The Holonomy Group

*I. The Curvature Tensor**

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Summary. This paper is the introductory part of a systematic study of over-determined systems of partial differential equations and their solutions obtained by algebraic methods. It deals with the mapping of connections. A more detailed description is given in the first section.

The second part (to be published later) deals with the Lie group induced by an arbitrary tensor. In the third part we shall apply the holonomy group (*i.e.* the Lie group induced by the curvature tensor) to the theory of covariant constant quadratic tensors. The next part deals with the same problem for Weyl's connection. Finally, in the last part we shall use algebraic methods to solve the field equations of general relativity.**

1. Description of the problems. (1) Let ${}^0\Gamma_{\lambda}{}^{\mu}$ be a given symmetric connection and $Q_{\lambda\mu}{}^{\nu} = Q_{\mu\lambda}{}^{\nu}$ a tensor. Then

$$(1.1) \quad \Gamma_{\lambda}{}^{\nu}{}_{\mu} \stackrel{\text{def}}{=} {}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu} + Q_{\lambda\mu}{}^{\nu}$$

is also a connection. Denote by $R_{\omega\mu\lambda}{}^{\nu}$ its curvature tensor. The first problem to be solved is *to find $Q_{\lambda\mu}{}^{\nu}$ for which $R_{\omega\mu\lambda}{}^{\nu}$ has a preassigned value.* We shall see that if this problem has a solution it can be obtained *by elementary algebraic methods.*† The sections 1–4 deal with this problem and with its applications.

(2) Assume now that ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ are Christoffel symbols of a metric tensor ${}^0g_{\lambda\mu}$. The second problem is *to find $Q_{\lambda\mu}{}^{\nu}$ for which $R_{\omega\mu\lambda}{}^{\nu}$ is the curvature tensor of some metric tensor $g_{\lambda\mu}$.* We shall see that if this problem has a solution it can be obtained *by elementary algebraic methods.* The sections 5–8 deal with this problem and its applications.

These two problems may be summarized as follows: *Given a metric tensor ${}^0g_{\lambda\mu}$ with the Christoffel symbols ${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ one has to find (1) a tensor $Q_{\lambda\mu}{}^{\nu}$ for which the*

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** Small Greek indices have the range of the first n Roman numerals.

† By “elementary algebraic methods” we mean the methods of solving either a system of algebraic equations or a completely integrable system of partial differential equations (of the first order) by means of power series.

$\Gamma_{\lambda}^{\nu}{}_{\mu}$, defined by (1.1), are Christoffel symbols and (2) the tensor $g_{\lambda\mu}$ whose Christoffel symbols are $\Gamma_{\lambda}^{\nu}{}_{\mu}$. We shall see that any solution $Q_{\lambda\mu}^{\nu}$, $g_{\lambda\mu}$ of this problem is obtained by elementary algebraic methods.

In the third part of this paper (to be published later) we shall deal with the problem of finding metric tensors having the same curvature tensor. Using the holonomy group of this curvature tensor we obtain the solution of this problem by elementary algebraic methods. Finally, we shall apply these results to the problem of finding by elementary algebraic methods all solutions of the field equations of the general relativity.

I.

2. Prerequisites. Let $g_{\lambda\mu}$ be the metric tensor of a Riemann space V_n ($n > 1$), $\{\Gamma_{\lambda}^{\nu}{}_{\mu}\}$ the Christoffel symbols of $g_{\lambda\mu}$ and $K_{\omega\mu\lambda}^{\nu}$ its curvature tensor:

$$(2.1) \quad \frac{1}{2}K_{\omega\mu\lambda}^{\nu} \stackrel{\text{def}}{=} \partial_{[\mu}\{\Gamma_{\omega]}^{\nu}{}_{\lambda}\} + \{\Gamma_{\alpha}^{\nu}{}_{[\mu}\}\{\Gamma_{\omega]}^{\alpha}{}_{\lambda}\}.$$

If

$$(2.2) \quad K_{\omega\mu\lambda}^{\nu} = \frac{2K}{n(n-1)} \delta_{[\mu}^{\nu} g_{\omega]\lambda}$$

then K is a constant for $n > 2$. If $K \neq 0$ then we say that the space V_n under consideration is of constant curvature and denote it by $\left\{ \begin{smallmatrix} S_n \\ E_n \end{smallmatrix} \right\}$. If

$$(2.3) \quad K_{\omega\mu\lambda}^{\nu} = 2(\delta_{[\mu}^{\nu} L_{\omega]\lambda} - g_{\lambda[\mu} L_{\omega]}^{\nu})$$

then we say that V_n is conformally flat (for $n > 3$) and, in any case, denote it by C_n .

The covariant derivative with respect to $\{\Gamma_{\lambda}^{\nu}{}_{\mu}\}$ will be denoted by ∇_{ω} . We have, in particular, for any tensor $T_{\lambda}^{\nu\cdots}$:

$$(2.4) \quad 2\nabla_{[\omega}\nabla_{\mu]}T_{\lambda}^{\nu\cdots} = -K_{\omega\mu\alpha}^{\nu}T_{\lambda}^{\alpha\cdots} - \cdots + K_{\omega\mu\lambda}^{\alpha}T_{\alpha}^{\nu\cdots} + \cdots.$$

Beltrami's operators will be defined by

$$(2.5) \quad \Delta_2(f) \stackrel{\text{def}}{=} g^{\lambda\mu}\nabla_{\lambda}\nabla_{\mu}f, \quad \Delta_1(f, g) \stackrel{\text{def}}{=} g^{\lambda\mu}(\nabla_{\lambda}f)(\nabla_{\mu}g).$$

In subsequent considerations we shall deal with an *a priori* given V_n , which we denote by 0V_n . If we want to stress that an object belongs to 0V_n we shall prefix it by "0" thus:

$${}^0g_{\lambda\mu}, \quad {}^0\{\Gamma_{\lambda}^{\nu}{}_{\mu}\}, \quad {}^0\nabla_{\mu}, \dots$$

In particular, we shall reserve the symbol ∇_{μ} for the covariant derivative with respect to a symmetric connection

$$(2.6) \quad \Gamma_{\lambda}^{\nu}{}_{\mu} \stackrel{\text{def}}{=} {}^0\{\Gamma_{\lambda}^{\nu}{}_{\mu}\} + Q_{\lambda\mu}^{\nu},$$

where $Q_{\lambda\mu}{}^\nu = Q_{\mu\lambda}{}^\nu$ is a tensor. The space with this connection will be denoted by L_n and termed an affine space. Its curvature tensor is given by

$$(2.7) \quad \frac{1}{2}R_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} \partial_{[\mu}\Gamma_{\omega]}{}^\nu{}_\lambda + \Gamma_{\alpha}{}^\nu{}_{[\mu}\Gamma_{\omega]}{}^\alpha{}_\lambda.$$

If $R_{\omega\mu\lambda}{}^\nu = 0$ then we say that L_n is flat and denote it by A_n .

The equation (2.6) will be referred to as the mapping ${}^0V_n \rightarrow L_n$. We shall say also that 0V_n is mapped on L_n .

3. The mapping ${}^0V_n \rightarrow L_n$. In the first theorem of this section we shall investigate the conditions for the curvature tensor $R_{\omega\mu\lambda}{}^\nu$ to be equal to an arbitrary given tensor.

Theorem (3.1). *Let $Q_{\omega\mu\lambda}{}^\nu$ be an arbitrary given tensor subject to the condition*

$$(3.1) \quad Q_{[\omega\mu\lambda]}{}^\nu = 0.$$

Necessary and sufficient conditions for the existence of a connection $\Gamma_{\lambda}{}^\nu{}_\mu = \Gamma_{\mu}{}^\nu{}_\lambda$, the curvature tensor (2.7) of which satisfies the requirement

$$(3.2) \quad R_{\omega\mu\lambda}{}^\nu = {}^0K_{\omega\mu\lambda}{}^\nu + Q_{[\omega\mu]\lambda}{}^\nu, *$$

are: There exists a tensor $Y_{\omega\mu\lambda}{}^\nu$ subject to the conditions

$$(3.3) \quad \begin{aligned} (a) \quad 2Y_{[\omega\mu]\lambda}{}^\nu &= Q_{[\omega\mu]\lambda}{}^\nu, \\ (b) \quad Y_{[\omega|\mu|\lambda]}{}^\nu &= 0 \end{aligned}$$

and a tensor $Q_{\lambda\mu}{}^\nu = Q_{\mu\lambda}{}^\nu$ which is a solution of

$$(3.4) \quad {}^0\nabla_{\mu}Q_{\lambda\omega}{}^\nu - Q_{\lambda\mu}{}^{\alpha}Q_{\omega\alpha}{}^\nu - Q_{\omega\mu}{}^{\alpha}Q_{\lambda\alpha}{}^\nu = Y_{\omega\mu\lambda}{}^\nu. **$$

If these conditions are satisfied then the connection $\Gamma_{\lambda}{}^\nu{}_\mu$ is given by (2.6).

