*A Deflation Map**

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1. The cohomology theory of finite groups, as formulated by ARTIN & TATE [1], is directed toward class-field-theoretic applications. Thus certain cohomological maps—namely, restriction, inflation and verlagerung (or transfer)—are of utmost importance. Both restriction and inflation are special cases of a general cohomological map that arises from homomorphisms of pairs; however, restriction is defined for all dimensions from $-\infty$ to $+\infty$, while inflation is defined only in dimensions ≥ 1 . The verlagerung, on the other hand, comes from a cochain map that is determined by a trace, and is defined in all dimensions. In a sense, restriction in positive dimensions and verlagerung in negative dimensions (verlagerung in positive dimensions and restriction in negative dimensions) are duals of each other. Our purpose here is to define a deflation map (in negative dimensions) dual to inflation and to determine some of its properties.

We shall use results from [1] freely and without further reference. Except for some slight modifications our notation and terminology will conform with that of [1].

2. G shall always represent a finite group written multiplicatively, and $\Gamma = Z[G]$ will denote the integer group ring of G . Concepts concerned with Γ may then be re-named with respect to G —for example, a left unitary Γ -module is called a G -module.

Suppose A and B are G -modules; then $\text{Hom}_G(A, B)$ (the set of all Γ -homomorphisms) is a group; moreover, $\text{Hom}_Z(A, B)$ (the set of all group homomorphisms) is a group which becomes a G -module when we put

$$f^\sigma = \sigma f \sigma^{-1} \quad \text{for } f \in \text{Hom}_Z(A, B), \quad \sigma \in G.$$

As a matter of fact, $\text{Hom}_G(A, B) = (\text{Hom}_Z(A, B))^G$ —where, for an arbitrary G -module M , we write $M^G = \{m \in M \mid \sigma m = m, \text{ all } \sigma \in G\}$.

Z can always be made into a G -module by taking the action of G to be trivial—that is, $\sigma n = n$ for every $\sigma \in G$ and $n \in Z$. For any G -module A , we denote the G -module $\text{Hom}_Z(A, Z)$ by $\hat{A}$. Note that Z and $\hat{Z}$ are canonically isomorphic as G -modules, and they are usually identified.

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Suppose that $\phi: A' \rightarrow A$ and $\psi: B \rightarrow B'$ are G -homomorphisms of G -modules. For $f \in \text{Hom}_Z(A, B)$ we put $(\phi, \psi)f = \psi f \phi \in \text{Hom}_Z(A', B')$, and then the map

$$(\phi, \psi) : \text{Hom}_Z(A, B) \rightarrow \text{Hom}_Z(A', B')$$

is a G -homomorphism of G -modules, and it takes $\text{Hom}_G(A, B)$ into $\text{Hom}_G(A', B')$. If in addition $\phi': A'' \rightarrow A'$ and $\psi': B' \rightarrow B''$ then $(\phi', \psi') \circ (\phi, \psi) = (\phi\phi', \psi'\psi)$. In general, the desired commutativity properties hold for (ϕ, ψ) .

Suppose that A is a G -module and that H is a normal subgroup of G with coset decomposition $G = \bigcup_i \sigma_i H = \bigcup_i H \sigma_i$. The trace map $S_{H \rightarrow G} : A^H \rightarrow A^G$ is given by $S_{H \rightarrow G} a = \sum_i \sigma_i a$. In particular, we have $S_G = S_{\{1\} \rightarrow G} : A \rightarrow A^G$ and $S_H = S_{\{1\} \rightarrow H} : A \rightarrow A^H$. Note that A^H becomes a G/H -module on which $S_{H \rightarrow G} = S_{G/H}$.

A G -complex $X = \sum_{-\infty}^{+\infty} \oplus X_r$ is perhaps best described by the diagram

$$(1) \quad \begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & X_2 & \xrightarrow{\partial} & X_1 & \xrightarrow{\partial} & X_0 \xrightarrow{\partial} X_{-1} \xrightarrow{\partial} X_{-2} \xrightarrow{\partial} \cdots \\ & & & & \searrow \epsilon & \nearrow \mu & \\ & & & & Z & & \\ & & \nearrow & & \searrow & & \\ & & O & & O & & \end{array}$$

Each X_r is a free module with a finite basis over Γ . All the ∂ 's (strictly speaking one should write $\partial_r : X_r \rightarrow X_{r-1}$), ϵ and μ are G -homomorphisms. The triangle commutes and exactness holds everywhere.

A G -complex does exist; in fact, the so-called standard complex (which is usually used for explicit computations, and which will be used *exclusively* from now on) is described in the following way: $X_0 = \Gamma[\cdot]$ is the free module over Γ with one free generator $[\cdot]$ (called the empty cell). For each $r \geq 1$, $X_r = \sum_{\sigma_i \in G} \oplus \Gamma[\sigma_1, \dots, \sigma_r]$ is the free Γ -module with basis consisting of the r -cells $[\sigma_1, \dots, \sigma_r]$ for all choices of $\sigma_1, \dots, \sigma_r$ from G . $X_{-1} = \Gamma(\cdot)$, so that (like X_0) X_{-1} has a single free generator. Finally, for $r > 1$ $X_{-r} = \sum_{\sigma_i \in G} \oplus \Gamma(\sigma_1, \dots, \sigma_{r-1})$. The G -homomorphisms ∂_r ($-\infty < r < \infty$), ϵ and μ —which are determined by their action on the basis elements—are given by

$$(2.1) \quad \partial_i([\sigma]) = \sigma[\cdot] - [\cdot],$$

$$(2.2) \quad \begin{aligned} \partial_r([\sigma_1, \dots, \sigma_r]) &= \sigma_1[\sigma_2, \dots, \sigma_r] \\ &+ \sum_{i=1}^{r-1} (-1)^i [\sigma_1, \dots, \sigma_i \sigma_{i+1}, \dots, \sigma_r] \\ &+ (-1)^r [\sigma_1, \dots, \sigma_{r-1}] \quad \text{for } r > 1, \end{aligned}$$

$$(2.3) \quad \partial_0([\cdot]) = S_G(\cdot) = \sum_{\rho \in G} \rho(\cdot) = \sum_{\rho \in G} (\cdot)^\rho,$$

$$(2.4) \quad \partial_{-1}((\cdot)) = \sum_{\rho \in G} (\rho)^{\rho^{-1}} - \sum_{\rho \in G} (\rho^{-1}),$$

$$\begin{aligned}
\partial_{-r}((\sigma_1, \dots, \sigma_{r-1})) &= \sum_{\rho \in G} (\rho, \sigma_1, \dots, \sigma_{r-1})^{\rho^{-1}} \\
(2.5) \quad &+ \sum_{\nu=1}^{r-1} (-1)^\nu \sum_{\rho \in G} (\sigma_1, \dots, \sigma_\nu \rho^{-1}, \rho, \sigma_{\nu+1}, \dots, \sigma_{r-1}) \\
&+ (-1)^r \sum_{\rho \in G} (\sigma_1, \dots, \sigma_{r-1}, \rho^{-1}) \quad \text{for } r > 1,
\end{aligned}$$

$$(2.6) \quad \epsilon([\cdot]) = 1 \quad \text{and} \quad \mu(1) = S_G(\cdot).$$

In fact, the negative part of the standard complex is gotten by “dualizing” the positive part. Thus we have, for every $r \geq 1$, $X_{-r} = \hat{X}_{r-1} = \text{Hom}_Z(X_{r-1}, Z)$ and $\partial_{-r} = (\partial_r, 1)$; and this accounts for the use of exponential notation for the action of G in negative dimensions. In particular, it is easy to see that

$$(3.1) \quad ((\cdot)^\rho)(\sigma[\cdot]) = \begin{cases} 1 & \text{if } \sigma = \rho \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned}
&((\tau_1, \dots, \tau_{r-1})^\rho)(\sigma[\sigma_1, \dots, \sigma_{r-1}]) \\
(3.2) \quad &= \begin{cases} 1 & \text{if } \sigma = \rho, \sigma_i = \tau_i, i = 1, \dots, r-1 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

If A is any G -module, then the cohomology groups $H^r(G, A)$ of G in A appear as kernels modulo images in the sequence

$$\dots \xrightarrow{\delta} \text{Hom}_G(X_{-1}, A) \xrightarrow{\delta} \text{Hom}_G(X_0, A) \xrightarrow{\delta} \text{Hom}_G(X_1, A) \rightarrow \dots$$

where $\delta = (\partial, 1)$.

