An Extension of a Theorem of Marcinkiewicz and Some of Its Applications

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1. Introduction. Let (M, μ) and (N, ν) be two σ -finite measure spaces, $\mathcal{S}(M)$ the class of all real-valued simple functions on M and $\mathfrak{M}(N)$ the class of all real-valued measurable functions on N .¹ If $p, q \geq 1$ we say that a linear operator T mapping $\mathcal{S}(M)$ into $\mathfrak{M}(N)$ is of *type* (p, q) if there exists a positive constant A such that for each simple function f in $\mathcal{S}(M)$

$$(1.1) \quad \|Tf\|_q \leq A \|f\|_p,$$

where

$$\|Tf\|_q = \left(\int_N |Tf|^q d\nu \right)^{1/q}$$

and

$$\|f\|_p = \left(\int_M |f|^p d\mu \right)^{1/p}.$$

If h is in $\mathfrak{M}(N)$, the *distribution function* of $|h|$, $\lambda(y)$, is defined for each $y > 0$ by

$$\lambda(y) = \nu\{x \text{ in } N; |h(x)| > y\}.$$

The operator T is of *weak type* (p, q) if there exists a positive number A such that, for each simple function f and $y > 0$,

$$(1.2) \quad \lambda(y) \leq \left[\frac{A}{y} \|f\|_p \right]^q,$$

where $\lambda(y)$ is the distribution function of $|h|$ and $h = Tf$.²

¹ Here M and N denote the point-sets and μ, ν the measures on the σ -rings of measurable subsets of M and N , respectively. Even though, for simplicity, we restrict ourselves to real-valued functions, our results are easily extensible to spaces of complex-valued functions.

² The condition of type (p, q) asserts that T , as an operator on $\mathcal{S}(M) \cap L^p(M)$ into $L^q(N)$, is bounded. It is easy to check that if T is of type (p, q) then it is of weak type (p, q) ; however, the converse is false (see the appendix of this paper).

Let $1 \leq p_k \leq q_k < \infty$ ($k = 0, 1$), $q_0 \neq q_1$, and define for $0 < t < 1$

$$\frac{1}{p_t} = \frac{(1-t)}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{(1-t)}{q_0} + \frac{t}{q_1}.$$

MARCINKIEWICZ ([2], [6]) proved the following theorem:

Theorem I. *If T is of weak types (p_0, q_0) and (p_1, q_1) then it is of type (p_t, q_t) .³*

The conclusion of this theorem, however, follows from weaker assumptions on the operator T . More precisely, we say that T is of *restricted weak type* (p, q) if there exists a positive number A such that for each measurable set $E \subset M$, of finite measure, and $y > 0$

$$(1.3) \quad \lambda_E(y) \leq \left[\frac{A}{y} ||\chi_E||_p \right]^q,$$

where χ_E is the characteristic function of E and $\lambda_E(y)$ is the distribution function of $|T\chi_E|$. Then, with the same restrictions on p_k and q_k ($k = 0, 1$), as well as $p_0 \neq p_1$, we have

Theorem II. *If T is of restricted weak types (p_0, q_0) and (p_1, q_1) then it is of type (p_t, q_t) .*

The purpose of this paper is to present a proof of Theorem II and to illustrate its use by giving new proofs of several well known results. We assume that Theorem I is known and, in the next section, we show how we can deduce Theorem II from it. Sections 3 and 4 are devoted to the inequalities of M. RIESZ for the Hilbert transform and the conjugate function (Theorems III and IV below). In the proofs of these theorems we make use of two new results (see Lemma 4 and Lemma 5) that are of interest in themselves. More specifically, we will establish the following:

Theorem. (a) *Let $E \subset (-\infty, \infty)$ be a Lebesgue measurable set of finite measure, $\tilde{\chi}_E$ the Hilbert transform of its characteristic function and $\lambda(y)$, $y > 0$, the distribution function of $|\tilde{\chi}_E|$; then*

$$\lambda(y) = \frac{2|E|}{\sinh y}.$$

(b) *Let $E \subset [0, 2\pi]$ be a Lebesgue measurable set, $\bar{\chi}_E$ the conjugate function of its characteristic function and $\lambda(y)$, $y > 0$, the distribution function of $|\bar{\chi}_E|$; then*

$$e^{i\lambda(y)} = \frac{\sinh \frac{1}{2}y + i \sin \frac{1}{2} |E|}{\sinh \frac{1}{2}y - i \sin \frac{1}{2} |E|}.$$

³ This theorem, here stated for linear operators, is valid for more general transformations. In this paper, however, we will be dealing only with linear operators. Furthermore, the bounds of the operator as a transformation on $L^{p'}$ into $L^{q'}$ depend only on t, p_0, q_0, p_1, q_1 and the constants of the weak type (p_0, q_0) and (p_1, q_1) inequalities. This fact is easily seen to extend to Theorem II and is used in §5.

In Section 5 we apply Theorem II to obtain two important results of PALEY (see Theorems V and VI), and the last section is devoted to establishing a well known inequality for fractional integrals (Theorem VIII). In order to obtain this last result we will make use of a slight variant of Theorem II (Theorem VII).

