdot g^2) \\ &= g_1^4[(x_{13} \cdot g^\alpha)(\bar{y}_{23} \cdot g^\beta) + (\bar{y}_{23} \cdot g^\beta)(x_{13} \cdot g^\alpha)](u_{12} \cdot g^2) = g_1^4(x_{13} \cdot g^\alpha)(\bar{y}_{23} \cdot g^\beta)(u_{12} \cdot g^2) \end{aligned}$$

by (36), where we have the decomposition

$$(40) \quad \lambda = \alpha + \beta,$$

for positive integers α and β . We also know that $x * u = x$, so that $x_{13} \cdot g^\alpha =$

$(x_{12} \cdot g^\gamma)(u_{23} \cdot g^\delta) + (u_{23} \cdot g^\delta)(x_{12} \cdot g^\gamma)$, and thus

$$\begin{aligned} \phi(x * y; \lambda) &= g_1^4[(x_{12} \cdot g^\gamma)(u_{23} \cdot g^\delta) + (u_{23} \cdot g^\delta)(x_{12} \cdot g^\gamma)](\bar{y}_{23} \cdot g^\delta)(u_{12} \cdot g^2) \\ &= g_1^4(x_{12} \cdot g^\gamma)(u_{23} \cdot g^\delta)(\bar{y}_{23} \cdot g^\delta)(u_{12} \cdot g^2) \end{aligned}$$

where we now have

$$(41) \quad \lambda = \gamma + \delta + \beta \quad (\gamma > 0, \delta > 0, \beta > 0).$$

We will also make the expansion $\bar{y}_{23} \cdot g^\beta = (y_{12} \cdot g^\rho)(u_{13} \cdot g^\sigma) + (u_{13} \cdot g^\sigma)(y_{12} \cdot g^\rho)$ and have

$$\phi(x * y; \lambda) = g_1^4(x_{12} \cdot g^\gamma)(u_{23} \cdot g^\delta)[(y_{12} \cdot g^\rho)(u_{13} \cdot g^\sigma) + (u_{13} \cdot g^\sigma)(y_{12} \cdot g^\rho)]$$

where

$$(42) \quad \lambda = \gamma + \delta + \rho + \sigma,$$

and the only restriction so far is that γ, δ, ρ , and σ are all positive integers. The term with $(y_{12} \cdot g^\rho)(u_{13} \cdot g^\sigma)(u_{12} \cdot g^2)$ vanishes by (38) if

$$(43) \quad \rho > 4, \quad \sigma > 4.$$

Also (39) implies that

$$(u_{23} \cdot g^\delta)(u_{13} \cdot g)(y_{12} \cdot g^\rho) = (u_{12} \cdot g^{\delta+\sigma})g_1^4(y_{12} \cdot g^{\rho-4}),$$

where we already have the condition $\rho > 4$. Thus

$$\phi(x * y; \lambda) = [g_1^4(x_{12} \cdot g^\gamma)(u_{12} \cdot g^{\sigma+\delta})][g_1^4(y_{12} \cdot g^{\rho-4})(u_{12} \cdot g^2)].$$

Take

$$(44) \quad \sigma = 5, \quad \delta = 1,$$

and obtain

$$(u_{12} \cdot g^6)g_1^4(y_{12} \cdot g^{\rho-4}) = (u_{12} \cdot g^2)(g_1^4 + g_2^4)g_1^4(y_{12} \cdot g^{\rho-4}) = (u_{12} \cdot g^2)g_1^4(y_{12} \cdot g^\rho).$$

Thus we have shown that if $\lambda = \gamma + \rho + 6$ and $\rho > 4$, then

$$\phi(x * y; \lambda) = [g_1^4(x_{12} \cdot g^\gamma)(u_{12} \cdot g^2)][g_1^4(y_{12} \cdot g^\rho)(u_{12} \cdot g^2)].$$

Let us replace γ by α and ρ by β in the following statement of the result just derived.

Lemma 2. *Let α be positive and $\beta > 4$. Then*

$$(45) \quad \phi(x; \alpha)\phi(y; \beta) = \phi(x * y; \alpha + \beta + 6).$$

We now compute

$$\begin{aligned} [\phi(x; \alpha)\phi(y; \beta)]\phi(z, \gamma) &= [\phi(x * y, \alpha + \beta + 6)][\phi(z, \gamma)] \\ &= \phi[(x * y) * z, \alpha + \beta + \gamma + 12] \end{aligned}$$

providing that $\beta > 4$ and $\gamma > 4$. We also have

$$\begin{aligned}\phi(x; \alpha)[\phi(y; \beta)\phi(z, \gamma)] &= \phi(x; \alpha)[\phi(y * z; \beta + \gamma + 6)] \\ &= \phi[x * (y * z), \alpha + \beta + \gamma + 12].\end{aligned}$$

Since $\mathfrak{A}$ is associative these products are equal and so $\phi(w; \alpha + \beta + \gamma + 12) = 0$ for $w = x * (y * z) - (x * y) * z$. But $\mathfrak{C}$ is not associative and so there exist elements x, y, z in $\mathfrak{C}$ such that $w \neq 0$. By Lemma 1 this cannot occur if $t > (\alpha + \beta + \gamma + 12) + 8$ and we take $\alpha = 1, \beta = \gamma = 5$ to see that $\mathfrak{G}$ is exceptional when $t + 1 > 32$. We will now derive the following generalization of this result.

Theorem. *Let $\mathfrak{S}$ be a simple exceptional Jordan algebra over any field $\mathfrak{F}$ of characteristic not two, and $\mathfrak{R} = \mathfrak{F}[g]$ be the associative polynomial algebra generated by a nilpotent element g of index at least 33. Then $\mathfrak{G} = \mathfrak{S} \times \mathfrak{R}$ is an exceptional solvable Jordan algebra.*

For it is well known that there exists a scalar extension Ω of $\mathfrak{F}$, obtained by the adjunction of indeterminates if necessary, such that there exists a Cayley division algebra $\mathfrak{C}$ over Ω . Let $\mathfrak{S}_0$ be the algebra of all three-rowed Hermitian matrices with elements in $\mathfrak{C}$, so that $\mathfrak{S}_0 \times \Omega[g]$ is exceptional. If $\mathfrak{S} \times \mathfrak{R}$ is special, so is $(\mathfrak{S} \times \mathfrak{R})_{\Omega}$. There also exists an extension field Λ of Ω such that $\mathfrak{S}_{\Lambda}$ and $(\mathfrak{S}_0)_{\Lambda}$ are isomorphic. Then $(\mathfrak{S}_0 \times \mathfrak{R})_{\Lambda}$ and $(\mathfrak{S} \times \mathfrak{R})_{\Lambda}$ are isomorphic and so $\mathfrak{S}_0 \times (\mathfrak{R})_{\Omega}$ can be imbedded in the associative algebra which contains $(\mathfrak{S} \times \mathfrak{R})_{\Omega}$. This contradicts our proof that $\mathfrak{S}_0 \times \mathfrak{R}_{\Omega}$ is exceptional, and completes the proof of our theorem.

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