The groups $H^0(G, A)$ and $H^{-1}(G, A)$ may be given simple and explicit interpretations as follows: Define group isomorphisms $\kappa: A \rightarrow \text{Hom}_G(X_0, A)$ and $\eta: A \rightarrow \text{Hom}_G(X_{-1}, A)$ by putting for every $a \in A$

$$(4) \quad (\kappa a)([\cdot]) = a \quad \text{and} \quad (\eta a)((\cdot)) = a.$$

These determine

$$\kappa: A^G \rightarrow Z^0(G, A), \kappa: SA \rightarrow B^0(G, A), \eta: A_S \rightarrow Z^{-1}(G, A), \eta: IA \rightarrow B^{-1}(G, A)$$

where Z and B denote cocycles and coboundaries respectively, and $A_S = \{a \in A \mid Sa = 0\}$, $IA = \{\sum_{\sigma \in G} (\sigma - 1)a_\sigma \mid a_\sigma \in A\}$. There are also induced in the natural way maps $\bar{\kappa}: A^G \rightarrow H^0(G, A)$, $\bar{\eta}: A_S \rightarrow H^{-1}(G, A)$, $\bar{\kappa}: A^G/SA \rightarrow H^0(G, A)$, $\bar{\eta}: A_S/IA \rightarrow H^{-1}(G, A)$.

Suppose next that H is a normal subgroup of G . The standard complex for G/H will be denoted by Y . Consider the diagram

$$\begin{array}{ccccccc}
(5) & & \dots & \xrightarrow{\partial^G} & X_2 & \xrightarrow{\partial^G} & X_1 & \xrightarrow{\partial^G} & X_0 & \xrightarrow{\epsilon^G} & Z \\
& & & & \downarrow \Lambda_2 & & \downarrow \Lambda_1 & & \downarrow \Lambda_0 & & \\
& & \dots & \xrightarrow{\partial^{G/H}} & Y_2 & \xrightarrow{\partial^{G/H}} & Y_1 & \xrightarrow{\partial^{G/H}} & Y_0 & \xrightarrow{\epsilon^{G/H}} & Z
\end{array}$$

Each Y_r is a G/H -module, and becomes a G -module via the canonical map $\lambda: \sigma \rightarrow \bar{\sigma} = \lambda(\sigma)$ of $G \rightarrow G/H$. Every $\partial^{G/H}$ is then a G -homomorphism, and so is $\epsilon^{G/H}$. The G -homomorphisms $\Lambda_0, \Lambda_1, \dots$ may be defined by putting

$$(6) \quad \begin{aligned} \Lambda_0([\cdot]_{\sigma}) &= [\cdot]_{G/H} \\ \Lambda_r([\sigma_1, \dots, \sigma_r]) &= [\bar{\sigma}_1, \dots, \bar{\sigma}_r] \quad \text{for } r \geq 1 \end{aligned}$$

and then extending linearly. Every Λ is an onto map, and all commutativities hold.

Let A be any G -module and let $i: A^H \rightarrow A$ denote the inclusion map (it is a G -homomorphism). Using the fact that if M and N are G/H -modules, then $\text{Hom}_{G/H}(M, N) = \text{Hom}_G(M, N)$ (because $\lambda: G \rightarrow G/H$ is onto), we get the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_G(X_0, A) & \xrightarrow{\delta^G} & \text{Hom}_G(X_1, A) & \xrightarrow{\delta^G} & \text{Hom}_G(X_2, A) \longrightarrow \\ & & (\Lambda_0, i) \uparrow & & (\Lambda_1, i) \uparrow & & (\Lambda_2, i) \uparrow \\ 0 & \longrightarrow & \text{Hom}_G(Y_0, A^H) & \xrightarrow{\delta^{G/H}} & \text{Hom}_G(Y_1, A^H) & \xrightarrow{\delta^{G/H}} & \text{Hom}_G(Y_2, A^H) \longrightarrow \\ & & \parallel & & \parallel & & \parallel \\ & & \text{Hom}_{G/H}(Y_0, A^H) & & \text{Hom}_{G/H}(Y_1, A^H) & & \text{Hom}_{G/H}(Y_2, A^H) \end{array}$$

where $\delta^G = (\partial^G, 1)$ and $\delta^{G/H} = (\partial^{G/H}, 1)$. Consequently the vertical maps induce homomorphisms

$$\inf_{G/H \rightarrow G} : H^r(G/H, A^H) \rightarrow H^r(G, A) \quad \text{for } r \geq 1.$$

(The case $r = 0$ is excluded because there one has a map between the unreduced cohomology groups.)

3. We are now in a position to define the deflation map. The diagram (5) may be dualized and attaching the result to the original gives

$$(7) \quad \begin{array}{ccccccc} \dots & \xrightarrow{\partial^G} & X_0 & \xrightarrow{\partial^G} & X_{-1} & \xrightarrow{\partial^G} & X_{-2} \xrightarrow{\partial^G} \dots \\ & & \downarrow \Lambda_0 & \nearrow \epsilon^G & \downarrow \mu^G & \nearrow \Lambda_{-1} & \downarrow \Lambda_{-2} \\ & & Y_0 & \xrightarrow{\partial^{G/H}} & Y_{-1} & \xrightarrow{\partial^{G/H}} & Y_{-2} \xrightarrow{\partial^{G/H}} \dots \end{array}$$

where $\Lambda_{-r} = (\Lambda_{r-1}, 1)$ for $r \geq 1$. The Λ_{-r} are G -homomorphisms; they are monomorphisms (because the positive dimensional Λ 's are onto), and all rectangles except the first commute. Because all the triangles commute, we have in the first rectangle $\Lambda_{-1} \partial^{G/H} \Lambda_0 = \Lambda_{-1} \mu^{G/H} \epsilon^{G/H} \Lambda_0 = \mu^G \epsilon^G = \partial^G$.

It will be necessary to know the action of the negative dimensional Λ 's explicitly. Since Λ_{-1} is a G -homomorphism, it suffices to know its behavior on a set of G -generators of Y_{-1} —that is, we want $\Lambda_{-1}(\cdot) \in X_{-1} = \text{Hom}_Z(X_0, Z)$.

Now $\Lambda_{-1}(\cdot)$ must be determined on the Z -basis $\{\sigma[\cdot] \mid \sigma \in G\}$ of X_0 . We have

$$(\Lambda_{-1}(\cdot))(\sigma[\cdot]) = ((\cdot)\Lambda_0)(\sigma[\cdot]) = ((\cdot))(\bar{\sigma}[\cdot]) = \begin{cases} 1 & \text{if } \bar{\sigma} = 1 \in G/H \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\Lambda_{-1}((\cdot)) = S_H(\cdot) = \sum_{\tau \in H} (\cdot)^\tau$; and since Λ_{-1} is a G -homomorphism, we have for every $\rho \in G$

$$(8.1) \quad \Lambda_{-1}((\cdot)^\rho) = (\rho S_H)(\cdot) = \sum_{\tau \in \rho H} (\cdot)^\tau.$$

In exactly the same way, denoting elements of G/H by α 's, we have for $r \geq 2$

$$(8.2) \quad \begin{aligned} \Lambda_{-r}((\alpha_1, \dots, \alpha_{r-1})^\rho) &= (\rho S_H) \sum_{\sigma_i \in \alpha_i} (\sigma_1, \dots, \sigma_{r-1}) \\ &= \sum_{\substack{\tau \in \rho H \\ \sigma_i \in \alpha_i}} (\sigma_1, \dots, \sigma_{r-1})^\tau. \end{aligned}$$

Of course, by making the obvious convention (8.2) includes (8.1).

Suppose now that A is any G -module; from (7) we have

$$(9) \quad \begin{array}{ccccccc} \dots & \xrightarrow{\delta^G} & \text{Hom}_G(X_{-2}, A) & \xrightarrow{\delta^G} & \text{Hom}_G(X_{-1}, A) & \xrightarrow{\delta^G} & \text{Hom}_G(X_0, A) \\ & & \downarrow (\Lambda_{-2}, 1) & & \downarrow (\Lambda_{-1}, 1) & & \uparrow (\Lambda_0, 1) \\ \dots & \xrightarrow{\delta^{G/H}} & \text{Hom}_G(Y_{-2}, A) & \xrightarrow{\delta^{G/H}} & \text{Hom}_G(Y_{-1}, A) & \xrightarrow{\delta^{G/H}} & \text{Hom}_G(Y_0, A) \end{array}$$

where $\delta^G = (\partial^G, 1)$ and $\delta^{G/H} = (\partial^{G/H}, 1)$. All the rectangles commute; in the last one $(\Lambda_0, 1)\delta^{G/H}(\Lambda_{-1}, 1) = \delta^G$ because $(\Lambda_0, 1)(\partial^{G/H}, 1)(\Lambda_{-1}, 1) = (\Lambda_{-1}\partial^{G/H}\Lambda_0, 1) = (\partial^G, 1) = \delta^G$.