2. Proof of Theorem II. We shall break up the proof into two lemmas. First, however, we need to introduce another notion weaker than type (p, q) . Using the notation introduced in (1.3), we say that T is of *restricted type* (p, q) if there exists a positive number A such that for each measurable set $E \subset M$, of finite measure,

$$(2.1) \quad \|T\chi_E\|_q \leq A \|\chi_E\|_p,$$

where χ_E is the characteristic function of E . This condition is easily seen to be stronger than “restricted weak type (p, q) ”. In case $p = 1$ “restricted type (p, q) ” is equivalent to “type (p, q) ”; however, this is not true in general. We postpone a discussion of this point to an appendix. Lemma 1 shows a relation between “restricted type (p, q) ” and “weak type (p, q) ”.

Suppose T is of restricted type (p, q) and let f be a function in $L^{q'}(N)$, where $(1/q) + (1/q') = 1$.⁴ Let ξ be the set-function, defined on the measurable subsets E of M of finite measure, such that

$$(2.2) \quad \xi(E) = \int_N (T\chi_E)f \, d\nu.$$

From (2.1) we immediately deduce that ξ is countably additive and absolutely continuous with respect to μ . Thus, by the Radon-Nikodym theorem, there exists a unique (almost everywhere) function h , on M , such that

$$(2.3) \quad \xi(E) = \int_E h \, d\mu$$

for each measurable set $E \subset M$ of finite measure. We define the operator T^* , acting on $L^{q'}(N)$, by letting $T^*f = h$. T^* is clearly linear; also, it behaves, at least formally, like the adjoint operator to T . That is, if s is in $S(M)$ and f is in $L^{q'}(N)$, we obtain from (2.2) and (2.3) and the linearity of T and T^*

$$(2.4) \quad \int_N (Ts)f \, d\nu = \int_M s(T^*f) \, d\mu.$$

In general, however, it is not true that T^*f is in $L^{p'}(M)$ for all f in $L^{q'}(N)$ (see the appendix of this paper).

Lemma 1. *Suppose T is of restricted type (p, q) , where $1 < p < \infty$ and $1 \leq q < \infty$, then T^* is of weak type (q', p') .*

⁴ Throughout this paper we will adhere to this notation; that is, if $s \geq 1$ then s' is that number (including ∞) satisfying $(1/s) + (1/s') = 1$.

Proof. Let f be in $L^{q'}(N)$ and $h = T^*f$. If $E_y = \{x \text{ in } M; |h(x)| > y\}$ and $\lambda(y) = \mu(E_y)$ is the distribution function of $|h|$, we must show the existence of a positive number B , independent of f in $L^{q'}(N)$, satisfying for each $y > 0$ ⁵

$$(2.5) \quad \lambda(y) \leq \left[\frac{B}{y} \|f\|_{q'} \right]^{p'}.$$

We put $E_y = E_y^+ \cup E_y^-$, where $E_y^+ = \{x \text{ in } M; h(x) > y\}$ and $E_y^- = \{x \text{ in } M; h(x) < -y\}$. Let $\lambda^+(y) = \mu(E_y^+)$, $\lambda^-(y) = \mu(E_y^-)$. Then, for $y > 0$, $E_y^+ \cap E_y^- = \emptyset$ and $\lambda(y) = \mu(E_y) = \mu(E_y^+) + \mu(E_y^-) = \lambda^+(y) + \lambda^-(y)$. Thus, by the definition of $\lambda^+(y)$, (2.4) with $s = \chi_{E_y^+}$ (= the characteristic function of E_y^+), Hölder's inequality and the assumption of restricted type (p, q) , we have

$$\begin{aligned} y\lambda^+(y) &= y \int_{E_y^+} d\mu \leq \int_{E_y^+} h \, d\mu = \int_M \chi_{E_y^+}(T^*f) \, d\mu = \int_N (T\chi_{E_y^+})f \, dv \\ &\leq \|T\chi_{E_y^+}\|_{q'} \|f\|_{q'} \leq A \|\chi_{E_y^+}\|_p \|f\|_{q'}. \end{aligned}$$

Since $\|\chi_{E_y^+}\|_p = [\lambda^+(y)]^{1/p}$, we have shown, for $y > 0$,

$$y\lambda^+(y) \leq A[\lambda^+(y)]^{1/p} \|f\|_{q'}.$$

Keeping in mind that $1 - (1/p) = (1/p')$, this last inequality can be transformed into

$$(2.6) \quad \lambda^+(y) \leq \left[\frac{A}{y} \|f\|_{q'} \right]^{p'}$$

for $y > 0$.

By a completely analogous argument we obtain, for $y > 0$,

$$(2.7) \quad \lambda^-(y) \leq \left[\frac{A}{y} \|f\|_{q'} \right]^{p'}.$$

Since $\lambda(y) = \lambda^+(y) + \lambda^-(y)$, (2.6) and (2.7) yield (2.5) with $B = 2^{1/p'} A$.

Lemma 2. Let T be of restricted weak types (p_0, q_0) and (p_1, q_1) , $q_0 \neq q_1$, then it is of restricted type (p_t, q_t) for $0 < t < 1$.