Proof. Substituting from (2.6) into (2.7) we obtain, by virtue of (2.1),

$$(3.5) \quad R_{\omega\mu\lambda}{}^\nu = {}^0K_{\omega\mu\lambda}{}^\nu + 2({}^0\nabla_{[\mu}Q_{\omega]\lambda}{}^\nu - Q_{\lambda[\mu}{}^{\alpha}Q_{\omega]\alpha}{}^\nu).$$

Therefore a necessary and sufficient condition for (3.2) to be satisfied is that there is a tensor $Q_{\lambda\mu}{}^\nu = Q_{\mu\lambda}{}^\nu$ which is a solution of

$$(3.6) \quad Q_{[\omega\mu]\lambda}{}^\nu = 2({}^0\nabla_{[\mu}Q_{\omega]\lambda}{}^\nu - Q_{\lambda[\mu}{}^{\alpha}Q_{\omega]\alpha}{}^\nu).$$

Hence, for our considerations, we may replace (3.2) by (3.6). Assume that there is a solution $Q_{\lambda\mu}{}^\nu = Q_{\mu\lambda}{}^\nu$ of (3.6) and define by means of it the tensor $Y_{\omega\mu\lambda}{}^\nu$ using (3.4). Then

$$(3.7) \quad Y_{[\omega\mu]\lambda}{}^\nu = {}^0\nabla_{[\mu}Q_{\omega]\lambda}{}^\nu - Q_{\lambda[\mu}{}^{\alpha}Q_{\omega]\alpha}{}^\nu,$$

* This condition can be satisfied only if (3.1) holds.

** This condition can be satisfied only if (3.3b) holds. If this is so, then

$$Y_{[\omega\mu\lambda]}{}^\nu = 0.$$

so that we obtain (3.3a) from (3.6) and (3.7). The equation (3.3b) is a consequence of (3.4). Assume now that the tensors $Y_{\omega\mu\lambda}{}^\nu$ and $Q_{\mu\lambda}{}^\nu$ satisfy the conditions (3.3) and (3.4). From (3.4) we obtain (3.7) and, substituting into this latter equation from (3.3a), we obtain (3.6).

In the next theorem we shall find conditions for the complete integrability of (3.4).

Theorem (3.2). *Necessary and sufficient conditions for (3.4) to be completely integrable are*

(1) 0V_n reduces to an E_n :

$$(3.8a) \quad {}^0K_{\omega\mu\lambda}{}^\nu = 0.$$

(2) The tensor Y vanishes:

$$(3.8b) \quad Y_{\omega\mu\lambda}{}^\nu = 0.$$

If these conditions are satisfied then L_n reduces to an A_n :

$$(3.9) \quad R_{\omega\mu\lambda}{}^\nu = 0.$$

Proof. A necessary and sufficient condition for (3.4) to be completely integrable is that its first integrability conditions are satisfied identically for any $Q_{\mu\lambda}{}^\nu = Q_{\lambda\mu}{}^\nu$. In order to obtain these integrability conditions one applies to (3.4) the operator ${}^0\nabla_\xi$, alternates the indices ξ and μ and eliminates derivatives of $Q_{\lambda\mu}{}^\nu$ by means of (3.4) itself. This method, based on the identity (2.4), yields

$$(3.10) \quad Q_{\alpha\beta}{}^\gamma T_{\xi\mu\lambda\omega}{}^\nu{}^{\alpha\beta\gamma} = {}^0\nabla_{[\xi} Y_{|\omega|\mu]\lambda}{}^\nu$$

where

$$(3.11) \quad T_{\xi\mu\lambda\omega}{}^\nu{}^{\alpha\beta\gamma} \stackrel{\text{def}}{=} {}^0K_{\xi\mu(\lambda}{}^{(\alpha} \delta_{\omega)}{}^{\beta)\gamma}{}^\nu - \frac{1}{2} {}^0K_{\xi\mu\gamma}{}^\nu \delta_{(\lambda\omega)}^{(\alpha\beta)} + 2Y_{[\xi\mu](\lambda}{}^{(\alpha} \delta_{\omega)}{}^{\beta)\gamma}{}^\nu - Y_{[\xi\mu]\gamma}{}^\nu \delta_{(\lambda\omega)}^{(\alpha\beta)} \\ - (\delta_{\omega}^{(\alpha} \delta_{[\xi} Y_{|\lambda|\mu]}{}^{\beta)\gamma}{}^\nu + \delta_{\lambda}^{(\alpha} \delta_{[\xi} Y_{|\omega|\mu]}{}^{\beta)\gamma}{}^\nu) - Y_{\gamma[\xi\mu]}{}^\nu \delta_{(\lambda\omega)}^{(\alpha\beta)}.$$

Hence necessary and sufficient conditions for the complete integrability of (3.4) are

$$(3.12) \quad \begin{aligned} (a) \quad & T_{\xi\mu\lambda\omega}{}^\nu{}^{\alpha\beta\gamma} = 0, \\ (b) \quad & {}^0\nabla_{[\xi} Y_{|\omega|\mu]\lambda}{}^\nu = 0. \end{aligned}$$

A lengthy elementary computation shows that (3.12a) is equivalent to (3.8a,b).^{*} Then, of course, (3.12b) is also satisfied. The proof of the last part of the theorem is obvious.

Let us now assume that (3.12) are not satisfied so that (3.4) is not completely integrable. Hence (3.10) is not identically satisfied for all $Q_{\lambda\mu}{}^\nu$. Denote this equation by E_1 . Apply to it the operator ${}^0\nabla$ and eliminate from the equation so obtained the unknowns $Q_{\lambda\mu}{}^\nu$ by means of (3.4). Denote the resulting equation

^{*} I am indebted to Professor R. S. MISHRA for carrying out this computation.

by E_2 . Repeating this operation we obtain finally an infinite set of equations

$$(3.13) \quad E_1, E_2, E_3, \dots, E_N, E_{N+1}, \dots$$

which do not contain derivatives of the unknowns $Q_{\lambda\mu}$.

Theorem (3.3). *Necessary and sufficient conditions for the system (3.4) to admit a solution are: There is a positive integer $N \leq \frac{1}{2}n^2(n+1)$ * such that the algebraic equations*

$$(3.14) \quad E_1, E_2, \dots, E_N$$

in the unknowns $Q_{\lambda\mu}$ are compatible for all values of the coordinates in a domain and that the equations of the set E_{N+1} are satisfied because of the former sets (3.14).

Proof. This is almost the exact wording of a theorem in EISENHART's book [2] (page 16) where the proof is given for the most general case.

Remark I. We see from this theorem that if (3.4) is not completely integrable and if it admits a solution $Q_{\lambda\mu}$, it may be obtained by resolving the algebraic system (3.14). If, however, (3.4) is completely integrable, then its solution may be obtained in the usual way by means of power series. *Hence if (3.4) admits a solution, this solution may be obtained by elementary algebraic methods.*

Remark II. If E_1 is solvable for $Q_{\lambda\mu}$ then we may rewrite it in the form

$$(3.15) \quad Q_{\lambda\mu} = Q_{\lambda\mu}(Y\dots, {}^0K\dots) **$$

where $Q_{\lambda\mu}$ is an explicit tensorial function of $Y_{\omega\mu\lambda}$ and ${}^0K_{\omega\mu\lambda}$. Then the system E_2 does not contain the unknowns $Q_{\lambda\mu}$ and represents a set of conditions imposed on $Y_{\omega\mu\lambda}$ and its first two derivatives and on ${}^0K_{\omega\mu\lambda}$ and its first derivatives. If this system E_2 is satisfied, all remaining systems of (3.13) are identically satisfied and (3.15) is a solution of (3.4). Hence in this case $N = 2$. If, however, E_2 is not satisfied, then (3.4) admits no solution. In the next section we shall illustrate this important remark.

The last theorem of this section is concerned with *volume preserving* spaces. We say that a space (or its connection) is volume preserving if its connection admits a covariant constant density. In particular, every V_n is volume preserving. We say that the mapping (2.6) is volume preserving if its resulting space L_n is also volume preserving.

Theorem (3.4). *A necessary and sufficient condition for the mapping (2.6) to be volume preserving is that the vector*

* $\frac{1}{2}n^2(n+1)$ is the number of algebraically different components of $Q_{\lambda\mu}$.

** $Y\dots$ stands here also for its first covariant derivatives.

$$(3.16a) \quad Q_\mu \stackrel{\text{def}}{=} Q_{\mu\lambda}{}^\lambda$$

is a gradient:

$$(3.16b) \quad \partial_{[\omega} Q_{\mu]} = 0.$$

Proof. The equation (3.6) yields

$$(3.17a) \quad Q_{[\omega\mu]\alpha}{}^\alpha = 2^0 \nabla_{[\mu} Q_{\omega]} = 2 \partial_{[\mu} Q_{\omega]}$$

where Q_μ is defined by (3.16a). Therefore we obtain from (3.2)

$$(3.17b) \quad R_{\omega\mu\alpha}{}^\alpha = 2 \partial_{[\mu} Q_{\omega]}.$$

We know that a necessary and sufficient condition for the connection $\Gamma_\lambda{}^\nu{}_\mu = \Gamma_\mu{}^\nu{}_\lambda$ to be volume preserving is

$$R_{\omega\mu\alpha}{}^\alpha = 0.$$

Comparing this equation with (3.17b) we obtain (3.16b).