For each $r \geq 1$ $(\Lambda_{-r}, 1)$ maps $f \in \text{Hom}_G(X_{-r}, A)$ into $(\Lambda_{-r}, 1)f = f\Lambda_{-r} \in \text{Hom}_G(Y_{-r}, A)$. Evaluating this on the generators $(\alpha_1, \dots, \alpha_{r-1})$ (with the convention in force when $r = 1$) gives

$$(f\Lambda_{-r})((\alpha_1, \dots, \alpha_{r-1})) = S_H(\sum_{\sigma_i \in \alpha_i} f(\sigma_1, \dots, \sigma_{r-1})) \in S_H A \subset A^H.$$

Extending linearly, it follows that $(\Lambda_{-r}, 1)f: Y_{-r} \rightarrow A^H$. Hence, in the second row of (9), $\text{Hom}_G(Y_{-r}, A)$ may be replaced by $\text{Hom}_G(Y_{-r}, A^H) = \text{Hom}_{G/H}(Y_{-r}, A^H)$ for $r \geq 1$. These latter are, of course, the cochains of G/H in A^H . Our diagram is now

$$(9.1) \quad \begin{array}{ccccc} \cdots & \xrightarrow{\delta^G} & \text{Hom}_G(X_{-2}, A) & \xrightarrow{\delta^G} & \text{Hom}_G(X_{-1}, A) & \xrightarrow{\delta^G} & \text{Hom}_G(X_0, A) \\ & & \downarrow (\Lambda_{-2}, 1) & & \downarrow (\Lambda_{-1}, 1) & & \uparrow (\Lambda_0, 1) \\ \cdots & \xrightarrow{\delta^{G/H}} & \text{Hom}_{G/H}(Y_{-2}, A^H) & \xrightarrow{\delta^{G/H}} & \text{Hom}_{G/H}(Y_{-1}, A^H) & \xrightarrow{\delta^{G/H}} & \text{Hom}_G(Y_0, A) \end{array}$$

Since this diagram commutes, it follows that for $r \geq 2$ $(\Lambda_{-r}, 1)$ maps cocycles into cocycles and coboundaries into coboundaries; also, $(\Lambda_{-1}, 1)$ takes coboundaries into coboundaries. Since Λ_0 maps X_0 onto Y_0 , $(\Lambda_0, 1)$ is a monomorphism, so that $(\Lambda_{-1}, 1)$ takes cocycles into cocycles. In conclusion, there is a cohomological map, called the *deflation*, such that

$$\text{defl}_{G \rightarrow G/H} : H^{-r}(G, A) \rightarrow H^{-r}(G/H, A^H) \quad \text{for } r \geq 1.$$

(By coherence, one may then carry over the deflation map to arbitrary G - and G/H -complexes.)

Remark. In case $A = Z$ there exists a useful map analogous to deflation. Since here the action of S_H is simply multiplication by $m = \#(H)$, we may define for $r \geq 1$ maps

$$\frac{(\Lambda_{-r}, 1)}{m} : \text{Hom}_G(X_{-r}, Z) \rightarrow \text{Hom}_{G/H}(Y_{-r}, Z) = \text{Hom}_G(Y_{-r}, Z^H)$$

by

$$\left(\frac{(\Lambda_{-r}, 1)}{m} f \right) (\alpha_1, \dots, \alpha_{r-1}) = \sum_{\sigma_1 \in \alpha_1} f(\sigma_1, \dots, \sigma_{r-1}).$$

By using $m(\Lambda_0, 1)$, we get a diagram similar to (9.1); and there are induced cohomological maps

$$\frac{\text{defl}}{m} : H^{-r}(G, Z) \rightarrow H^{-r}(G/H, Z), \quad r \geq 1.$$

We shall also have occasion to use maps of the $\frac{\text{defl}}{m}$ type when A is a module on which the action of G is trivial.

In some proofs of a theorem of TATE (which gives sufficient conditions that the G module A be cohomologically trivial—see [2; p. 53]) one has recourse to a somewhat accidental exact sequence

$$H^0(H, A) \rightarrow H^0(G, A) \rightarrow H^0(G/H, A^H) \rightarrow 0.$$

This sequence is derived as follows. Start with the sequence $A^H \xrightarrow{S_{H \rightarrow G}} A^G \xrightarrow{1} A^G \rightarrow 0$ which is not exact. Since $S_{H \rightarrow G} : S_H A \rightarrow S_G A$ and $S_G A = S_{H \rightarrow G} S_H A \subset S_{H \rightarrow G} A^H$ we get an induced sequence

$$A^H / S_H A \xrightarrow{\tilde{S}_{H \rightarrow G}} A^G / S_G A \xrightarrow{\tilde{1}} A^G / S_{H \rightarrow G} A^H \rightarrow 0$$

which is exact. Because these are canonically isomorphic to the 0-dimensional groups in question and since the trace $S_{H \rightarrow G}$ corresponds to verlagerung in dimension 0, we have the commutative diagram

$$(10) \quad \begin{array}{ccccccc} \frac{A^H}{S_H A} & \xrightarrow{\tilde{S}_{H \rightarrow G}} & \frac{A_G}{S_G A} & \xrightarrow{\tilde{1}} & \frac{A^G}{S_{H \rightarrow G} A^H} & \longrightarrow & 0 \\ \downarrow \tilde{K}_H & & \downarrow \tilde{K}_G & & \downarrow \tilde{K}_{G/H} & & \\ H^0(H, A) & \xrightarrow{V_{H \rightarrow G}} & H^0(G, A) & \xrightarrow{\text{defl}} & H^0(G/H, A^H) & \longrightarrow & 0 \end{array}$$

where deflation is defined to give commutativity.

This deflation map may be described explicitly through a cochain map—also denoted by defl. In fact, if for $a \in A^G$ we put

$$\text{defl}(\kappa_G a) = \kappa_{G/H} a$$

then the following diagram commutes:

$$\begin{array}{ccc} A^G & \xrightarrow{1} & A^G \\ \downarrow K_G & & \downarrow K_{G/H} \\ \mathcal{Z}^0(G, A) & \xrightarrow{\text{defl}} & \mathcal{Z}^0(G/H, A^H) \end{array}$$

From it we get the corresponding rectangle of (10), and

$$(11) \quad \begin{array}{ccc} A^G & \xrightarrow{1} & A^G \\ \downarrow \bar{K}_G & & \downarrow \bar{K}_{G/H} \\ H^0(G, A) & \xrightarrow{\text{defl}} & H^0(G/H, A^H) \end{array}$$

4. In this section some of the standard types of properties are verified for deflation.

Proposition 1. *Let A be a G -module and H a normal subgroup of G ; then the following diagram commutes:*

$$\begin{array}{ccc} A_{S_G} & \xrightarrow{S_H} & (A^H)_{S_{G/H}} \\ \downarrow \bar{\eta}_G & & \downarrow \bar{\eta}_{G/H} \\ H^{-1}(G, A) & \xrightarrow{\text{defl}} & H^{-1}(G/H, A^H) \end{array}$$

Proof. Starting with $a \in A_{S_G}$ the lower path gives

$$\begin{aligned} a \rightarrow \tilde{\eta}_G a &= \eta_G a + B^{-1}(G, A) \rightarrow (\Lambda_{-1,1})(\eta_G a) + B^{-1}(G/H, A^H) \\ &= (\eta_G a) \Lambda_{-1} + B^{-1}(G/H, A^H). \end{aligned}$$

The upper path yields

$$a \rightarrow S_H a \rightarrow \tilde{\eta}_{G/H}(S_H a) = \eta_{G/H}(S_H a) + B^{-1}(G/H, A^H).$$

But $(\eta_G a) \Lambda_{-1} = \eta_{G/H}(S_H a)$ since

$$((\eta_G a) \Lambda_{-1}) \left(\frac{\cdot}{G/H} \right) = (\eta_G a)(S_H \left(\frac{\cdot}{G} \right)) = S_H a = (\eta_{G/H}(S_H a)) \left(\frac{\cdot}{G/H} \right).$$

(Of course, S_H maps $I_G \cdot A \rightarrow I_{G/H} A^H$ and the corresponding diagram with the $\tilde{\eta}$'s holds.)