Proof. Let $p = p_t$, $q = q_t$ for a fixed t between 0 and 1. Suppose $E \subset M$ is a measurable set of finite measure, χ_E its characteristic function, $h = T\chi_E$ and $\lambda(y)$ the distribution function of $|h|$. One can easily verify that

$$(2.8) \quad \int_N |h|^q \, dv = q \int_0^\infty y^{q-1} \lambda(y) \, dy.$$

We can assume, without loss of generality, that $q_0 < q_1$. Then, using the restricted weak type relations (with constants A_0 and A_1), we obtain for any $C > 0$

⁵ According to our definitions we need only consider T^* restricted to the class of simple functions on N , but the proof of (2.5) yields the result for all functions in $L^{q'}(N)$.

$$\begin{aligned}
\int_N |h|^q d\nu &= q \int_0^\infty y^{q-1} \lambda(y) dy = q \int_0^C y^{q-1} \lambda(y) dy + q \int_C^\infty y^{q-1} \lambda(y) dy \\
&\leq q \int_0^C y^{q-1} \left(\frac{A_0}{y} [\mu(E)]^{1/p_0} \right)^{q_0} dy + q \int_C^\infty y^{q-1} \left(\frac{A_1}{y} [\mu(E)]^{1/p_1} \right)^{q_1} dy \\
&= \left[q \frac{A_0^{q_0}}{(q - q_0)} \right] [\mu(E)]^{q_0/p_0} C^{q-q_0} + \left[q \frac{A_1^{q_1}}{(q_1 - q)} \right] [\mu(E)]^{q_1/p_1} C^{q-q_1}.
\end{aligned}$$

Letting $C = [\mu(E)]^s$, where

$$s = \frac{1}{q - q_0} \left(\frac{q}{p} - \frac{q_0}{p_0} \right) = \frac{1}{q - q_1} \left(\frac{q}{p} - \frac{q_1}{p_1} \right),$$

we have

$$[\mu(E)]^{q_0/p_0} C^{q-q_0} = [\mu(E)]^{q/p} = [\mu(E)]^{q_1/p_1} C^{q-q_1}.$$

Thus we have shown that

$$(2.9) \quad \int_N |h|^q d\nu \leq A^q [\mu(E)]^{q/p},$$

where

$$A = \left\{ \frac{q A_0^{q_0}}{q - q_0} + \frac{q A_1^{q_1}}{q_1 - q} \right\}^{1/q}.$$

Taking q^{th} roots, (2.9) reduces to (2.1) and the lemma is proved.

Theorem II is now an easy consequence of these two lemmas and Theorem I: Since T is of restricted weak types (p_0, q_0) and (p_1, q_1) , by Lemma 2, it is of restricted type (p_t, q_t) for $0 < t < 1$. Thus, by Lemma 1, T^* is of weak type (p'_t, q'_t) . Suppose $0 < t_0 < s < t_1 < 1$. The conditions $p_0 \neq p_1$, $1 \leq p_k \leq q_k$ ($k = 0, 1$) imply $p'_0 \neq p'_1$, $1 \leq q'_k \leq p'_k$ ($k = 0, 1$). Thus we have $p'_0 \neq p'_1$, $1 < q'_{t_0} \leq p'_{t_0}$ ($k = 0, 1$). Hence we can apply Theorem I to T^* , with (p_k, q_k) replaced by (q'_k, p'_k) ($k = 0, 1$), obtaining the result that T^* is of type (q'_s, p'_s) . It follows from our previous discussion, in fact, that T^* is defined on all of $L^{q''}(N)$ and is a bounded operator into $L^{p''}(M)$. Thus the adjoint of T^* is also bounded. But it follows from (2.4) and the representation theorem of F. RIESZ that this adjoint must agree with T on $\mathcal{S}(M)$. Thus T is of type (p_s, q_s) . By making t_0 tend to 0 and t_1 to 1, if necessary, we see that s can be any number satisfying $0 < s < 1$.

3. The Hilbert transform. We now consider the special case where (M, μ) and (N, ν) are both the real line with Lebesgue measure. We shall apply Theorem II to the operator $\tilde{f} = Tf$ defined by the Cauchy principal value integral

$$(3.1) \quad \tilde{f}(x) = P \int_{-\infty}^{\infty} \frac{f(t)}{t - x} dt = \lim_{\epsilon \rightarrow 0+} \left\{ \int_{-\infty}^{x-\epsilon} \frac{f(t)}{t - x} dt + \int_{x+\epsilon}^{\infty} \frac{f(t)}{t - x} dt \right\}.$$

This operator is known as the *Hilbert transform*. It is a well known fact (see [4]) that $\tilde{f}(x)$ exists for almost all x if f is in $L^p(-\infty, \infty)$, $1 \leq p < \infty$. We shall not require this knowledge here, however. Our purpose will be to give a meaning to $\tilde{f}$ for simple functions f and obtain a new proof of the following theorem of M. RIESZ ([4], p. 133):

Theorem III. *The Hilbert transform is of type (p, p) for $1 < p < \infty$.*

We first observe that if f is the characteristic function of the interval (a, b) , it follows directly from (3.1) that

$$(3.2) \quad \tilde{f}(x) = \log \left| \frac{x - b}{x - a} \right|.$$

For the remainder of this paper we shall adhere to the following notation: If E is a Lebesgue measurable subset of the real line, $|E|$ will denote its measure and χ_E its characteristic function.