4. Applications. Let p_λ and q_λ be two arbitrary vectors. Put

$$(4.1) \quad \begin{aligned} (a) \quad P_{\mu\lambda} &\stackrel{\text{def}}{=} {}^0 \nabla_\mu p_\lambda - p_\mu p_\lambda + \tfrac{1}{2} p^\alpha q_\alpha {}^0 g_{\mu\lambda}, \\ (b) \quad Q_{\mu\lambda} &\stackrel{\text{def}}{=} {}^0 \nabla_\mu q_\lambda - q_\mu q_\lambda + \tfrac{1}{2} p^\alpha q_\alpha {}^0 g_{\mu\lambda} \end{aligned}$$

and

$$(4.2) \quad \begin{aligned} Y_{\omega\mu\lambda}{}^\nu &\stackrel{\text{def}}{=} \delta_\omega^\nu P_{\mu\lambda} - {}^0 g_{\lambda\omega} Q_\mu{}^\nu + \delta_\lambda^{\nu 0} \nabla_\mu p_\omega + \delta_{(\lambda}{}^\nu [g_{\mu)\lambda} p^\alpha p_\alpha - 4 p_\mu p_\lambda] \\ &\quad + 3 q^\nu (2 p_{(\lambda} - q_{(\lambda)} {}^0 g_{\omega\mu)} - \delta_\lambda^\nu (3 p_\mu p_\omega - {}^0 g_{\omega\mu} p^\alpha q_\alpha). \end{aligned}$$

An easy inspection shows that (3.3b) is satisfied. Moreover,

$$(4.3) \quad 2 Y_{[\omega\mu]\lambda}{}^\nu = Q_{[\omega\mu]\lambda}{}^\nu = 2 (\delta_{[\omega}^\nu P_{\mu]\lambda} - {}^0 g_{\lambda[\omega} Q_{\mu]}{}^\nu + \delta_\lambda^{\nu 0} \nabla_{[\mu} p_{\omega]}).$$

Substituting from (4.2) into (3.10) and using (2.4), we see that (3.10) admits a solution

$$(4.4) \quad Q_{\mu\lambda}{}^\nu = \delta_\lambda^\nu p_\mu + \delta_\mu^\nu p_\lambda - q^{\nu 0} g_{\lambda\mu}.$$

Substituting from (4.4) into E_2 we obtain an identity. Hence E_2 is satisfied, $N = 1$, and therefore (3.4) admits a solution (4.4).

Substituting from (4.4) into (2.6) we obtain the mapping

$$(4.5) \quad \Gamma_\lambda{}^\nu{}_\mu = {}^0 \{ \lambda{}^\nu{}_\mu \} + \delta_\lambda^\nu p_\mu + \delta_\mu^\nu p_\lambda - q^{\nu 0} g_{\lambda\mu}.$$

This mapping is called

(1) *general minimal-geodesic* if

$$(4.5a) \quad p^\nu q^\lambda (p^\mu - q^\mu) \neq 0,$$

(2) *special minimal-geodesic* if

$$(4.5b) \quad p^\nu = q^\nu \neq 0,$$

(3) *restricted minimal-geodesic* if

$$(4.5c) \quad q' \neq p' = 0,$$

(4) *geodesic* if

$$(4.5d) \quad p' \neq q' = 0.$$

The mappings (4.5a,b,c) preserve only minimal geodesics (with the possible exception of the curves q'^* , i.e., only the minimal geodesics of 0V_n are at the same time autoparallel lines of L_n . The mapping (4.5d) preserves all geodesics: Every geodesic of 0V_n is an autoparallel line of L_n .

Theorem (4.1). *A necessary and sufficient condition for the mapping (4.5) to be volume preserving is: The vector*

$$(4.6a) \quad (n+1)p_\mu - q_\mu$$

is a gradient:

$$(4.6b) \quad (n+1){}^0\nabla_{[\mu}p_{\lambda]} = {}^0\nabla_{[\mu}q_{\lambda]}.$$

The proof follows from

$$Q_\mu = (n+1)p_\mu - q_\mu$$

and from Theorem (3.3).

II.

5. The mapping ${}^0V_n \rightarrow E_n$. In this part of the paper we shall investigate the conditions which make out of (2.6) a ${}^0V_n \rightarrow V_n$ mapping. We shall assume that $Q_{\lambda\mu}$ is a given tensor (found by elementary algebraic methods devised in the previous part).

The following two theorems will be proved simultaneously.

Theorem (5.1). *A necessary and sufficient condition for the mapping (2.6) to be a ${}^0V_n \rightarrow V_n$ mapping is: There is a tensor $g_{\lambda\mu}$ of rank n which is a solution of*

$$(5.1) \quad {}^0\nabla_\mu g_{\lambda\nu} = Q_{\lambda\mu}{}^\alpha g_{\alpha\nu} + Q_{\nu\mu}{}^\alpha g_{\lambda\alpha}.$$

If this condition is satisfied, then L_n reduces to a V_n with the metric tensor $g_{\lambda\mu}$.

Theorem (5.2) *A necessary condition for the mapping (2.6) to be a ${}^0V_n \rightarrow V_n$ mapping is: The vector (3.16a) satisfies the equation*

$$(5.2) \quad 2Q_\omega = \frac{1}{g} {}^0\nabla_\omega g \quad (= \partial_\omega \ln g / {}^0g)$$

* A curve whose tangent vector is q' is called a q' -curve. Geodesics of 0V_n and autoparallel lines of L_n are given by similar equations, i.e.

$$v^\alpha ({}^0\nabla_\alpha v^{\lambda'}) v^{\lambda'} = 0, \quad v^\alpha (\nabla_\alpha v^{\lambda'}) v^{\lambda'} = 0.$$

where g is the determinant of the tensor $g_{\lambda\mu}$ mentioned in the previous theorem and 0g is the determinant of ${}^0g_{\lambda\mu}$.

Proof. A necessary and sufficient condition that L_n , resulting from the mapping (2.6), be a V_n is: There is a tensor $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n which is covariant constant with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$:

$$(5.3a) \quad \nabla_{\mu} g_{\lambda\nu} = 0.$$

If this condition is satisfied, then

$$(5.3b) \quad \nabla_{\mu} g = 0.$$

The equation (5.3a) [(5.3b)] is equivalent to (5.1) [to (5.2)].

We shall investigate first the case when (5.1) is completely integrable.

Theorem (5.2) *A necessary and sufficient condition for (5.1) to be completely integrable is: The curvature tensor ${}^0K_{\omega\mu\lambda}{}^{\nu}$ is of the form*

$$(5.4) \quad {}^0K_{\omega\mu\lambda}{}^{\nu} = 2[{}^0\nabla_{[\omega} Q_{\mu]\lambda}{}^{\nu} + Q_{\lambda[\mu} {}^{\alpha}Q_{\omega]\alpha}{}^{\nu}].$$

If this condition is satisfied, then our L_n reduces to an E_n with the metric tensor $g_{\lambda\mu}$ which is a solution of the completely integrable system (5.1).

Proof. The equations (5.3a) and (5.1) being equivalent, their integrability conditions must be equivalent. The first integrability conditions of (5.3a) are

$$(5.5) \quad R_{\xi\mu(\lambda} {}^{\alpha}\delta_{\omega)} g_{\alpha\beta} = 0.$$

The system (5.3a) (and, therefore, also the system (5.1)) is completely integrable if and only if (5.5) is an identity for all $g_{\lambda\mu}$'s, i.e.,

$$(5.6a) \quad R_{\xi\mu(\lambda} ({}^{\alpha}\delta_{\omega)}^{\beta}) = 0.$$

The condition (5.6a) is equivalent to

$$(5.6b) \quad R_{\xi\mu\lambda}{}^{\nu} = 0.$$

Substituting from (5.6b) into (3.2) one obtains (5.4). The remaining part of the theorem is obvious.

In the next theorem we shall use the first normal tensor of the 0V_n . Put

$${}^0\Gamma_{\lambda}{}^{\nu}{}_{\mu\omega} \stackrel{\text{def}}{=} 2[\partial_{(\omega} {}^0\{\lambda}{}^{\nu}{}_{\mu)}\} - 2{}^0\{\alpha}{}^{\nu}{}_{(\lambda}\} {}^0\{\mu}{}^{\alpha}{}_{\omega)}\}].$$

Then the first normal tensor ${}^0N_{\omega\mu\lambda}{}^{\nu}$ of our 0V_n is defined by

$$(5.7) \quad {}^0N_{\omega\mu\lambda}{}^{\nu} \stackrel{\text{def}}{=} \partial_{\mu} {}^0\{\omega}{}^{\nu}{}_{\lambda}\} - 2{}^0\{\alpha}{}^{\nu}{}_{(\omega}\} {}^0\{\lambda)}{}^{\alpha}{}_{\mu)} - {}^0\Gamma_{\lambda\mu\omega}{}^{\nu} . *$$

Theorem (5.3). *Assume that our 0V_n is not flat. A sufficient condition for (5.1) to be completely integrable is: The conditions of Theorem (3.3) are satisfied for*

$$(5.8) \quad Y_{\omega\mu\lambda}{}^{\nu} \stackrel{\text{def}}{=} -{}^0N_{\omega\mu\lambda}{}^{\nu} .$$

Proof. We see from (5.7) that the tensor Y defined by (5.8) satisfies (3.3b).