Proposition 2. *Deflation commutes with verlagerung. More precisely, let $N \subset H \subset G$ with N normal in G ; then for any G -module A the following diagram commutes:*

$$\begin{array}{ccc} H^{-r}(H, A) & \xrightarrow{V_{H \rightarrow G}} & H^{-r}(G, A) \\ \text{defl} \downarrow & & \downarrow \text{defl} \\ H^{-r}(H/N, A^N) & \xrightarrow{V_{H/N \rightarrow G/N}} & H^{-r}(G/N, A^N) \end{array} \quad r \geq 0$$

Proof. First let us dispose of the cases $r = 0$ and 1. Recall that in dimension 0 verlagerung $V_{H \rightarrow G}$ corresponds to the trace $S_{H \rightarrow G}$, and in dimension -1 $V_{H \rightarrow G}$ corresponds to inclusion. Making use of (11), the assertion for $r = 0$ follows from the commutativity of

$$\begin{array}{ccc} A^H & \xrightarrow{S_{H \rightarrow G}} & A^G \\ \downarrow 1 & & \downarrow 1 \\ (A^N)^{H/N} = A^H & \xrightarrow{S_{H/N \rightarrow G/N}} & A^G = (A^N)^{G/N} \end{array}$$

By using Proposition 1, the result for $r = 1$ follows from

$$\begin{array}{ccc} A_{S_H} & \xrightarrow{\text{incl}} & A_{S_G} \\ S_N \downarrow & & \downarrow S_N \\ A_{S_{H/N}} & \xrightarrow{\text{incl}} & A_{S_{G/N}} \end{array}$$

For the remaining dimensions, it surely suffices to show that commutativity

holds for cochains. Since the cochain map $V_{H \rightarrow G}$ is given by

$$(V_{H \rightarrow G}t)(\sigma_1, \dots, \sigma_{r-1}) = \begin{cases} t(\sigma_1, \dots, \sigma_{r-1}) & \text{if all } \sigma_i \in H \\ 0 & \text{otherwise,} \end{cases}$$

it is not too painful to verify this directly. Starting with a $-r$ -dimensional cochain t of H in A (and writing $\beta_i \in G/N$, $\sigma_i \in G$, $h_i \in H$) we have

$$\begin{aligned} (V_{H/N \rightarrow G/N}(\text{defl } t))(\beta_1, \dots, \beta_{r-1}) &= \begin{cases} \text{defl } t(\beta_1, \dots, \beta_{r-1}) & \text{if all } \beta_i \in H/N \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} S_N\left\{\sum_{h_i \in \beta_i} t(h_1, \dots, h_{r-1})\right\} & \text{if all } \beta_i \in H/N \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} (\text{defl } (V_{H \rightarrow G}t))(\beta_1, \dots, \beta_{r-1}) &= S_N\left\{\sum_{\sigma_i \in \beta_i} (V_{H \rightarrow G}t)(\sigma_1, \dots, \sigma_{r-1})\right\} \\ &= S_N\left\{\sum_{\sigma_i \in \beta_i \cap H} t(\sigma_1, \dots, \sigma_{r-1})\right\}. \end{aligned}$$

Since $\beta_i \cap H \neq \emptyset$ if and only if $\beta_i \subset H$, the two expressions are the same. This completes the proof.

Proposition 3. *Deflation commutes with restriction; more precisely, if $N \subset H \subset G$ with N normal in G , then for any G -module A the following diagram commutes:*

$$\begin{array}{ccc} H^{-r}(G, A) & \xrightarrow{\text{res}_{G \rightarrow H}} & H^{-r}(H, A) \\ \text{defl} \downarrow & & \downarrow \text{defl} \\ H^{-r}(G/N, A^N) & \xrightarrow{\text{res}_{G/N \rightarrow H/N}} & H^{-r}(H/N, A^N) \end{array} \quad r \geq 0$$

Proof. In dimension 0, res corresponds to incl so the assertion follows from the commutativity of

$$\begin{array}{ccc} A^G & \xrightarrow{\text{incl}} & A^H \\ 1 \downarrow & & \downarrow 1 \\ (A^N)^{G/N} = A^G & \xrightarrow{\text{incl}} & A^H = (A^N)^{H/N} \end{array}$$

As for dimension -1 —if $G = \bigcup \tau_i H$ is a left coset decomposition then $G = \bigcup H \tau_i^{-1}$ is a right coset decomposition and $\text{res}_{G \rightarrow H}$ corresponds to multiplication by $\sum_i \tau_i^{-1}$. It follows that $\text{res}_{G/N \rightarrow H/N}$ also corresponds to multiplication by

$\sum_i \tau_i^{-1}$. The result now follows from the commutativity of

$$\begin{array}{ccc} A_{S_G} & \xrightarrow{\sum \tau_i^{-1}} & A_{S_H} \\ S_N \downarrow & & \downarrow S_N \\ (A^N)_{S_{G/N}} & \xrightarrow{\sum \tau_i^{-1}} & (A^N)_{S_{H/N}} \end{array}$$

For dimensions ≤ -2 , using the fact that

$$(\text{res}_{G \rightarrow H} t)(h_1, \dots, h_{r-1}) = \sum_{\sigma_i, \dots, \sigma_r \in \{\tau_i\}} \sigma_1^{-1} t(\sigma_1 h_1 \sigma_2^{-1}, \sigma_2 h_2 \sigma_3^{-1}, \dots, \sigma_{r-1} h_{r-1} \sigma_r^{-1}),$$

a straightforward computation shows that the corresponding cochain (or cocycle) diagram commutes. In fact, for $\alpha_1, \dots, \alpha_{r-1} \in H/N$ and $t \in Z^{-r}(G, A)$

$$\begin{aligned} (\text{defl}(\text{res } t))(\alpha_1, \dots, \alpha_{r-1}) &= \sum_{h_i \in \alpha_i} \sum_{\sigma_1, \dots, \sigma_r \in \{\tau_i\}} S_N \sigma_1^{-1} t(\sigma_1 h_1 \sigma_2^{-1}, \dots, \sigma_{r-1} h_{r-1} \sigma_r^{-1}) \\ &= (\text{res}(\text{defl } t))(\alpha_1, \dots, \alpha_{r-1}). \end{aligned}$$

Proposition 4. *Deflation commutes with the coboundary arising from exact sequences. More precisely, suppose that $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$ is an exact G -sequence and that for the normal subgroup H the G/H sequence $0 \rightarrow A^H \xrightarrow{i} B^H \xrightarrow{j} C^H \rightarrow 0$ is exact. Then the following diagram commutes:*

$$\begin{array}{ccc} H^{-r}(G, C) & \xrightarrow{\delta_*} & H^{-r+1}(G, A) \\ \text{defl} \downarrow & & \downarrow \text{defl} \\ H^{-r}(G/H, C^H) & \xrightarrow{\delta_*} & H^{-r+1}(G/H, A^H) \end{array} \quad r \geq 1$$

Proof. First we take care of the case $r = 1$. In the diagram

$$(12) \quad \begin{array}{ccc} C_{S_G} & & A^G \\ \bar{\eta}_G \downarrow & & \downarrow \bar{\kappa}_G \\ H^{-1}(G, C) & \xrightarrow{\delta_*} & H^0(G, A) \end{array}$$

we know that, for $c \in C_{S_G}$, $\delta_*(\bar{\eta}_G c) = \bar{\kappa}_G a$ where $j\bar{b} = c$ and $i\bar{a} = S_G b$. Thus from (11) $\text{defl} \{ \delta_*(\bar{\eta}_G c) \} = \bar{\kappa}_{G/H} a$. Going the other way, and using the diagram corresponding to (12) for G/H , we have

$$\delta_* \{ \text{defl}(\bar{\eta}_G c) \} = \delta_* \{ \bar{\eta}_{G/H}(S_H c) \} = \bar{\kappa}_{G/H} a'$$

where $jb' = S_H a$ and $ia' = S_{G/H} b'$. Since

$$j(S_H b) = S_H j b = S_H a \quad \text{and} \quad ia' = S_{G/H}(S_H b) = S_G b \quad \text{if} \quad b' = S_H b,$$

we may take $a' = a$.

For the remaining dimensions, we write

$$X_- = \sum_{i < 0} \oplus X_i, \quad Y_- = \sum_{i < 0} \oplus Y_i, \quad \Lambda_- = \sum_{i < 0} \oplus \Lambda_i$$

and $\text{Hom}_G(X_-, A) = \sum_{i < 0} \oplus \text{Hom}_G(X_i, A)$. Under our hypotheses, the following diagram commutes and has exact rows:

$$(13) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_G(X_-, A) & \xrightarrow{(1,i)} & \text{Hom}_G(X_-, B) & \xrightarrow{(1,j)} & \text{Hom}_G(X_-, C) \longrightarrow 0 \\ & & (\Lambda_-, 1) \downarrow & & (\Lambda_-, 1) \downarrow & & (\Lambda_-, 1) \downarrow \\ 0 & \longrightarrow & \text{Hom}_{G/H}(Y_-, A^H) & \xrightarrow{(1,i)} & \text{Hom}_{G/H}(Y_-, B^H) & \xrightarrow{(1,j)} & \text{Hom}_{G/H}(Y_-, C^H) \longrightarrow 0 \end{array}$$

From this we get in the usual way the commutative diagram with exact rows

$$(14) \quad \begin{array}{ccccccccc} \dots & \longrightarrow & H^{-3}(G, C) & \xrightarrow{\delta_*} & H^{-2}(G, A) & \xrightarrow{i_*} & H^{-2}(G, B) & \xrightarrow{j_*} & H^{-2}(G, C) & \xrightarrow{\delta_*} & H^{-1}(G, A) \\ & & \text{defl} \downarrow & & \text{defl} \downarrow & & \text{defl} \downarrow & & \text{defl} \downarrow & & \text{defl} \downarrow \\ \dots & \longrightarrow & H^{-3}(G/H, C^H) & \xrightarrow{\delta_*} & H^{-2}(G/H, A^H) & \xrightarrow{i_*} & H^{-2}(G/H, B^H) & \xrightarrow{j_*} & H^{-2}(G/H, C^H) & \xrightarrow{\delta_*} & H^{-1}(G/H, A^H) \end{array}$$

As we have this diagram, exactness does not hold at the last place, and the commutativity of the last rectangle must be handled separately—it follows from the fact that in (13) in each case the 0-dimensional cochains can be attached and that $(\Lambda_-, 1)$ takes cocycles into cocycles.