The following lemma is basic:

Lemma 3. *Let E be the union of a finite number of mutually disjoint intervals (a_k, b_k) , $k = 1, 2, \dots, n$; then $\tilde{\chi}_E(x)$ exists, except for a finite number of values of x , and the distribution function of $|\tilde{\chi}_E|$ is*

$$(3.3) \quad \lambda(y) = \frac{2|E|}{\sinh y}, \quad y > 0.$$

Proof. From (3.2) we have $\tilde{\chi}_E(x) = \log \prod_{k=1}^n |(x - b_k)/(x - a_k)|$. Thus $\tilde{\chi}_E(x)$ exists except when $x = a_k$ or $x = b_k$, $k = 1, 2, \dots, n$.

By renumbering the intervals if necessary, we may suppose that

$$a_1 < b_1 < a_2 < b_2 < \dots < a_n < b_n.$$

The distribution function $\lambda(y)$ of $|\tilde{\chi}_E|$ is the measure of the set

$$E_y = \left\{ x : \left| \log \prod_{k=1}^n \left| \frac{x - b_k}{x - a_k} \right| \right| > y \right\}, \quad y > 0.$$

Let

$$E_y^+ = \left\{ x : \log \prod_{k=1}^n \left| \frac{x - b_k}{x - a_k} \right| > y \right\} = \left\{ x : \prod_{k=1}^n \left(\frac{x - b_k}{x - a_k} \right)^2 > e^{2y} \right\},$$

and

$$E_y^- = \left\{ x : \log \prod_{k=1}^n \left| \frac{x - b_k}{x - a_k} \right| < -y \right\} = \left\{ x : \prod_{k=1}^n \left(\frac{x - b_k}{x - a_k} \right)^2 < e^{-2y} \right\}$$

Also, let $\lambda^+(y) = |E_y^+|$ and $\lambda^-(y) = |E_y^-|$. Then $\lambda(y) = \lambda^+(y) + \lambda^-(y)$.

We shall first examine $\lambda^+(y)$. We begin by observing that the polynomial equation

$$(3.4) \quad \exp \{2\tilde{\chi}_E(x)\} = \prod_{k=1}^n \left(\frac{x - b_k}{x - a_k} \right)^2 = e^{2y}$$

has at most $2n$ distinct roots. On the other hand, $\tilde{\chi}_E(x) \rightarrow -\infty$ as $x \rightarrow b_k$ and $\tilde{\chi}_E(x) \rightarrow +\infty$ as $x \rightarrow a_k$, $k = 1, 2, 3, \dots, n$. Thus, since $\tilde{\chi}_E(x) \rightarrow 0$ as $x \rightarrow -\infty$, we find different roots of (3.4) to the right and left of each a_k in each of the intervals $(-\infty, b_1)$, (b_1, b_2) , (b_2, b_3) , $\dots$, (b_{n-1}, b_n) . Since there are $2n$ such roots, these are all of them. Let us denote these roots by $s_1, t_1, s_2, t_2, \dots, s_n, t_n$, where

$$(3.5) \quad s_1 < a_1 < t_1 < b_1 < s_2 < a_2 < t_2 < b_2 < \dots < s_n < a_n < t_n < b_n.$$

Clearly, E_v^+ is the union of the intervals (s_k, t_k) , $k = 1, 2, \dots, n$. Thus

$$\lambda^+(y) = |E_v^+| = \sum_{k=1}^n (t_k - s_k) = \sum_{k=1}^n t_k - \sum_{k=1}^n s_k.$$

Taking square roots in (3.4) we obtain the two equations

$$(3.6) \quad \prod_{k=1}^n \frac{x - b_k}{x - a_k} = e^y$$

and

$$(3.7) \quad \prod_{k=1}^n \frac{x - b_k}{x - a_k} = -e^y.$$

On the other hand, it follows from (3.5) that $(s_k - b_i)/(s_k - a_i) > 0$ for all j and k , while $(t_k - b_i)/(t_k - a_i) > 0$ for $j \neq k$, but $(t_k - b_k)/(t_k - a_k) < 0$. Thus $\{s_1, s_2, \dots, s_n\}$ are the roots of equation (3.6), while $\{t_1, t_2, \dots, t_n\}$ are the roots of equation (3.7). We rephrase this in the following way: Let

$$p(x) = \frac{\prod_{k=1}^n (x - b_k) - e^y \prod_{k=1}^n (x - a_k)}{1 - e^y} = x^n + \left[\frac{e^y \sum_{k=1}^n a_k - \sum_{k=1}^n b_k}{1 - e^y} \right] x^{n-1} + \dots$$