* See, for instance, [2] p. 70.

Therefore, we may consider solutions of (3.4) where Y is defined by (5.8). At least one solution exists if the conditions of Theorem (3.3) are satisfied. Moreover, we obtain from (5.7), (5.8), (3.3a) and (2.1)

$$(5.9) \quad {}^0K_{\omega\nu\lambda}{}^\nu = 2{}^0N_{[\omega\mu]\lambda}{}^\nu = -2Y_{[\omega\mu]\lambda}{}^\nu = -Q_{[\omega\mu]\lambda}{}^\nu$$

and, by virtue of (3.6), this equation is equivalent to (5.4).

In the next theorem we shall describe the construction of the metric tensor $g_{\lambda\mu} = g_{\mu\lambda}$.

Theorem (5.4). *If and only if (5.4) is satisfied, there exists a set of linearly independent vectors $\hat{u}_\lambda^*$ which have the following properties:*

(1) *They are n linearly independent gradient-solutions of the completely integrable system*

$$(5.10) \quad \begin{aligned} (a) \quad & {}^0\nabla_\mu u_\lambda = Q_{\lambda\mu}{}^\alpha u_\alpha, \\ (b) \quad & u_\lambda = \partial_\lambda u. \end{aligned}$$

(2) *They satisfy the initial conditions*

$$(5.10c) \quad \hat{u}_\lambda(P) = \hat{c}_\lambda$$

where $\hat{c}_\lambda$ is an arbitrary system of n^2 constants with

$$(5.10d) \quad \text{Det}((\hat{c}_\lambda)) \neq 0.$$

(3) *If $g_{ab} = g_{ba}$ is an arbitrary system of $\frac{1}{2}n(n+1)$ constants with*

$$\text{Det}((g_{ab})) \neq 0,$$

then

$$(5.11) \quad g_{\lambda\mu} = g_{ab}\hat{u}_\lambda^a\hat{u}_\mu^b = g_{\mu\lambda}$$

is a tensor of rank n which is a solution of the completely integrable system (5.1). This tensor satisfies the initial conditions

$$(5.12) \quad g_{\lambda\mu}(P) = g_{ab}\hat{c}_\lambda^a\hat{c}_\mu^b.$$

Proof. First integrability conditions of

$$(5.13a) \quad \nabla_\mu u_\lambda = 0$$

are

$$(5.13b) \quad R_{\omega\mu\lambda}{}^\nu u_\nu = 0.$$

They are identically satisfied if and only if (5.6b) holds. This equation is equivalent to (5.4) while (5.13a) is equivalent to (5.10a). This proves the statements

* Latin indices run from 1 to n .

(1) and (2). Using (5.11) and (5.13a) we obtain (5.3a) and this equation is equivalent to (5.1).

Remark I. The tensor $g_{\lambda\mu}$, defined by (5.11), does not define necessarily the same orthogonality as the tensor ${}^0g_{\lambda\mu}$. In fact, if $g_{ab} = 0$ for $a \neq b$ then the vectors $\vec{u}_\lambda$ are perpendicular if measured by $g_{\lambda\mu}$. On the other hand, they need not be perpendicular in the metric of ${}^0g_{\lambda\mu}$.

Remark II. The equations (5.4) impose some conditions on the structure of the tensors ${}^0K_{\omega\mu\lambda}{}^\nu$ and $Q_{\mu\lambda}{}^\nu$. In the next section we shall study these conditions in more detail.

6. Applications. In this section we shall study in more detail the conditions of Theorem (5.2) in the case of a general minimal-geodesic mapping*

$$(6.1) \quad \begin{aligned} (a) \quad \Gamma_\lambda{}^\nu{}_\mu &= {}^0\{\lambda{}^\nu{}_\mu\} + Q_{\lambda\mu}{}^\nu, \\ (b) \quad Q_{\lambda\mu}{}^\nu &= p_\lambda \delta_\mu^\nu + p_\mu \delta_\lambda^\nu - q^\nu {}^0g_{\lambda\mu}, \\ (c) \quad p^\nu q^\mu (p^\lambda - q^\lambda) &\neq 0, \end{aligned}$$

assuming that q_μ is a gradient:

$$(6.2a) \quad q_\mu = \partial_\mu q.$$

Lemma (6.1). The vector p_μ is a gradient:

$$(6.2b) \quad p_\mu = \partial_\mu p,$$

and**

$$(6.3) \quad \begin{aligned} (a) \quad P &\stackrel{\text{def}}{=} P_{\lambda\mu}{}^0 g^{\lambda\mu} = {}^0\Delta_2(p) - {}^0\Delta_1(p, p - \tfrac{1}{2}nq), \\ (b) \quad Q &\stackrel{\text{def}}{=} Q_{\lambda\mu}{}^0 g^{\lambda\mu} = {}^0\Delta_2(q) - {}^0\Delta_1(q, q - \tfrac{1}{2}np). \end{aligned}$$

Proof. The equation (6.2b) follows from (6.2a) and from Theorem (3.4) (or from Theorem (4.1)). The equations (6.3) follow then from (4.1) and (6.2).

In the next theorem we shall use the abbreviations

$$(6.4) \quad \begin{aligned} (a) \quad {}^0K_{\mu\lambda} &\stackrel{\text{def}}{=} {}^0K_{\alpha\mu\lambda}{}^\alpha = {}^0K_{\lambda\alpha\beta\mu}{}^0 g^{\alpha\beta} \quad (= {}^0K_{\lambda\mu}), \\ (b) \quad {}^0K &\stackrel{\text{def}}{=} {}^0K_{\mu\lambda}{}^0 g^{\mu\lambda}, \\ (c) \quad c_{\lambda\mu} &\stackrel{\text{def}}{=} {}^0K_{\mu\lambda} + \frac{{}^0K}{2(1-n)} {}^0g_{\mu\lambda} \end{aligned}$$

and assume $n > 2$.

Theorem (6.1). Necessary and sufficient conditions that (5.1) be completely

* The remaining cases (4.5b, c, d) will be studied later in this paper.

** Cf. (4.1).

integrable are:

I. Conditions imposed on ${}^0K_{\omega\mu\lambda}{}^{\nu}$. If $n > 3$ then ${}^0K_{\omega\mu\lambda}{}^{\nu}$ has to be of the form

$$(6.5a) \quad \frac{1}{2}(n-2){}^0K_{\omega\mu\lambda}{}^{\nu} = \delta_{[\omega}^{\nu}({}^0K_{\mu]\lambda} + \frac{{}^0K}{1-n}g_{\mu]\lambda}) - {}^0g_{\lambda[\omega}{}^0K_{\mu]}{}^{\nu},$$

equivalent to

$$(6.5b) \quad \frac{1}{2}(n-2){}^0K_{\omega\mu\lambda}{}^{\nu} = \delta_{[\omega}^{\nu}c_{\mu]\lambda} - {}^0g_{\lambda[\omega}c_{\mu]}{}^{\nu},$$

i.e., 0V_n has to be conformally flat. If $n = 3$ then (6.5) is always satisfied and the condition imposed on ${}^0K_{\omega\mu\lambda}{}^{\nu}$ is

$$(6.6) \quad {}^0\nabla_{[\omega}c_{\mu]\lambda} = 0.$$

II. Two conditions imposed on the gradient vectors p_{μ} , q_{μ} .

(a) These vectors are related by

$$(6.7a) \quad c_{[\omega}{}^{\nu}[p_{\mu]}(3p_{\nu} + q_{\nu}) - q_{\mu]}(3q_{\nu} + p_{\nu})] = 0.$$

If this condition is satisfied then the vector X_{μ} defined by

$$(6.7b) \quad \frac{1}{2n}X_{\mu} \stackrel{\text{def}}{=} \frac{1}{2-n}c_{\mu}{}^{\nu}(q_{\nu} - p_{\nu}) + \frac{1}{2n}(q_{\mu} + p_{\mu})(P - Q) - \frac{1}{2}(q_{\mu} - p_{\mu})p^{\alpha}q_{\alpha}$$

is a gradient:

$$(6.7c) \quad X_{\mu} = \partial_{\mu}X.$$

(b) The system (6.7b) admits at least one solution p, q provided that X is defined by

$$(6.7d) \quad X = P - Q + \text{const.}^*$$

If these conditions are satisfied then L_n reduces to an E_n with the metric tensor $g_{\lambda\mu}$ given by (5.11).