As a supplement to Proposition 4, and to provide a bridge between deflation and inflation, we have

Proposition 5. *Let $0 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$ be an exact G -sequence, and suppose that $0 \rightarrow A^H \xrightarrow{i} B^H \xrightarrow{j} C^H \rightarrow 0$ is an exact G/H -sequence; then the following diagram commutes:*

$$\begin{array}{ccc} H^0(G, C) & \xrightarrow{\delta_*} & H^1(G, A) \\ \text{defl} \downarrow & & \uparrow \text{inf} \\ H^0(G/H, C^H) & \xrightarrow{\delta_*} & H^1(G/H, A^H) \end{array}$$

Proof. The δ_* 's are defined from the exact sequences

$$0 \rightarrow \text{Hom}_G(X, A) \xrightarrow{(1,i)} \text{Hom}_G(X, B) \xrightarrow{(1,j)} \text{Hom}_G(X, C) \rightarrow 0,$$

$$0 \rightarrow \text{Hom}_{G/H}(Y, A^H) \xrightarrow{(1,i)} \text{Hom}_{G/H}(Y, B^H) \xrightarrow{(1,j)} \text{Hom}_{G/H}(Y, C^H) \rightarrow 0.$$

Consider any $\alpha \in H^0(G, C)$. We may take $c \in C^G$ such that $\bar{\kappa}_G(c) = \alpha$. Therefore $\text{defl } \alpha = \bar{\kappa}_{G/H}(c)$. Choose $b \in B^H$ such that $jb = c$; thus $(1, j)\{\kappa_{G/H}(b)\} = j\{\kappa_{G/H}(b)\} = \kappa_{G/H}(c)$ ($\kappa_{G/H}(c)$ represents α and is in $\text{Hom}_{G/H}(Y_0 C^H)$, $\kappa_{G/H}(b) \in \text{Hom}_{G/H}(Y_0, B^H)$). Now there exists $f \in \text{Hom}_{G/H}(Y_1, A^H)$ such that $(1, i)f = \delta\{\kappa_{G/H}(b)\}$. Thus f represents δ_* $\text{defl } \alpha$, and $if[\bar{\sigma}] = \kappa_{G/H}(b)\partial[\bar{\sigma}] = \kappa_{G/H}(b)\{\bar{\sigma}[\cdot] - [\cdot]\} = (\bar{\sigma} - 1)b$. Finally $\inf_{G/H} \delta_* \text{defl } \alpha$ is represented by $\inf_{G/H} f \in \text{Hom}_G(X_1, A)$.

On the other hand, $(1, j)\{\kappa_G(b)\} = \kappa_G(c)$; and $(1, i) \inf f = \delta\{\kappa_G(b)\}$ since

$$(1, i) \inf f[\sigma] = if[\bar{\sigma}] = (\bar{\sigma} - 1)b = (\sigma - 1)b = (\delta\{\kappa_G(b)\})[\sigma].$$

Therefore $\inf f$ represents $\delta_* \alpha$ and the proof is complete.

Remark. Suppose that in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' \longrightarrow 0 \end{array}$$

the top row is an exact G -sequence, the bottom row is an exact G/H -sequence (hence also a G -sequence) and f, g, h are G -homomorphisms. Then the following diagram has exact rows and commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_G(X_-, A) & \longrightarrow & \text{Hom}_G(X_-, B) & \longrightarrow & \text{Hom}_G(X_-, C) \longrightarrow 0 \\ & & (\Lambda_-, f) \downarrow & & (\Lambda_-, g) \downarrow & & (\Lambda_-, h) \downarrow \\ 0 & \longrightarrow & \text{Hom}_{G/H}(Y_-, A') & \longrightarrow & \text{Hom}_{G/H}(Y_-, B') & \longrightarrow & \text{Hom}_{G/H}(Y_-, C') \longrightarrow 0 \end{array}$$

From it we get the obvious analogue of (14) with defl being replaced by the induced maps $(\Lambda_-, f)_*$, $(\Lambda_-, g)_*$ and $(\Lambda_-, h)_*$. Of course, if the maps

$$\frac{(\Lambda_-, f)}{m}, \quad \frac{(\Lambda_-, g)}{m}, \quad \frac{(\Lambda_-, h)}{m}$$

have meaning a similar result holds for them.

Proposition 6. *Deflation is transitive; in other words, if $N \subset H \subset G$ with N and H both normal in G , then*

$$\text{defl}_{G/N \rightarrow G/N/H/N \cong G/H} \circ \text{defl}_{G \rightarrow G/N} = \text{defl}_{G \rightarrow G/H}$$

Proof. In dimension 0 this is clear from (11). For dimensions ≤ -1 the result may be chased through by noting that the positive dimensional Λ 's (in terms of which inflation is defined) have the transitivity property, and hence so do their duals the negative dimensional Λ 's.

Proposition 7. Let A be a G -module and H be a normal subgroup of G ; then in

$$H^{-r}(H, A) \xrightarrow{V} H^{-r}(G, A) \xrightarrow{\text{defl}} H^{-r}(G/H, A^H), \quad r \geq 0,$$

we have $\text{defl} \circ V = 0$.

Proof. This is immediate from the fact that $H^{-r}(\{1\}, A^H) = (0)$ and from the commutativity of

$$\begin{array}{ccc} H^{-r}(H, A) & \xrightarrow{V} & H^{-r}(G, A) \\ \text{defl} \downarrow & & \downarrow \text{defl} \\ H^{-r}(H/H, A^H) & \xrightarrow{V} & H^{-r}(G/H, A^H) \end{array}$$

The reader will have no difficulty in verifying

Proposition 8. Deflation commutes with induced homomorphisms. More precisely, let $f: A \rightarrow B$ be a G -homomorphism of G -modules, and let $f: A^H \rightarrow B^H$ be the corresponding G/H -homomorphism of G/H -modules; then the following diagram commutes:

$$\begin{array}{ccc} H^{-r}(G, A) & \xrightarrow{f_*} & H^{-r}(G, B) \\ \text{defl} \downarrow & & \downarrow \text{defl} \\ H^{-r}(G/H, A^H) & \xrightarrow{f_*} & H^{-r}(G/H, B^H) \end{array} \quad r \geq 0$$

Remark. It is clear that $\frac{\text{defl}}{m}$ (which has been defined only for $A = Z$) has properties like those of defl that have been treated in the preceding propositions.

5. As a dual of the standard theorem about the exactness of the inflation-restriction sequence, we prove

Theorem 1. Let A be a G -module and H be a normal subgroup of G ; let $r \geq 0$. If $H^{-v}(H, A) = (0)$ for $v = 0, 1, 2, \dots, r-1$ then the sequence

$$H^{-r}(H, A) \xrightarrow{V} H^{-r}(G, A) \xrightarrow{\text{defl}} H^{-r}(G/H, A^H) \rightarrow 0$$

is exact.

Proof. By induction on r . If $r = 0$ (the hypotheses are then vacuous) the sequence is surely exact (by (10)). Suppose then that the result holds for $r = k \geq 1$ (and for all G -modules that satisfy the conditions). First of all, there exists an exact G -sequence $0 \rightarrow C \rightarrow B \rightarrow A \rightarrow 0$ such that B is G -regular. (In the terminology of [2] this says that the G -module B splits. What concerns us is that $H^n(H, B) = H^n(G, B) = (0)$ for all n .) Thus, for every n , $\delta_*: H^n(H, A) \rightarrow H^{n+1}(H, C)$ and $\delta_*: H^n(G, A) \rightarrow H^n(G, C)$ are isomorphisms onto. If we are trying to prove our result for $r = k+1$ and for the G -module A , our hypotheses

therefore give $H^{-\nu}(H, C) = (0)$ for $\nu = -1, 0, 1, \dots, k-1$. Because $H^1(H, C) = (0)$, it follows that $0 \rightarrow C^H \rightarrow B^H \rightarrow A^H \rightarrow 0$ is an exact G/H -sequence; of course, B^H is G/H -regular. Consider now the diagram

$$\begin{array}{ccccccc} H^{-k}(H, C) & \xrightarrow{V} & H^{-k}(G, C) & \xrightarrow{\text{defl}} & H^{-k}(G/H, C^H) & \longrightarrow & 0 \\ \delta_* \uparrow & & \delta_* \uparrow & & \delta_* \uparrow & & \\ H^{-k-1}(H, A) & \xrightarrow{V} & H^{-k-1}(G, A) & \xrightarrow{\text{defl}} & H^{-k-1}(G/H, A^H) & \longrightarrow & 0 \end{array}$$

The δ_* 's are isomorphisms onto, and the rectangles commute (using Proposition 4 and the fact that V commutes with δ_*). By the induction hypothesis the top row is exact, hence so is the bottom row. This completes the proof.