and

$$q(x) = \frac{\prod_{k=1}^n (x - b_k) + e^y \prod_{k=1}^n (x - a_k)}{1 + e^y} = x^n + \left[\frac{-e^y \sum_{k=1}^n a_k - \sum_{k=1}^n b_k}{1 + e^y} \right] x^{n-1} + \dots;$$

then $\{s_1, s_2, \dots, s_n\}$ are the roots of the polynomial $p(x)$ and $\{t_1, t_2, \dots, t_n\}$ are the roots of the polynomial $q(x)$. Since both polynomials have leading coefficients 1, we have

$$p(x) = \prod_{k=1}^n (x - s_k) = x^n + (-s_1 - s_2 - \dots - s_n)x^{n-1} + \dots$$

and

$$q(x) = \prod_{k=1}^n (x - t_k) = x^n + (-t_1 - t_2 - \dots - t_n)x^{n-1} + \dots.$$

Equating the coefficient of x^{n-1} in each case, we have

$$-\sum_{k=1}^n s_k = \frac{e^y \sum_{k=1}^n a_k - \sum_{k=1}^n b_k}{1 - e^y}, \quad \sum_{k=1}^n t_k = \frac{e^y \sum_{k=1}^n a_k + \sum_{k=1}^n b_k}{1 + e^y}.$$

Thus

$$\begin{aligned} \lambda^+(y) = |E_y^+| &= \sum_{k=1}^n (t_k - s_k) = \frac{e^y \sum_{k=1}^n a_k + \sum_{k=1}^n b_k}{1 + e^y} + \frac{e^y \sum_{k=1}^n a_k - \sum_{k=1}^n b_k}{1 - e^y} \\ &= \frac{2 \sum_{k=1}^n (b_k - a_k)}{e^y - e^{-y}} = |E| \frac{1}{\sinh y}. \end{aligned}$$

A completely analogous argument shows that $\lambda^-(y) = |E|/\sinh y$. Thus $\lambda(y) = \lambda^+(y) + \lambda^-(y) = 2|E|/\sinh y$ and (3.3) is established.

We now observe that, if $1 < p < \infty$, then $\int_0^\infty (y^{p-1}/\sinh y) dy < \infty$. Hence, if E is the union of a finite number of mutually disjoint intervals, we have by Lemma 3 and (2.8)

$$\begin{aligned} \|\tilde{\chi}_E\|_p &= \left\{ \int_{-\infty}^\infty |\tilde{\chi}_E(x)|^p dx \right\}^{1/p} = \left\{ p \int_0^\infty y^{p-1} \lambda(y) dy \right\}^{1/p} \\ &= \left\{ p 2 |E| \int_0^\infty \frac{y^{p-1} dy}{\sinh y} \right\}^{1/p} = \left\{ 2p \int_0^\infty \frac{y^{p-1} dy}{\sinh y} \right\}^{1/p} \|\chi_E\|_p. \end{aligned}$$

Letting

$$A_p = \left\{ 2p \int_0^\infty \frac{y^{p-1} dy}{\sinh y} \right\}^{1/p},$$

we thus have

$$(3.8) \quad \|\tilde{\chi}_E\|_p = A_p \|\chi_E\|_p$$

for each set E that is the union of a finite number of mutually disjoint intervals.

Now suppose that E is an arbitrary measurable set of real numbers, of finite measure. We can then find a sequence of sets $\{E_n\}$, each E_n being a finite union of mutually disjoint intervals, such that $\|\chi_E - \chi_{E_n}\|_p \rightarrow 0$ as $n \rightarrow \infty$. But (3.1) defines $\tilde{\chi}_{E_n}(x) - \tilde{\chi}_{E_m}(x) = \tilde{\chi}_{E_n - E_m}(x) - \tilde{\chi}_{E_m - E_n}(x)$ for each pair of numbers n and m , except for a finite number of values of x . Thus, by (3.8), we have

$$\begin{aligned} \|\tilde{\chi}_{E_n} - \tilde{\chi}_{E_m}\|_p &= \|\tilde{\chi}_{E_n - E_m} - \tilde{\chi}_{E_m - E_n}\|_p \\ &\leq A_p (\|\chi_{E_n - E_m}\|_p + \|\chi_{E_m - E_n}\|_p) \leq 2A_p \|\chi_{E_n} - \chi_{E_m}\|_p. \end{aligned}$$

Since the sequence $\{\chi_{E_n}\}$ is convergent in the L^p -norm, it is Cauchy with respect to this norm. The above inequalities, therefore, show that the sequence $\{\tilde{\chi}_{E_n}\}$ is Cauchy with respect to the L^p -norm. By the Riesz-Fischer theorem, therefore,

this last sequence has a limit which we denote by $\tilde{\chi}_E$. It is easy to check that $\tilde{\chi}_E$ is uniquely defined by this process. The following lemma extends (3.3) to arbitrary measurable sets of finite measure.