Proof. This theorem is an application of Theorems (5.2) and (5.4) to (6.1). According to the statement of the theorem to be proved, we deal only with cases for which $n > 2$. Substituting into (5.4) from (6.1b), (6.2), we obtain, by virtue of (4.1),

$$(6.8) \quad {}^0K_{\omega\mu\lambda}{}^{\nu} = 2(\delta_{[\mu}^{\nu}P_{\omega]\lambda} - {}^0g_{\lambda[\mu}Q_{\omega]}{}^{\nu}).^{**}$$

This equation yields

$$(6.9) \quad \begin{aligned} (a) \quad {}^0K_{\alpha\mu\lambda}{}^{\alpha} &= {}^0K_{\mu\lambda} = (1-n)P_{\mu\lambda} + Q_{\mu\lambda} - Q^0g_{\lambda\mu}, \\ (b) \quad {}^0K_{\mu\alpha\beta\lambda}{}^0g^{\alpha\beta} &= {}^0K_{\mu\lambda} = (1-n)Q_{\mu\lambda} + P_{\mu\lambda} - P^0g_{\lambda\mu}, \end{aligned}$$

* See (6.2) and (6.3).

** Cf. also (4.3).

so that

$$\begin{aligned}
 (6.10) \quad (a) \quad (2-n)P_{\mu\lambda} &= {}^0K_{\mu\lambda} + \frac{1}{n} {}^0g_{\lambda\mu}[P + Q(n-1)], \\
 (b) \quad (2-n)Q_{\mu\lambda} &= {}^0K_{\mu\lambda} + \frac{1}{n} {}^0g_{\lambda\mu}[Q + P(n-1)], \\
 (c) \quad P_{\mu\lambda} - Q_{\mu\lambda} &= \frac{1}{n} {}^0g_{\lambda\mu}(P - Q), \\
 (d) \quad {}^0K &= (1-n)(P + Q),
 \end{aligned}$$

and consequently, by virtue of (6.4c),

$$\begin{aligned}
 (6.11) \quad (a) \quad \frac{1}{2-n} c_{\mu\lambda} &= P_{\mu\lambda} + \frac{1}{2n} {}^0g_{\lambda\mu}(Q - P) = Q_{\mu\lambda} + \frac{1}{2n} {}^0g_{\lambda\mu}(P - Q), \\
 (b) \quad P_{\mu\lambda} &= \frac{1}{2-n} c_{\mu\lambda} - \frac{1}{2n} {}^0g_{\lambda\mu}(Q - P), \\
 (c) \quad Q_{\mu\lambda} &= \frac{1}{2-n} c_{\mu\lambda} - \frac{1}{2n} {}^0g_{\lambda\mu}(P - Q).
 \end{aligned}$$

Substituting from (6.10a,b,d) into (6.8) one obtains (6.5a). Substituting from (6.11b,c) and (6.10d) into (6.8) one obtains (6.5b). One of the consequences of Bianchi's identity is

$$(6.12) \quad {}^0\nabla_{[\nu} {}^0K_{\omega\mu]\lambda}{}^\nu = 0.$$

Substituting into this equation from (6.5b) we have (for $n > 2$)

$$(6.13a) \quad (n-3) {}^0\nabla_{[\omega} c_{\mu]\lambda} = 0.$$

If $n > 3$ then (6.13a) reduces to

$$(6.13b) \quad {}^0\nabla_{[\omega} c_{\mu]\lambda} = 0,$$

while for $n = 3$ the equation (6.13a) reduces to an identity. One knows also that for $n = 3$ the equation (6.5b) is always satisfied.

These results are necessary for the remaining part of the proof. The equation (6.8) (equivalent to (6.5)) imposes some conditions on the structure of the curvature tensor ${}^0K_{\omega\mu\lambda}{}^\nu$ by means of the two tensors $P_{\mu\lambda}$ and $Q_{\mu\lambda}$ defined by (4.1). These latter equations may be regarded as differential equations for p_μ and q_μ . As such they yield the first integrability conditions in the form

$$\begin{aligned}
 (6.14) \quad \frac{1}{2-n} {}^0\nabla_{[\omega} c_{\mu]\lambda} + \frac{1}{2n} {}^0g_{\lambda[\mu} {}^0\nabla_{\omega]}(P - Q) \\
 = {}^0g_{\lambda[\mu} \{P_{\omega]}{}^\nu q_\nu - Q_{\omega]}{}^\nu p_\nu + \tfrac{1}{2}(p_{\omega]} - q_{\omega])p^\alpha q_\alpha\} \\
 = -\frac{1}{2-n} {}^0\nabla_{[\omega} c_{\mu]\lambda} + \frac{1}{2n} {}^0g_{\lambda[\mu} {}^0\nabla_{\omega]}(P - Q).
 \end{aligned}$$

If (6.1c) holds, then the system (6.14) splits into two. The first set is (6.13b) which, for $n > 3$, is satisfied because of (6.12), while for $n = 3$ it represents a condition imposed on $c_{\mu\lambda}$. The second system reduces to

$$(6.15) \quad {}^0g_{\lambda[\mu} \left\{ \frac{1}{2n} {}^0\nabla_{\omega]}(P - Q) - [P_{\omega]}{}^\nu q_\nu - Q_{\omega]}{}^\nu p_\nu + \frac{1}{2}(p_{\omega]} - q_{\omega])p^\alpha q_\alpha \right\} = 0,$$

and this condition is equivalent to

$$(6.16) \quad \frac{1}{2n} {}^0\nabla_\omega(P - Q) = P_\omega{}^\nu q_\nu - Q_\omega{}^\nu p_\nu + \frac{1}{2}(p_\omega - q_\omega)p^\alpha q_\alpha = 0.$$

Using (6.11b,c) one can rewrite (6.16) in the form (6.7b) where we put

$$(6.17a) \quad X_\mu = {}^0\nabla_\mu(P - Q).$$

Substituting from (6.7b) into

$$(6.17b) \quad \partial_{[\omega} X_{\mu]} = {}^0\nabla_{[\omega} X_{\mu]} = 0$$

we obtain (6.7a). If this condition is satisfied X_μ is a gradient, so that (6.7c) holds. The equation (6.17a) shows that the function X has to satisfy the condition (6.7d) in order that (6.16) be fulfilled. If this is so, then (6.16) may be regarded as an overdetermined system of n differential equations (of the third order) for two unknowns p and q . If this system admits at least one solution, then further derivatives of (6.16) will not add new integrability conditions of (4.1). Substituting the solution p, q into (4.1) we obtain two tensors $Q_{\mu\lambda}, P_{\mu\lambda}$ which satisfy (6.8) (equivalent to (6.5)). If (6.16) does not admit any solution p, q , then the integrability conditions of (4.1) are not satisfied and there are no tensors $P_{\mu\lambda}, Q_{\mu\lambda}$ satisfying (6.8) (equivalent to (6.5)).

Remark I. We see from (6.1) that all minimal geodesics of 0V_n are mapped into straight lines of the resulting E_n and that only these geodesics have this property if q^ν are not geodesic curves. If q^ν are geodesics (minimal or not) they are also mapped on straight lines of the resulting E_n . In the next theorem we shall deal with this case.

Theorem (6.2). *A necessary and sufficient condition for the curves q^ν to be mapped into straight lines of the resulting E_n is: q^ν is an eigenvector of $C_\lambda{}^\nu$ (of ${}^0K_\lambda{}^\nu$).*

Proof. A necessary and sufficient condition for the curves q^ν to be mapped into the straight lines of E_n is that q^ν be geodesics. Using (4.1b) and (6.11c) we obtain

$$q^{\mu 0} \nabla_\mu q^\nu = q^\nu \left[q^\alpha q_\alpha - \frac{1}{2} p^\alpha q_\alpha - \frac{1}{2n} (P - Q) \right] + \frac{1}{2-n} c_\mu{}^\nu q^\mu.$$

This equation shows that a necessary and sufficient condition for the curves q^ν to be geodesic curves is that q^ν be an eigenvector of $c_\mu{}^\nu$ or, according to (6.4c), an eigenvector of ${}^0K_\mu{}^\nu$.

7. Continuation. In this section we shall consider first the special minimal-geodesic mappings characterized by (6.1a,b) and

$$(7.1) \quad p^{\nu} = q^{\nu} \neq 0.$$

Although the results in this case are not all new, we present them here because they are special cases of the more general Theorems (5.2), (5.4) and (6.1). We assume again $n > 2$.

Theorem (7.1). *For a special minimal-geodesic mapping (6.1a,b), (7.1) the first set of conditions of Theorem (6.1) remains unchanged. The second set is identically satisfied and has to be replaced by (6.2a).*

Proof. The equation (6.2a) follows at once from Theorem (3.4) and from the equation (7.1). (In fact, according to Theorem (5.2) our L_n must reduce to an E_n which is always volume preserving.) In order to prove the remaining statements, let us observe that in the proof of Theorem (6.1) we used the condition $p^{\nu} \neq q^{\nu}$ only for deriving all three terms of the equations (6.14). Therefore the part of the proof which does not depend on the assumption $p^{\nu} \neq q^{\nu}$ holds also for the case described by (7.1). Then we have, according to (4.1),

$$(7.2) \quad (a) \quad P_{\mu\lambda} = Q_{\mu\lambda}, \quad (b) \quad P = Q,$$

so that (6.10) and (6.11) reduce to

$$(7.3) \quad \begin{aligned} (a) \quad & (2 - n)Q_{\mu\lambda} = {}^0K_{\mu\lambda} + {}^0g_{\mu\lambda}Q, \\ (b) \quad & {}^0K = 2(1 - n)Q, \\ (c) \quad & \frac{1}{2 - n} c_{\mu\lambda} = Q_{\mu\lambda}. \end{aligned}$$

Using these equations instead of (6.10) and (6.11) we obtain the first set of conditions of Theorem (6.1). In particular, (6.6) follows from the first two terms of (6.14) and from (7.2a). The conditions (6.7a) are identically satisfied by virtue of (7.1), while (7.2) and (7.1) show that (6.7b) are identically satisfied for $X = \text{const.}$, so that $X_{\mu} = 0$.