6. It is well known that $H^{-2}(G, Z)$ is isomorphic to the factor commutator group G/C^c . In this section we interpret defl and $\frac{\text{defl}}{m} : H^{-2}(G, Z) \rightarrow H^{-2}(G/H, Z)$ as maps of $G/G^c \rightarrow G/H/(G/H)^c$. First some preliminaries are needed.

Put $\Gamma_G = Z[G]$ and define $\epsilon^G : \Gamma_G \rightarrow Z$ by $\epsilon^G(\sum_{\sigma \in G} n_\sigma \sigma) = \sum_{\sigma \in G} n_\sigma$. The kernel is

$$I_G = \{\gamma \in \Gamma_G \mid \epsilon^G(\gamma) = 0\} = \left\{ \sum_{\sigma \neq 1} \oplus Z(\sigma - 1) \right\}.$$

Since in the exact G -sequence $0 \rightarrow I_G \rightarrow \Gamma_G \rightarrow Z \rightarrow 0$ Γ_G is G -regular, it follows that $H^{-2}(G, Z) \xrightarrow[\cong]{\delta_*} H^{-1}(G, I_G)$. Furthermore,

$$H^{-1}(G, I_G) \xleftarrow{\eta^G} (I_G)_{s_G} / I_G \cdot I_G = I_G / I_G^2$$

and $I_G / I_G^2 \approx G/G^c$ under the correspondence $(\sigma - 1) + I_G^2 \leftrightarrow \sigma G^c$. Combining all of these gives a canonical isomorphism of $H^{-2}(G, Z)$ and G/G^c .

Theorem 2. *Let H be a normal subgroup of G with $m = \#(H)$, then the following diagram commutes:*

$$\begin{array}{ccc} H^{-2}(G, Z) & \approx & \frac{G}{G^c} \\ \downarrow \frac{\text{defl}}{m} & & \downarrow \text{canonical} \\ H^{-2}(G/H, Z) & \approx & \frac{G/H}{(G/H)^c} \end{array}$$

Proof. Consider the diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_G & \xrightarrow{i} & \Gamma_G & \xrightarrow{\epsilon^G} & Z \longrightarrow 0 \\ & & \phi' \downarrow & & \phi \downarrow & & 1 \downarrow \\ 0 & \longrightarrow & I_{G/H} & \xrightarrow{i} & \Gamma_{G/H} & \xrightarrow{\epsilon^{G/H}} & Z \longrightarrow 0 \end{array}$$

The top row is an exact G -sequence; the bottom row is an exact G/H -sequence. ϕ is defined by mapping $\sigma \rightarrow \bar{\sigma} \in G/H$ and extending linearly, and ϕ' is the restriction of ϕ to I_G . Thus ϕ' , ϕ , 1 are G -homomorphisms, and by the remark after Proposition 4 we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_G(X_-, I_G) & \xrightarrow{(1, i)} & \text{Hom}_G(X_-, \Gamma_G) & \xrightarrow{(1, \epsilon^G)} & \text{Hom}_G(X_-, Z) \longrightarrow 0 \\ & & \downarrow \frac{(\Lambda_-, \phi')}{m} & & \downarrow \frac{(\Lambda_-, \phi)}{m} & & \downarrow \frac{(\Lambda_-, 1)}{m} \\ 0 & \longrightarrow & \text{Hom}_{G/H}(Y_-, I_{G/H}) & \longrightarrow & \text{Hom}_{G/H}(Y_-, \Gamma_{G/H}) & \longrightarrow & \text{Hom}_{G/H}(Y_-, Z) \longrightarrow 0 \end{array}$$

The vertical maps are well defined—since H acts trivially on $I_{G/H}$, $\Gamma_{G/H}$ and Z , and when an element of one of these modules is divisible by m it is uniquely divisible. From this (as in that same remark) it follows that the rectangle in the following diagram commutes:

$$\begin{array}{ccc} H^{-2}(G, Z) & \xrightarrow[\approx]{\delta_*} & H^{-1}(G, I_G) \approx \frac{I_G}{I_G^2} \approx \frac{G}{G^c} \\ \downarrow \frac{\text{defl}}{m} & & \downarrow \frac{(\Lambda_-, \phi')}{m} * \\ H^{-2}(G/H, Z) & \xrightarrow[\approx]{\delta_*} & H^{-1}(G/H, I_{G/H}) \approx \frac{I_{G/H}}{I_{G/H}^2} \approx \frac{G/H}{(G/H)^c} \end{array}$$

Now we take an element of G/G^c and follow it through.

$$\begin{aligned} \sigma G^c &\rightarrow (\sigma - 1) + I_G^2 \rightarrow \eta_G(\sigma - 1) + B^{-1}(G, I_G) \\ &\rightarrow \frac{(\Lambda_-, \phi')}{m} [\eta_G(\sigma - 1)] + B^{-1}(G/H, I_{G/H}) \\ &= \frac{\phi'}{m} \eta_G(\sigma - 1) \Lambda_{-1} + B^{-1}(G/H, I_{G/H}) = \eta_{G/H}(\bar{\sigma} - \bar{1}) + B^{-1}(G/H, I_{G/H}) \\ &\left(\text{since } \left(\frac{\phi'}{m} \eta_G(\sigma - 1) \Lambda_{-1} \right) \Big|_{G/H} = \frac{\phi'}{m} S_H(\sigma - 1) = \phi'(\sigma - 1) = \bar{\sigma} - \bar{1} \right) \\ &\rightarrow (\bar{\sigma} - \bar{1}) + I_{G/H}^2 \rightarrow \bar{\sigma}(G/H)^c. \end{aligned}$$

Of course, defl maps $\sigma G^c \rightarrow \bar{\sigma}^m(G/H)^c$.

7. In this section we give a somewhat sketchy treatment (due to the large number of details involved) of the behavior of deflation with respect to the cup product.

Let $\theta: A \times B \rightarrow C$ be a G -pairing of G -modules (see [3]), and let $\bar{\theta}: A \otimes B \rightarrow C$ be the corresponding G -homomorphism. If H is a normal subgroup of G , there

is an induced G/H -pairing of G/H -modules (which is also denoted by θ), $\theta: A^H \times B^H \rightarrow C^H$.

Let X be any G -complex; we put $\partial' = \partial \otimes 1: X_p \otimes X_q \rightarrow X_{p-1} \otimes X_q$ and $\partial'' = 1 \otimes \partial: X_p \otimes X_q \rightarrow X_p \otimes X_{q-1}$ for every pair (p, q) . For all (p, q) there exist G -homomorphisms $\phi_{p,q}: X_{p+q} \rightarrow X_p \otimes X_q$ such that

- (i) $\phi_{p,q} \partial = \partial' \phi_{p+1,q} + (-1)^p \partial'' \phi_{p,q+1}$,
- (ii) $\epsilon = (\epsilon \otimes \epsilon) \phi_{0,0}$.

Let f be a p -cochain of G in A and g be a q -cochain of G in B —that is, $f \in \text{Hom}_G(X_p, A)$ and $g \in \text{Hom}_G(X_q, B)$; then defining

$$f \cup_\theta g = \bar{\theta} \circ (f \otimes g) \circ \phi_{p,q}$$

we have $f \cup_\theta g \in \text{Hom}_G(X_{p+q}, C)$. This cochain cup product induces the cup product of cohomology classes (also written as $\cup_\theta$),

$$\cup_\theta: H^p(G, A) \otimes H^q(G, B) \rightarrow H^{p+q}(G, C) \quad \text{all } (p, q).$$

In order to be more explicit about the $\phi_{p,q}$'s, we need to recall two facts about any G -complex (1). Firstly, there exist contracting homotopies—that is Z -homomorphisms $D_r: X_r \rightarrow X_{r+1}$ for every r such that $D\partial + \partial D = 1$ in all dimensions. Secondly, by virtue of the G -regularity of X_r , there exist Z -homomorphisms $\pi_r: X_r \rightarrow X_r$ such that $S(\pi_r) = \sum_{\sigma \in G} \pi_r^\sigma = 1$ (the identity map of X_r).