Lemma 4. *Let E be a measurable set of real numbers having finite measure and let $\lambda(y)$, $y > 0$, be the distribution function of $|\tilde{\chi}_E|$; then*

$$(3.9) \quad \lambda(y) = \frac{2|E|}{\sinh y}.^6$$

Proof. Let $y > 0$ be a point of continuity of λ . Let $\{E_n\}$ be a sequence of sets, each a finite union of mutually disjoint intervals, such that (a) $\|\chi_{E_n} - \chi_E\|_2 \rightarrow 0$ and, consequently, (b) $\|\tilde{\chi}_{E_n} - \tilde{\chi}_E\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Let $F_y = \{x: |\tilde{\chi}_E(x)| > y\}$ and $F_y^{(n)} = \{x: |\tilde{\chi}_{E_n}(x)| > y\}$; then $|F_y| = \lambda(y)$ and $|F_y^{(n)}| = \lambda_n(y)$ (= distribution function of $|\tilde{\chi}_{E_n}|$). We shall show that $|\lambda(y) - \lambda_n(y)| \rightarrow 0$ as $n \rightarrow \infty$. Since, by (a), $|E_n| \rightarrow |E|$ as $n \rightarrow \infty$, this fact, together with Lemma 3, implies (3.9) for each point of continuity of λ . On the other hand, since $\lambda(y)$ is non-increasing and agrees with $2|E|/\sinh y$ except for, possibly, a countable number of values of y (the points of discontinuity of λ), we must have agreement for *all* positive y .

We first observe that $|\lambda(y) - \lambda_n(y)| \leq |F_y^{(n)} - F_y| + |F_y - F_y^{(n)}|$. Furthermore, if $0 < \epsilon < y$,

$$F_y^{(n)} - F_y = \{F_y^{(n)} - F_{y-\epsilon}\} \cup \{F_y^{(n)} \cap (F_{y-\epsilon} - F_y)\}$$

and

$$F_y - F_y^{(n)} = \{F_y - F_{y-\epsilon}^{(n)}\} \cup \{F_y \cap (F_{y-\epsilon}^{(n)} - F_y^{(n)})\};$$

thus

$$\begin{aligned} |F_y^{(n)} - F_y| &= |F_y^{(n)} - F_{y-\epsilon}| + |F_y^{(n)} \cap (F_{y-\epsilon} - F_y)| \\ &\leq |F_y^{(n)} - F_{y-\epsilon}| + |F_{y-\epsilon}| - |F_y|, \end{aligned}$$

and, similarly,

$$|F_y - F_y^{(n)}| \leq |F_y - F_{y-\epsilon}^{(n)}| + |F_{y-\epsilon}^{(n)}| - |F_y^{(n)}|.$$

Since λ is continuous at y , $|F_{y-\epsilon}| - |F_y| \rightarrow 0$ as $\epsilon \rightarrow 0$. Similarly

$$|F_{y-\epsilon}^{(n)}| - |F_y^{(n)}| = 2|E_n| \left(\frac{1}{\sinh(y-\epsilon)} - \frac{1}{\sinh y} \right) \rightarrow 0$$

as $\epsilon \rightarrow 0$, independently of n , since, by (a), the sequence $\{|E_n|\}$ is convergent and hence bounded. Thus it suffices to show that, for $\epsilon > 0$ fixed, $|F_y^{(n)} - F_{y-\epsilon}| \rightarrow 0$ and $|F_y - F_{y-\epsilon}^{(n)}| \rightarrow 0$ as $n \rightarrow \infty$. On the other hand, if x is in $F_y^{(n)} - F_{y-\epsilon}$, we have

$$|\tilde{\chi}_{E_n}(x) - \tilde{\chi}_E(x)| \geq |\tilde{\chi}_{E_n}(x)| - |\tilde{\chi}_E(x)| > y - (y - \epsilon) = \epsilon,$$

⁶ We only need $\lambda(y) \leq 2|E|/\sinh y$ in order to obtain Theorem III; but the fact that we have equality, as was mentioned in the introduction, is of interest in itself.

while if x is in $F_\nu - F_{\nu-\epsilon}^{(n)}$,

$$|\tilde{\chi}_E(x) - \tilde{\chi}_{E_n}(x)| \geq |\tilde{\chi}_E(x)| - |\tilde{\chi}_{E_n}(x)| > y - (y - \epsilon) = \epsilon.$$

Thus, letting $G = (F_\nu^{(n)} - F_{\nu-\epsilon}^{(n)}) \cup (F_\nu - F_{\nu-\epsilon}^{(n)})$, we obtain

$$\epsilon^2 |G| = \int_G \epsilon^2 \leq \int_G |\tilde{\chi}_E - \tilde{\chi}_{E_n}|^2 \leq \|\tilde{\chi}_{E_n} - \tilde{\chi}_E\|_2^2 \rightarrow 0$$

as $n \rightarrow \infty$, by (b).

Since $|F_\nu^{(n)} - F_{\nu-\epsilon}^{(n)}| + |F_\nu - F_{\nu-\epsilon}^{(n)}| = |G|$ the lemma is proved.