Remark. If $n = 2$, then we may write, for the linear element of our 0V_2 , $ds^2 = f^{-1}(u, v)(du^2 \pm dv^2)$, so that $d's^2 = f ds^2$ has the curvature zero.

The following two theorems concerning the special minimal-geodesic mapping will be proved simultaneously.

Theorem (7.2). *If $n > 2$ and if (7.1) holds then the function involved in (6.2) is a solution of*

$$(7.4a) \quad {}^0K = (1 - n)[2{}^0\Delta_2(q) + (n - 2)\Delta_1(q, q)]$$

satisfying the equation

$$(7.4b) \quad \frac{1}{2 - n} c_{\mu\lambda} = {}^0\nabla_{\mu}q_{\lambda} - q_{\mu}q_{\lambda} + \frac{1}{2}q^{\alpha}q_{\alpha}{}^0g_{\lambda\mu}.$$

Theorem (7.3). *Among all possible solutions $g_{\mu\lambda}$ of the completely integrable system (5.1) there are tensors proportional to ${}^0g_{\lambda\mu}$:*

$$(7.5) \quad g_{\lambda\mu} = e^{2q} g_{\lambda\mu} \quad \text{const.},$$

where, for $n > 2$, q is a solution of (7.4).

Proof. Substituting from (7.3c) into (4.1b) we obtain

$$\frac{1}{2-n} c_{\mu\lambda} = {}^0\nabla_\mu q_\lambda - q_\mu q_\lambda + \frac{1}{2} q^\alpha q_\alpha {}^0g_{\lambda\mu}$$

and consequently, according to (6.4c),

$$\frac{1}{2-n} \left[{}^0K - \frac{n}{2(n-1)} {}^0K \right] = {}^0\Delta_2(q) - \frac{2-n}{2} {}^0\Delta_1(q, q).$$

This equation is equivalent to (7.4).^{*} The equation (7.5) yields

$$(7.6a) \quad {}^0\nabla_\mu g_{\lambda\nu} = 2e^{2q} q_\mu {}^0g_{\lambda\nu} \quad \text{const.}$$

while, according to (6.1b), (7.1) and (7.5),

$$(7.6b) \quad Q_{\lambda\mu}{}^\alpha g_{\alpha\nu} + Q_{\nu\mu}{}^\alpha g_{\lambda\alpha} = 2e^{2q} q_\mu {}^0g_{\nu\lambda} \quad \text{const.}$$

The equations (7.6) show that $g_{\lambda\mu}$, defined by (7.5), is a solution of the completely integrable system (5.1).

Remark. Theorem (7.3) justifies the name “conformal mapping” which is often used for the special minimal-geodesic mapping. The same theorem explains also why the name “special minimal-geodesic” mapping is more appropriate. In fact, the solution $g_{\lambda\mu}$ of (5.1) need not have the form (7.5). In any case, a simple inspection shows that according to (5.2) we always have, even if (7.5) does not hold,

$$(7.7) \quad g = {}^0g e^{2qn} \quad \text{const.}$$

In the next theorem we shall discuss the restricted minimal-geodesic mapping characterized by (6.1a,b) and (4.5c), as well as the geodesic mapping characterized by (6.1a,b) and (4.5d). For the sake of brevity we shall refer to the restricted minimal-geodesic mapping as the q -mapping, while the geodesic mapping will be referred to as the p -mapping. *We shall not limit ourselves to the case $n > 2$.*

Theorem (7.4). *For a $\left. \begin{smallmatrix} p \\ q \end{smallmatrix} \right\}$ -mapping the first set of conditions in Theorem (5.1) reduces to the requirement, for all $n \geq 2$, that 0V_n be of constant curvature: ${}^0V_n = S_n$, i.e.,*

$$(7.8) \quad (a) \quad {}^0K_{\omega\mu\lambda}{}^\nu = \frac{2{}^0K}{n(1-n)} \delta_{[\mu}{}^\nu {}^0g_{\omega]\lambda}, \quad (b) \quad {}^0K = \text{const.}$$

^{*} The same result could have been obtained from (7.3 a, b).

The second set does not exist for $n = 2$ and is identically satisfied for $n > 2$.^{*}
This set has to be replaced by

$$(7.9) \quad \begin{array}{ll} \text{(a)} & p_\mu = \partial_\mu p \\ \text{(b)} & q_\mu = \partial_\mu q \end{array} \quad \text{for a } \left. \begin{array}{l} p \\ q \end{array} \right\} \text{-mapping.}$$

If these conditions are satisfied, then the function $\left. \begin{array}{l} p \\ q \end{array} \right\}$ is a solution of

$$(7.10) \quad \frac{{}^0K}{1-n} = {}^0\Delta_2(\xi) - {}^0\Delta_1(\xi, \xi)$$

satisfying for $\xi = \left\{ \begin{array}{l} p \\ q \end{array} \right\}$

$$\frac{{}^0K}{n(1-n)} {}^0g_{\lambda\mu} = {}^0\nabla_\mu p_\lambda - p_\mu p_\lambda \quad (p\text{-mapping}),$$

$$\frac{{}^0K}{n(1-n)} {}^0g_{\lambda\mu} = {}^0\nabla_\mu q_\lambda - q_\mu q_\lambda \quad (q\text{-mapping})$$

for all $n \geq 2$.

Proof. Let us start with the p -mapping

$$(7.11) \quad \text{(a)} \quad q = Q = 0, \quad \text{(b)} \quad Q_{\lambda\mu} = 0,$$

and assume first $n > 2$. The equations (7.9a) follow from Theorem (3.4) and from (7.11a). The equation (6.10d) reduces to

$$(7.12a) \quad {}^0K = (1-n)P$$

and therefore, according to (6.11a),

$$(7.12b) \quad c_{\mu\lambda} = \frac{2-n}{2n(1-n)} {}^0K {}^0g_{\lambda\mu}.$$

Substituting this value into (6.5b) we obtain (7.8a),[†] so that (7.8b) must hold. Using (7.8b), (7.11) and (7.12a) we see that (6.14) reduces to (6.6) which is satisfied identically for any ${}^0K = \text{const.}$, even for $n = 3$. Substituting the value of $P_{\mu\lambda}$ from (7.12c) into (4.1a) we obtain

$$(7.13) \quad \frac{{}^0K}{n(1-n)} {}^0g_{\lambda\mu} = {}^0\nabla_\mu p_\lambda - p_\mu p_\lambda,$$

^{*} The statement has to be understood as follows: Applying the method of previous proofs to $n > 2$ the second set results in an identity. Another method for $n = 2$ does not require the second set.

[†] The same result follows from substituting

$$(7.12c) \quad P_{\mu\lambda} = \frac{1}{2-n} c_{\mu\lambda} + \frac{{}^0K}{2n(1-n)} {}^0g_{\lambda\mu} = \frac{{}^0K}{n(1-n)} {}^0g_{\lambda\mu}.$$

and (7.11b) into (6.8).

and this equation yields (7.10) for $\xi = p$. The statement about the second set for $n > 2$ follows at once from (7.12), (7.11) and (7.8b). Assume now $n = 2$ so that

$$(7.14) \quad \begin{aligned} {}^0K_{\omega\mu\lambda}{}^\nu &= {}^0K\delta_{[\omega}{}^\nu g_{\mu]\lambda} , \\ {}^0K_{\mu\lambda} &= \tfrac{1}{2}{}^0K{}^0g_{\lambda\mu} . \end{aligned}$$

According to (7.11), (6.9a) and (7.14), we have

$$(7.15) \quad P_{\mu\lambda} = -\tfrac{1}{2}{}^0K{}^0g_{\lambda\mu} .$$

Substituting this value into (4.1) we obtain

$$(7.16) \quad {}^0\nabla_\mu p_\lambda = p_\mu p_\lambda - \tfrac{1}{2}{}^0K{}^0g_{\lambda\mu} .$$

Hence the integrability conditions (6.14) of (4.1a,b) have to be replaced here by the integrability conditions of (7.16), *i.e.*, by

$$(7.17) \quad {}^0g_{\lambda[\mu}{}^0\nabla_{\omega]}{}^0K = 0 .$$

They are satisfied only by ${}^0K = \text{const.}$ Moreover, the equations (7.17) represent all integrability conditions of (4.1a). Hence the second set of conditions (derived from integrability conditions of (4.1)) does not exist for $n = 2$. Moreover, (7.16) yields also (7.10) for $n = 2$. The proof for a q -mapping follows a similar pattern.