Writing $D'_{p,q} = S(D_p \otimes \pi_q)$ and $D''_{p,q} = S(\pi_p \otimes D_q)$, the existence of the $\phi_{p,q}$'s is then proved according to the following procedure: Choose any $\xi \in X_0$ such that $\epsilon(\xi) = 1 \in Z$; for each q , define $\phi_{0,q}: X_q \rightarrow X_0 \otimes X_q$ by

$$(15) \quad \phi_{0,q}(x) = \sum_{\sigma \in G} (\sigma \xi \otimes \pi_q^\sigma x), \quad x \in X_q.$$

Calling p the row index and q the column index, a row by row inductive procedure may be used to define all the $\phi_{p,q}$. If $p \geq 0$ and $\phi_{p,q}$ is already defined for the entire p^{th} row, we put

$$(16) \quad \phi_{p+1,q} = D' \phi_{p,q} \partial - (-1)^p D' \partial'' \phi_{p,q+1} \quad \text{every } q.$$

While if $p \leq 0$ and the p^{th} row is known, the $(p-1)^{\text{th}}$ row is defined by

$$(17) \quad \phi_{p-1,q} = (-1)^p D'' \partial' \phi_{p,q-1} \quad \text{every } q.$$

Now, let us specialize to the case of the standard complex. Here it is natural to take $\xi = [\cdot]$ and π_q as the projection on the $\rho = 1$ component of X_q expressed as a G -regular module—for example, $X_1 = \sum_{\rho \in G} \oplus \rho(\sum_{\sigma \in G} \oplus Z[\sigma])$ and π_1 is the projection of X_1 on $\sum_{\sigma \in G} \oplus Z[\sigma]$. We also know that in positive dimensions D is given by $D(\sigma[\sigma_1, \dots, \sigma_r]) = [\sigma, \sigma_1, \dots, \sigma_r]$.

Since for each q and every $\rho \in G$, π_q^ρ is the projection of X_q on the ρ -component, it follows immediately from (15) that

$$\begin{aligned}
& \phi_{0,0} : [\cdot] \rightarrow [\cdot] \otimes [\cdot], \\
(18) \quad & \phi_{0,q} : [\sigma_1, \dots, \sigma_q] \rightarrow [\cdot] \otimes [\sigma_1, \dots, \sigma_q], \quad q > 0, \\
& \phi_{0,-1} : (\cdot) \rightarrow [\cdot] \otimes (\cdot), \\
& \phi_{0,-q} : (\sigma_1, \dots, \sigma_{q-1}) \rightarrow [\cdot] \otimes (\sigma_1, \dots, \sigma_{q-1}), \quad q > 1.
\end{aligned}$$

By using (16) and (17) one sees that

$$(19) \quad \phi_{1,1} : [\sigma, \tau] \rightarrow [\sigma] \otimes \sigma[\tau] - \sigma \partial[\sigma^{-1}, \sigma] \otimes \sigma[\tau],$$

$$(20) \quad \phi_{-1,1} : [\cdot] \rightarrow \sum_{\rho \in G} ((\cdot)^\rho \otimes \rho[\rho^{-1}]).$$

(A rather messy computation shows that

$$\phi_{-2,2} : [\cdot] \rightarrow \sum_{\rho, \tau \in G} ((\tau)^\rho \otimes \rho[1, \rho^{-1}]) - \sum_{\rho, \tau \in G} ((\tau^{-1}\rho)^\tau \otimes \tau[\tau^{-1}\rho, \rho^{-1}]).$$

For each integer r , let $H^r(G, A) \xrightleftharpoons[\text{inf}]{\text{defl}} H^r(G/H, A^H)$ refer to the one map that exists.

Theorem 3. *Let $\theta: A \times B \rightarrow C$ be a G -pairing of G -modules; then for all integers p and q , the following diagram commutes:*

$$\begin{array}{ccc}
H^p(G, A) \times H^q(G, B) & \xrightarrow{\cup_\theta} & H^{p+q}(G, C) \\
\text{defl} \downarrow \uparrow \text{inf} & & \text{defl} \downarrow \uparrow \text{inf} \\
H^p(G_H, A^H) \times H^q(G_H, B^H) & \xrightarrow{\cup_\theta} & H^{p+q}(G_H, C^H)
\end{array}$$

Proof. There are six classes to be distinguished:

- I. $p \geq 1, \quad q \geq 1.$
- II. $p \geq 1, \quad q \leq 0, \quad p + q \geq 1.$
- III. $p \geq 1, \quad q \leq 0, \quad p + q \leq 0.$
- IV. $p \leq 0, \quad q \leq 0.$
- V. $p \leq 0, \quad q \geq 1, \quad p + q \leq 0.$
- VI. $p \leq 0, \quad q \geq 1, \quad p + q \geq 1.$

To reduce the number of straightforward verifications, we apply a dimension shifting argument. Suppose the theorem is true for the pair (p, q) ; we show that if $(p+1, q)$ is in the same class as (p, q) then the theorem holds for $(p+1, q)$.

Putting as usual $\Gamma = Z[G]$ and $J = \Gamma/Z(\sum_{\sigma \in G} \sigma)$, we have the sequence

$$(21) \quad 0 \rightarrow Z \rightarrow \Gamma \rightarrow J \rightarrow 0.$$

This is an exact G -sequence which splits, and Γ is G -regular. Tensoring with A and identifying $Z \otimes A \approx A$ gives

$$(22) \quad 0 \rightarrow A \rightarrow \Gamma \otimes A \rightarrow J \otimes A \rightarrow 0.$$

This is an exact G -sequence, and $\Gamma \otimes A$ is G -regular.

Viewing (21) as an exact H -sequence, and noting that $H^1(H, Z) = (0)$ gives

$$(23) \quad 0 \rightarrow Z \rightarrow \Gamma^H \rightarrow J^H \rightarrow 0.$$

This is an exact G/H -sequence with Γ^H G/H -regular; and since J^H is Z -free this sequence splits. Therefore, tensoring by A^H gives the exact G/H -sequence

$$0 \rightarrow A^H \rightarrow \Gamma^H \otimes A^H \rightarrow J^H \otimes A^H \rightarrow 0.$$

However, $\Gamma^H \otimes A^H = (\Gamma \otimes A)^H$ and $J^H \otimes A^H = (J \otimes A)^H$ —so that

$$(24) \quad 0 \rightarrow A^H \rightarrow (\Gamma \otimes A)^H \rightarrow (J \otimes A)^H \rightarrow 0$$

is an exact G/H -sequence whose middle term is G/H -regular.

From (22) and (24) there arises the commutative diagram

$$(25) \quad \begin{array}{ccc} H^p(G, J \otimes A) & \xrightarrow{\delta_*} & H^{p+1}(G, A) \\ \text{defl} \downarrow & \uparrow \text{inf} & \downarrow \text{defl} \uparrow \text{inf} \\ H^p(\mathcal{G}_H, (J \otimes A)^H) & \xrightarrow{\delta_*} & H^{p+1}(\mathcal{G}_H, A^H) \end{array}$$

in which the δ_* 's are isomorphisms onto. Of course, A may be replaced by C in (25).

A well defined G -pairing $\omega: (J \otimes A) \times B \rightarrow J \otimes C$ exists and may be described by

$$\omega((j \otimes a), b) = j \otimes \theta(a, b),$$

and it induces a G/H -pairing $\omega: (J \otimes A)^H \times B^H \rightarrow (J \otimes C)^H$. Now our inductive hypothesis is that the theorem holds for all modules A, B, C and all G -pairings in dimension (p, q) ; hence the following diagram commutes:

$$(26) \quad \begin{array}{ccc} H^p(G, J \otimes A) \times H^q(G, B) & \xrightarrow{\cup \omega} & H^{p+q}(G, J \otimes C) \\ \text{defl} \downarrow \uparrow \text{inf} & & \downarrow \text{defl} \uparrow \text{inf} \\ H^p(\mathcal{G}_H, (J \otimes A)^H) \times H^q(\mathcal{G}_H, B^H) & \xrightarrow{\cup \omega} & H^{p+q}(\mathcal{G}_H, (J \otimes C)^H) \end{array}$$

Taking the diagram

$$(27) \quad \begin{array}{ccccc} H^{p+1}(G, A) \times H^q(G, B) & \xrightarrow{\wr_{\theta}} & H^{p+q+1}(G, C) \\ \text{defl} \downarrow \uparrow \text{inf} & & \downarrow \uparrow \text{inf} \\ H^{p+1}(G/H, A^H) \times H^q(G/H, B^H) & \xrightarrow{\wr_{\theta}} & H^{p+q+1}(G/H, C^H) \end{array}$$

and connecting it to (26) by the δ_* 's for A and C and by the identity map for B , we get a three-dimensional diagram. Since the connections are all isomorphisms onto, and the requisite commutativities hold, the commutativity of (27) follows from that of (26).