We can now define the Hilbert transform $\tilde{f}$ of a simple function $f = \sum_{n=1}^N c_n \chi_{E_n}$, where $c_1, c_2, \dots, c_N$ are real numbers and $E_1, E_2, \dots, E_N$ are mutually disjoint measurable sets of finite measure, by letting

$$(3.10) \quad \tilde{f} = \sum_{n=1}^N c_n \tilde{\chi}_{E_n}.$$

It is easily established that this operator is then well defined on the class of simple functions and is linear. Lemma 4 immediately implies that the Hilbert transform is of restricted weak type (p, p) for $1 \leq p < \infty$. In fact, the argument preceding Lemma 4 shows that it is of restricted type (p, p) for $1 < p < \infty$. We can thus apply Theorem II to this operator, letting (p_0, q_0) and (p_1, q_1) be any pair of the types (p_0, p_0) and (p_1, p_1) with $1 \leq p_0 < p_1 < \infty$. This application of Theorem II yields the result that the Hilbert transform is of the type (p, p) for $p_0 < p < p_1$; letting p_0 tend to 1 and p_1 tend to ∞ , if necessary, we obtain Theorem III.⁷

4. The conjugate function. Another application of Theorem II, very similar to the one presented in the previous section, yields the inequality of M. RIESZ for conjugate functions. We begin by making this statement more precise. In this section, we consider the special case where (M, μ) and (N, ν) are both the interval $(0, 2\pi)$ with Lebesgue measure. Theorem II will then be applied to the operator $\tilde{f} = Tf$ defined by the Cauchy principal value integral

$$(4.1) \quad \begin{aligned} \tilde{f}(\varphi) &= P \int_0^{2\pi} f(\theta) \frac{\sin(\varphi - \theta)}{1 - \cos(\varphi - \theta)} d\theta \\ &= \lim_{\epsilon \rightarrow 0+} \left\{ \int_0^{\varphi-\epsilon} + \int_{\varphi+\epsilon}^{2\pi} \right\} f(\theta) \frac{\sin(\varphi - \theta)}{1 - \cos(\varphi - \theta)} d\theta. \end{aligned}$$

This operator differs from the conjugate function operator, [5], by a constant multiple.⁸ It is well known that $\tilde{f}(\varphi)$ exists for almost all φ if f is in $L^p(0, 2\pi)$,

⁷ It is of interest to note that L. H. LOOMIS proves that the Hilbert transform is of weak type $(1, 1)$ by also analyzing the spacing of the roots of a certain polynomial (see *Bull. A.M.S.*, 1946, p. 1083).

⁸ Using the standard definition of the conjugate function, this is most easily shown by integrating by parts.

$1 \leq p \leq \infty$. Here again, however, this knowledge is not required for our purposes. In fact, we can proceed in a manner completely analogous to that in §3. We first observe that if f is the characteristic function of the interval (α, β) , where $0 \leq \alpha < \beta \leq 2\pi$, then it follows from (4.1) that

$$(4.2) \quad \bar{f}(\varphi) = \log \frac{1 - \cos(\varphi - \alpha)}{1 - \cos(\varphi - \beta)}.$$

Thus $\bar{f}$ ($= \bar{\chi}_E$) is defined whenever f is the characteristic function χ_E of a set $E = (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) \cup \dots \cup (\alpha_n, \beta_n)$, where $0 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_n < \beta_n \leq 2\pi$. The basic lemma (Lemma 5, below), the analogue to Lemma 3, will then give us an expression for the distribution function of $|\bar{f}|$. This result can then be extended to all characteristic functions of measurable subsets of $(0, 2\pi)$ (this extension being the analogue to Lemma 4), which enables us to define $\bar{f}$ for all simple functions f and to apply Theorem II. We thus obtain the inequality of M. RIESZ for conjugate functions, which can be stated in the following manner:

Theorem IV. *The conjugate function operator is of type (p, p) for $1 < p < \infty$.*

We shall devote the remainder of this section to the statement and proof of Lemma 5. We leave it to the reader to fill in the remaining details for the proof of Theorem IV which we have just outlined. These details are essentially the same as their analogues in §3.

Lemma 5. *Let E be the union of a finite number of mutually disjoint intervals (α_k, β_k) , $k = 1, 2, \dots, n$, where $0 \leq \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \alpha_n < \beta_n \leq 2\pi$, and $\lambda(y)$, $y > 0$, the distribution function of $|\bar{\chi}_E|$. Then $\lambda(y)$ satisfies the equation*

$$(4.3) \quad \exp \left\{ \frac{1}{2} i \lambda(y) \right\} = \frac{\sinh \frac{1}{2} y + i \sin \frac{1}{2} |E|}{\sinh \frac{1}{2} y - i \sin \frac{1}{2} |E|};$$

consequently there exists an absolute constant $A > 0$ such that

$$\lambda(y) \leq \frac{A}{\sinh \frac{1}{2} y} |E|.$$

Proof. For each $y > 0$, $\lambda(y)$ is the measure of the set $E_y = \{\varphi; |\bar{\chi}_E(\varphi)| > y\}$. Let $E_y^+ = \{\varphi; \bar{\chi}_E(\varphi) > y\}$, $E_y^- = \{\varphi; \bar{\chi}_E(\varphi) < -y\}$, $\lambda^+(y) = |E_y^+|$ and $\lambda^-(y) = |E_y^-|$. Then $\lambda(y) = \lambda^+(y) + \lambda^-(y)$. We begin by examining $\lambda^+(y)$.