Remark. The case $q^\nu = 0$, dealt with in the previous theorem, is a slightly modified version of the famous Beltrami theorem* according to which geodesics of an S_n are mapped on straight lines of an E_n .

III.

8. The mapping ${}^0V_n \rightarrow V_n$. In this part of the paper we shall investigate the system (5.1) which we assume *not to be completely integrable*. We shall also assume that $Q_{\lambda\mu}{}^\nu$ is a given tensor (found by elementary algebraic methods devised in the first part of this paper). Therefore the tensor $R_{\omega\mu\lambda}{}^\nu$, defined by (3.2), is known and so is the connection $\Gamma_\lambda{}^\nu{}_\mu$, defined by (2.6).

We shall use these objects in the following abbreviations:

$$(8.1) \quad \begin{aligned} T_{\xi_0\mu\lambda\nu}{}^{\alpha\beta} &\stackrel{\text{def}}{=} R_{\xi_0\mu(\lambda}{}^{(\alpha} \delta_{\nu)}^{\beta)} , \\ (a) \quad T_{\xi_1\omega\mu\lambda\nu}{}^{\alpha\beta} &\stackrel{\text{def}}{=} \nabla_{\xi_1} T_{\omega\mu\lambda\nu}{}^{\alpha\beta} , \\ &\vdots \\ T_{\xi_p\cdots\xi_1\omega\mu\lambda\nu}{}^{\alpha\beta} &\stackrel{\text{def}}{=} \nabla_{\xi_p}\cdots\nabla_{\xi_1} T_{\omega\mu\lambda\nu}{}^{\alpha\beta} , \end{aligned}$$

* See, for instance, [1] p. 492.

is a solution of (8.2a) and (8.3), and every tensor $g_{\lambda\mu}$ of the $(s - 1)$ -parameter set

$$(8.5) \quad g_{\lambda\mu} = {}^0g_{\lambda\mu} + X_{\lambda\mu}$$

is a solution (5.1).

(2) There is at least one set of constants ${}^{(u)}c$ such that at least one of the tensors $g_{\lambda\mu}$ defined by (8.5) is of rank n . This tensor is then a solution (5.1) with the rank n .

Proof. The tensor $g_{\lambda\mu}$ we are looking for may always be written in the form (8.5), where $X_{\lambda\mu}$ is an unknown tensor to be found. The equation (5.1) is equivalent to (5.3a) and this latter equation reduces, by virtue of (8.5), to (8.2a). Its integrability conditions are

$$(8.6) \quad E'_1, E'_2, \dots, E'_N, E'_{N+1}, \dots$$

where E'_{p+1} stands for (8.3). Hence, according to EISENHART's theorem,* there has to be an integer $N \leq \frac{1}{2}n(n+1)$ such that

$$(8.7) \quad E'_1, E'_2, \dots, E'_N$$

is a compatible set and the remaining equations of the set (8.6) are satisfied by virtue of (8.7). This amounts to the requirement that both matrices $((T))$ and $((T, Q))$ have the same rank $r \leq \frac{1}{2}n(n+1)$. In order to prove that $((T))$ and $((T, Q))$ actually have the same rank let us consider the integrability conditions

$$(8.8a) \quad {}^0g_{\alpha\beta} T_{\xi_p \dots \xi_1 \omega \mu \lambda \nu}^{\alpha\beta} = -Q_{\xi_p \dots \xi_1 \omega \mu \lambda \nu}$$

of the system satisfied by ${}^0g_{\lambda\mu}$, namely

$$(8.8b) \quad \nabla_\mu {}^0g_{\lambda\nu} = -2Q_{\mu(\lambda\nu)}.$$

Applying to (8.8b) the Eisenhart theorem and remembering that (8.8a) admits a solution ${}^0g_{\lambda\mu}$, we see that (8.8a) must admit ${}^0g_{\lambda\mu}$ as a solution. This is possible if and only if $((T))$ and $((T, Q))$ have the same rank $r \leq \frac{1}{2}n(n+1)$. If $r = \frac{1}{2}n(n+1)$, then ${}^0g_{\lambda\mu}$ is the only solution of (8.8). Comparing (8.8), (8.2a) and (8.3) we see that (8.2a) and (8.3) always admit a solution (8.2b). Hence if $r = \frac{1}{2}n(n+1)$ this is the only solution of (8.2a), (8.3). According to (8.5) this solution leads to the trivial solution $g_{\lambda\mu} = 0$ of (5.1). Therefore a necessary condition for the existence of a non-trivial solution of (5.1) is $r \leq \frac{1}{2}n(n+1) - 1 = \frac{1}{2}(n-1)(n+2)$. If this condition is satisfied the system

(8.3) admits s linearly independent solutions $Y_{\lambda\mu}^w$. Put

$$(8.8c) \quad {}^{(u)}X_{\lambda\mu} = \sum_{w=1}^s {}^{(u)}\varphi_w^w Y_{\lambda\mu}^w.$$

${}^{(u)}X_{\lambda\mu}$ is a solution of (8.2a) if and only if ${}^{(u)}\varphi_w^w$ is a solution of

* *Loc. cit.* p. 16.

$$\sum_{w=1}^s (Y_{\lambda\nu} \partial_\mu \varphi_w + \varphi_w \nabla_\mu Y_{\lambda\nu}) = 2Q_{\mu(\lambda\nu)} .$$

This system for the unknowns φ is completely integrable. If one denotes by ${}^{(u)}\varphi_w$ s solutions with $\text{Det} ({}^{(u)}\varphi_w) \neq 0$, then ${}^{(u)}X_{\lambda\mu}$, given by (8.8c), are s linearly independent solutions of (8.2a) and (8.3). From (8.4a) we obtain

$$(8.8d) \quad 2Q_{\mu(\lambda\nu)} \left(1 - \sum_{u=1}^s {}^{(u)}c \right) = \sum_{u=1}^s {}^{(u)}X_{\lambda\nu} \partial_\mu {}^{(u)}c .$$

According to our assumption the last column of $((T, Q))$ does not consist of all zeros, hence we must have $\nabla_{[\omega} Q_{\mu](\lambda\nu)}$ and therefore the integrability conditions of (8.8d) reduce to (8.4b) and (8.8d) reduces to

$$\sum {}^{(u)}X_{\lambda\nu} \partial_\mu {}^{(u)}c = 0 .$$

Because ${}^{(u)}X_{\lambda\mu}$ are linearly independent this equation can be satisfied only by constant ${}^{(u)}c$'s.

The remaining part of the proof of Theorem (8.3) is obvious.

Remark. According to Theorem (8.3) we see that if the system (5.1) admits a solution *it is obtainable by elementary algebraic methods*. The following illustration will show how to proceed.

Illustration. Assume that our 0V_n admits a unit vector u_λ which is a solution of

$$(8.9a) \quad {}^0\nabla_\mu u_\lambda = u_\mu v_\lambda .$$

Hence v_μ must be perpendicular to u_λ :

$$(8.9b) \quad u_\lambda v_\mu g^{\lambda\mu} = 0$$

and, moreover,

$$(8.9c) \quad \frac{1}{2} {}^0K_{\omega\mu\lambda}{}^\nu u_\nu = u_{[\omega} v_{\mu]} v_\lambda + u_{[\mu} {}^0\nabla_{\omega]} v_\lambda .$$

Put

$$(8.10) \quad Q_{\lambda\mu}{}^\nu \stackrel{\text{def}}{=} c u_\lambda u_\mu v^\nu \quad (c = \text{const.}) .$$

Then we have, according to (8.9a,b) and (3.6),

$$(8.11a) \quad Q_{[\mu\omega]\lambda}{}^\nu = 2{}^0\nabla_{[\mu} Q_{\omega]\lambda}{}^\nu = 2c u_{[\mu} (v_{\omega]} v^\nu - {}^0\nabla_{\omega]} v^\nu) u_\lambda ,$$

so that, by virtue of (8.9c), (3.5) and (3.6),

$$(8.12) \quad \frac{1}{2} R_{\omega\mu\lambda}{}^\alpha u_\alpha = u_{[\omega} (v_{\mu]} v_\lambda - {}^0\nabla_{\mu]} v_\lambda)$$

and, according to (8.9) and (8.10),

$$(8.13) \quad 2\nabla_{[\mu} Q_{\omega](\lambda\nu)} = 2{}^0\nabla_{[\mu} Q_{\omega](\lambda\nu)} = 2c u_{[\mu} (v_{\omega]} v_{(\nu} - {}^0\nabla_{\omega]} v_{(\nu)}) u_{\lambda)} .$$

Therefore the set E_1 , *i.e.*,

$$(8.14a) \quad \frac{1}{2}(R_{\omega\mu\lambda}{}^{\alpha}X_{\alpha\nu} + R_{\omega\mu\nu}{}^{\alpha}X_{\lambda\alpha}) = 2\nabla_{[\omega}Q_{\mu](\lambda\nu)} ,$$

reduces to

$$(8.14b) \quad \frac{1}{2}(R_{\omega\mu\lambda}{}^{\alpha}X_{\alpha\nu} + R_{\omega\mu\nu}{}^{\alpha}X_{\lambda\alpha}) = 2cu_{[\omega}(\nu_{\mu]}v_{(\lambda} - {}^0\nabla_{\mu]}v_{(\lambda})u_{\nu)} .$$