The passage from case (p, q) to $(p, q + 1)$ is effected in exactly the same way; the fact that in the three-dimensional diagram corresponding to (26) and (27) the faces that contain δ_* have commutativity character $(-1)^p$ does not affect the end result.

A bit more can be squeezed out of the procedure used above. Suppose that $p \neq 0, p + q = 0$; thus going from (p, q) to $(p + 1, q)$ (or $(p, q + 1)$) involves a change of class. However, in this situation it is still true that (p, q) implies $(p + 1, q)$. In fact, from (26) and (27) we get on the right side the diagram

$$\begin{array}{ccc} H^0(G, J \otimes C) & \xrightarrow{\delta_*} & H^1(G, C) \\ \text{defl} \downarrow & & \uparrow \text{inf} \\ H^0(G/H, (J \otimes C)^H) & \xrightarrow{\delta_*} & H^1(G/H, C^H) \end{array}$$

The desired result then follows immediately from Proposition 5.

Next, we show that (p, q) implies $(p - 1, q)$. Suppose $A = I \otimes A_0, C = I \otimes C_0$; then starting from

$$(28) \quad 0 \rightarrow I \rightarrow \Gamma \rightarrow Z \rightarrow 0,$$

we have

$$(29) \quad 0 \rightarrow I \otimes I \rightarrow \Gamma \otimes I \rightarrow I \rightarrow 0$$

and tensoring with A_0 yields

$$(30) \quad 0 \rightarrow I \otimes A \rightarrow \Gamma \otimes A \rightarrow A \rightarrow 0.$$

Furthermore, since $H^1(H, I \otimes I) \approx H^0(H, I) \approx H^{-1}(H, Z) = (0)$ the sequence (29) gives

$$(31) \quad 0 \rightarrow (I \otimes I)^H \rightarrow (\Gamma \otimes I)^H \rightarrow I^H \rightarrow 0;$$

and tensoring this with A_0^H gives

$$(32) \quad 0 \rightarrow (I \otimes A)^H \rightarrow (\Gamma \times A)^H \rightarrow A^H \rightarrow 0.$$

Therefore, from (30) and (32) the following diagram (and the corresponding one for C) commutes:

$$\begin{array}{ccc} H^{p-1}(G, A) & \xrightarrow{\delta_*} & H^p(G, I \otimes A) \\ \text{defl} \downarrow & \uparrow \text{inf} & \downarrow \text{defl} \\ & & \uparrow \text{inf} \\ H^{p-1}(G/H, A^H) & \xrightarrow{\delta_*} & H^p(G, (I \otimes A)^H) \end{array}$$

The theorem then holds in dimension $(p-1, q)$ for $A = I \otimes A_0$ and $C = I \otimes C_0$.

Now consider any A, B, C ; from (28) we get

$$(33) \quad 0 \rightarrow I \otimes J \rightarrow \Gamma \otimes J \rightarrow J \rightarrow 0;$$

and since $H^1(H, I \otimes J) = 0$,

$$(34) \quad 0 \rightarrow (I \otimes J)^H \rightarrow (\Gamma \otimes J)^H \rightarrow J^H \rightarrow 0.$$

Tensor (33) with A and (34) with A^H . The left rectangle of the following diagram therefore commutes:

$$\begin{array}{ccccc} H^{p-1}(G, I \otimes J \otimes A) & \xrightarrow{\delta_*^{-1}} & H^{p-2}(G, J \otimes A) & \xrightarrow{\delta_*} & H^{p-1}(G, A) \\ \text{defl} \downarrow & \uparrow \text{inf} & \downarrow \text{defl} & \uparrow \text{inf} & \downarrow \text{defl} \\ H^{p-1}(G/H, (I \otimes J \otimes A)^H) & \xrightarrow{\delta_*^{-1}} & H^{p-2}(G/H, (J \otimes A)^H) & \xrightarrow{\delta_*} & H^{p-1}(G/H, A^H) \end{array}$$

The right rectangle commutes by (25). Of course, the corresponding diagram holds for C . Since the theorem holds for $(p-1, q)$ and $I \otimes J \otimes A, B, I \otimes J \otimes C$, it then carries over to A, B, C . In this last step, we needed to know that $(p-2, q)$ is in the same class as (p, q) and $(p-1, q)$. Similarly for the passage from (p, q) to $(p, q-1)$ we must have $(p, q-2)$ in the same class.

Defining the G -pairing $\theta': B \times A \rightarrow C$ by $\theta'(b, a) = \theta(a, b)$, it is known that for $\alpha \in H^p(G, A)$, $\beta \in H^q(G, B)$

$$\alpha \cup_\theta \beta = (-1)^{pq} \beta \cup_{\theta'} \alpha.$$

The analogous statement holds for G/H ; consequently, if the theorem is true for (p, q) then it also holds for (q, p) . In particular, the classes II and VI are "equivalent"; as are classes III and V.

Because cupping with a 0-dimensional cohomology class has the same action as a certain induced homomorphism, we shall be able to give a simple conceptual proof of the validity of the theorem for all $(0, q)$ (and hence for all $(p, 0)$ too).

Then as soon as the theorem is verified for $(1, 1)$ and $(-1, 1)$ we are finished. (Note that (p, q) with $p + q = 1$, $p \neq 0, 1$, can be reached only by a change of class.)

Case $(0, q)$. Let $q \leq 0$ (the case $q \geq 1$ goes the same way) and let $\alpha \in H^0(G, A)$, $\beta \in H^q(G, B)$. There exists $a \in A^q$ such that $\bar{\kappa}_G(a) = \alpha$. Then

$$\text{defl}(\alpha \cup_\theta \beta) = \text{defl}(\bar{\kappa}_G(a) \cup_\theta \beta) = \text{defl}\{(\theta^a)_* \beta\}$$

(where $\theta^a : B \rightarrow C$ is the G -homomorphism such that $\theta^a(b) = \theta(a, b)$)

$$= (\theta^a)_* \text{defl } \beta$$

(by Proposition 8)

$$= \bar{\kappa}_{G/H}(a) \cup_\theta \text{defl } \beta = \text{defl } \alpha \cup_\theta \text{defl } \beta.$$

It should be noted that we actually need just $(0, 0)$ —this follows also from $\bar{\kappa}_G(a) \cup_\theta \bar{\kappa}_G(b) = \bar{\kappa}_G\{\theta(a, b)\}$ and $\bar{\kappa}_{G/H}(a) \cup_\theta \bar{\kappa}_{G/H}(b) = \bar{\kappa}_{G/H}\{\theta(a, b)\}$.

Case $(1, 1)$. Let $\alpha \in H^1(G/H, A^H)$ and $\beta \in H^1(G/H, B^H)$ and choose representing cocycles f for α and g for β . Therefore $\inf \alpha \cup_\theta \inf \beta$ is represented by $\bar{\theta} \circ (\inf f \otimes \inf g) \circ \phi_{1,1}^G$, while $\inf (\alpha \cup_\theta \beta)$ is represented by $\inf \{\bar{\theta} \circ (f \otimes g) \circ \phi_{1,1}^{G/H}\}$. Since $\delta f = 0 = \delta(\inf f)$ these cocycles are the same—in fact, using (19), one sees easily that their common value on a 2-cell $[\sigma, \tau]$ is $\bar{\theta}\{f[\bar{\sigma}] \otimes \bar{g}[\bar{\tau}]\}$.

Case $(-1, 1)$. Let f represent $\alpha \in H^{-1}(G, A)$ and let g represent $\beta \in H^1(G/H, B^H)$. Since

$$\begin{aligned} (\text{defl } f \otimes g) \phi_{-1,1}^{G/H}[\cdot] &= (f \Lambda_{-1} \otimes g) \sum_{x \in G/H} ((\cdot)^x \otimes x[x^{-1}]) \\ &= \sum_{x \in G/H} \left\{ \sum_{\rho \in x} (\rho f(\cdot) \otimes xg[x^{-1}]) \right\} = \sum_{x \in G/H} \left\{ \sum_{\rho \in x} (\rho f(\cdot) \otimes \rho(\inf g)[\rho^{-1}]) \right\} \\ &= \sum_{\rho \in G} (\rho f(\cdot) \otimes \rho(\inf g)[\rho^{-1}]) = (f \otimes \inf g) \phi_{-1,1}^G[\cdot] \end{aligned}$$

it follows that $(\text{defl } \alpha) \cup_\theta \beta = \text{defl } (\alpha \cup_\theta \inf \beta)$. This finally completes the proof.

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