Using (8.12) we see that this set admits a solution

$$(8.15) \quad X_{\lambda\mu} = cu_{\lambda}u_{\mu} .$$

This equation may be taken for E_1 .^{*} The equation E_2 reduces then to

$$(8.16) \quad 2Q_{\mu(\lambda\nu)} = c\nabla_{\mu}u_{\lambda}u_{\nu} .$$

Because, by virtue of (8.9a) and (8.10),

$$c\nabla_{\mu}u_{\lambda}u_{\nu} = c[{}^0\nabla_{\mu}u_{\lambda}u_{\nu} - 2Q_{\mu(\lambda}{}^{\alpha}u_{\nu)}u_{\alpha}] = c{}^0\nabla_{\mu}u_{\lambda}u_{\nu} = c(u_{\mu}v_{\lambda}u_{\nu} + u_{\lambda}u_{\mu}v_{\nu}) = 2Q_{\mu(\lambda\nu)}$$

(8.16) is identically satisfied. Hence we have $N = 1$ and certainly the rank r of $((T))$ and $((T, Q))$ is $r \leq \frac{1}{2}(n-1)(n+2)$, for otherwise the system (8.2a) would not admit a solution (8.15). On the other hand, if we write $\hat{u}_{\lambda} \stackrel{\text{def}}{=} u_{\lambda}$ we may put ${}^0g_{\lambda\mu}$ in the form

$$(8.17) \quad {}^0g_{\lambda\mu} = \sum_x \hat{u}_{\lambda}^x \hat{u}_{\mu}^x \epsilon_x$$

where $\hat{u}_{\lambda}$ are mutually perpendicular unit vectors:

$${}^0g^{\lambda\mu} \hat{u}_{\lambda}^a \hat{u}_{\mu}^b = \begin{cases} \epsilon_a & \text{for } \begin{cases} a = b \\ a \neq b \end{cases}, \quad \epsilon_a = \pm 1. \\ 0 \end{cases}$$

Therefore, by virtue of (8.5) and (8.15),

$$(8.18) \quad \begin{aligned} g_{\lambda\mu} &= {}^0g_{\lambda\mu} + X_{\lambda\mu} = \sum_x \hat{u}_{\lambda}^x \hat{u}_{\mu}^x \epsilon_x + c\hat{u}_{\lambda}^n \hat{u}_{\mu}^n \\ &= \sum_{x'=1}^{n-1} \hat{u}_{\lambda}^{x'} \hat{u}_{\mu}^{x'} \epsilon_{x'} + (c + \epsilon_n) \hat{u}_{\lambda}^n \hat{u}_{\mu}^n . \end{aligned}$$

Hence for $c \neq -\epsilon_n$ the tensor $g_{\lambda\mu}$ defined by (8.18) is a solution of (5.3a) (equivalent to (5.1)) of rank n .

In the next theorem we shall replace the non-homogeneous system (8.3) by another homogeneous system.

Theorem (8.4). *Necessary and sufficient conditions that the system (5.1) admits at least one solution $g_{\lambda\mu}$ of rank n are:*

(1) *The matrix $((T))$ is of rank $r \leq \frac{1}{2}(n-1)(n+2)$ so that the homogeneous systems (5.3a) and*

$$(8.19) \quad g_{\alpha\beta} T_{\xi_p \dots \xi_1 \omega \mu \lambda \nu}^{\alpha\beta} = 0, \quad p = 1, 2, \dots ,$$

admit $s' = \frac{1}{2}n(n+1) - r$ linearly independent solutions ${}^{(u')}g_{\lambda\mu}^$, $u' = 1, \dots, s'$.*

^{*} In doing so, we deprive ourselves of other solutions which might result from (8.14).

If this condition is satisfied then there is an s' -parameter set of tensors

$$(8.20) \quad g_{\lambda\mu} = \sum_{u'=1}^{s'} {}^{(u')}S {}^{(u')}g_{\lambda\mu}^* *$$

(defined by means of s' constants ${}^{(u')}S$) which satisfy ((8.19) and) (5.1).

(2) The constants ${}^{(u')}S$ may be chosen in each a way that at least one of the tensors (8.20) is of rank n .

This tensor is then a solution (5.1) with the rank n .

Proof. The system (8.19) constitutes the complete set of integrability conditions of (5.3a) (equivalent to (5.1)). Applying again EISENHART's theorem we see that $((T))$ must be of rank $r \leq \frac{1}{2}(n-1)(n+2)$ in order to admit solutions other than $g_{\lambda\mu} = 0$. If this condition is satisfied then we have $s' = \frac{1}{2}n(n+1) - r$ linearly independent solutions ${}^{w'}h_{\lambda\mu}$ of (8.19) and we may put

$$g_{\lambda\mu}^* = \sum_{w'=1}^{s'} \varphi_{w'} {}^{w'}h_{\lambda\mu}.$$

The system

$$\sum_{w'=1}^{s'} ({}^{w'}h_{\lambda\mu} \partial_{\nu} \varphi_{w'} + \varphi_{w'} \nabla_{\nu} {}^{w'}h_{\lambda\mu}) = 0$$

for the unknowns φ 's is completely integrable. If ${}^{(u')}_{w'}\varphi$ are s' solutions of this system with $\text{Det} (({}^{(u')}_{w'}\varphi)) \neq 0$, then

$${}^{(u')}g_{\lambda\nu}^* = \sum_{w'=1}^{s'} {}^{(u')}_{w'}\varphi {}^{w'}h_{\lambda\nu}$$

are s' linearly independent solutions of (5.3a). The remaining part of the proof is obvious.

Remark. The results of this paper may now be summarized as follows: Starting with a given metric tensor ${}^0g_{\lambda\mu}$ and its Christoffel symbols ${}^0\{\lambda{}^{\nu}{}_{\mu}\}$ we investigated the conditions for $Q_{\lambda\mu}{}^{\nu}$ which yield for the connection (2.6) the curvature tensor $R_{\omega\mu\lambda}{}^{\nu}$, whose value is given *a priori* by (3.2). We found that if this problem has a solution $Q_{\lambda\mu}{}^{\nu}$, it may be obtained by elementary algebraic methods. Taking this solution we investigated the conditions imposed on $R_{\omega\mu\lambda}{}^{\nu}$ under which the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, given by (2.6), is a metric connection based on a tensor $g_{\lambda\mu} = g_{\mu\lambda}$ of rank n . We found again that if such a tensor exists it may be obtained by elementary algebraic methods. A more detailed statement concerning these results is contained in the following.

* The number s' used here and the number s used in Theorem (8.3) are related by

$$s' = s - 1.$$

Theorem (8.5). Let ${}^0g_{\lambda\mu}$ be a metric tensor with a non-vanishing curvature tensor ${}^0K_{\omega\mu\lambda}{}^\nu$. Necessary and sufficient conditions that the connection (2.6) be a metric connection based on a tensor $g_{\lambda\mu}$ with non-vanishing preassigned curvature tensor are: There is a tensor $Y_{\omega\mu\lambda}{}^\nu = Y_{\lambda\mu\omega}{}^\nu$ with the following properties:

(a) The conditions of Theorem (3.3) are satisfied. If this is so, then the set of algebraic equations (3.14) admits at least one solution

$$(8.21) \quad Q_{\lambda\mu}{}^\nu = Q_{\lambda\mu}{}^\nu({}^0K\dots, Y\dots) \cdot *$$

(b) The conditions of Theorem (8.3) or Theorem (8.4) are satisfied if we take for $Q_{\lambda\mu}{}^\nu$ the tensor defined by (8.21) and replace $R_{\omega\mu\lambda}{}^\nu$ by

$$R_{\omega\mu\lambda}{}^\nu = {}^0K_{\omega\mu\lambda}{}^\nu + 2{}^0Y_{[\omega\mu]\lambda}{}^\nu,$$

the covariant derivative $\nabla_\mu A_\lambda^{\nu\dots}$ of any tensor $A_\lambda^{\nu\dots}$ being given by

$$\nabla_\mu A_\lambda^{\nu\dots} = {}^0\nabla_\mu A_\lambda^{\nu\dots} + Q_{\alpha\mu}{}^\nu A_\mu^{\alpha\dots} + \dots - Q_{\lambda\mu}{}^\alpha A_\alpha^{\nu\dots} - \dots.$$

If these two conditions are satisfied, then the tensor $g_{\lambda\mu} = g_{\lambda\mu}({}^0K\dots, Y\dots)^{**}$ is obtained as a solution of an algebraic system by the elementary algebraic methods described in Theorems (8.3) and/or (8.4).†

Proof. This theorem is only a restatement of results proved previously.

Remark. The system (8.19) is considerably simpler than the system (8.3). In the second part of this paper we shall see that even the system (8.19) may be considerably simplified by means of the holonomy group.

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* ${}^0K\dots(Y\dots)$ stands also for its covariant derivatives.

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† A more general problem of this kind has been dealt with in [3] Chapter IV.