As an immediate consequence of (4.2) we have

$$(4.4) \quad \bar{\chi}_E(\varphi) = \log \prod_{k=1}^n \frac{1 - \cos(\varphi - \alpha_k)}{1 - \cos(\varphi - \beta_k)} = \log \prod_{k=1}^n \frac{\sin^2 \frac{1}{2} [\varphi - \alpha_k]}{\sin^2 \frac{1}{2} [\varphi - \beta_k]}.$$

Thus

$$(4.5) \quad \exp \{ \bar{\chi}_E(\varphi) \} = \prod_{k=1}^n \frac{2e^{i\varphi} - e^{2i\varphi} e^{-i\alpha_k} - e^{i\alpha_k}}{2e^{i\varphi} - e^{2i\varphi} e^{-i\beta_k} - e^{i\beta_k}},$$

and

$$E_y^+ = \left\{ \varphi; \prod_{k=1}^n \frac{2e^{i\varphi} - e^{2i\varphi}e^{-i\alpha_k} - e^{i\alpha_k}}{2e^{i\varphi} - e^{2i\varphi}e^{-i\beta_k} - e^{i\beta_k}} > e^y \right\}.$$

Consider the polynomial

$$Q(z) = \prod_{k=1}^n (2z - z^2e^{-i\alpha_k} - e^{i\alpha_k}) - e^y \prod_{k=1}^n (2z - z^2e^{-i\beta_k} - e^{i\beta_k}).$$

$Q(z)$ has order $2n$ and thus has $2n$ roots. We now claim that these roots are distinct and all lie on the unit circle; furthermore, they can be divided into two classes

$$S = \{\zeta_k = e^{i\alpha_k}; k = 1, 2, \dots, n\}, \quad T = \{\tau_k = e^{i\beta_k}; k = 1, 2, \dots, n\},$$

where

$$0 \leq \alpha_1 < s_1 < \beta_1 < t_1 < \alpha_2 < s_2 < \beta_2 < t_2 < \dots$$

$$< \alpha_n < s_n < \beta_n < t_n < \alpha_1 + 2\pi.$$

Toward this end, we first note that, by (4.4), $\bar{\chi}_E(\varphi) \rightarrow -\infty$ as $\varphi \rightarrow \alpha_k$ and $\bar{\chi}_E(\varphi) \rightarrow +\infty$ as $\varphi \rightarrow \beta_k$, $k = 1, 2, \dots, n$. Thus, since $\bar{\chi}_E$ is continuous in each of the open intervals (α_1, β_1) , (α_2, β_2) , $\dots$, (α_n, β_n) , (β_1, α_2) , (β_2, α_3) , $\dots$, (β_{n-1}, α_n) , $(\beta_n, \alpha_1 + 2\pi)$, there exists a number in each of these intervals which is a solution of the equation $\bar{\chi}_E(\varphi) = y$. But it follows from (4.5) and the expression for Q that, for each such solution φ , $e^{i\varphi}$ is a root of Q . Since there are $2n$ such intervals, and these intervals are mutually disjoint if regarded as intervals of arc on the unit circle, we have exhibited $2n$ distinct roots of Q and have established the above claim.

Thus, if z is a root of Q , $z = e^{i\varphi}$ with either $e^{i\varphi}$ in S or $e^{i\varphi}$ in T , and by (4.4)

$$(4.6) \quad e^y = \prod_{k=1}^n \frac{\sin^2(\frac{1}{2}[\varphi - \alpha_k])}{\sin^2(\frac{1}{2}[\varphi - \beta_k])}.$$

If $e^{i\varphi} = e^{i\alpha_k}$ is in S , then we must have

$$1^\circ \quad 0 < \frac{s_k - \alpha_i}{2} < \pi, \quad 0 < \frac{s_k - \beta_i}{2} < \pi \quad \text{for } j = k-1, k-2, \dots, 1;$$

$$2^\circ \quad 0 < \frac{s_k - \alpha_k}{2} < \pi, \quad 0 > \frac{s_k - \beta_k}{2} > -\pi;$$

$$3^\circ \quad 0 > \frac{s_k - \alpha_i}{2} > -\pi, \quad 0 > \frac{s_k - \beta_i}{2} > -\pi \quad \text{for } j = k+1, k+2, \dots, n.$$

In cases 1° and 3° , therefore, $\sin \frac{1}{2}(s_k - \alpha_i) / \sin \frac{1}{2}(s_k - \beta_i) > 0$ while, in case 2° , we have $\sin \frac{1}{2}(s_k - \alpha_k) / \sin \frac{1}{2}(s_k - \beta_k) < 0$.

Thus, for $z = e^{i\varphi}$ in S , φ is a root of

$$(4.7) \quad e^{iy} = - \prod_{k=1}^n \frac{\sin(\frac{1}{2}[\varphi - \alpha_k])}{\sin(\frac{1}{2}[\varphi - \beta_k])}.$$

Consequently, if $z = e^{i\varphi}$ is in T then φ is a root of

$$(4.8) \quad e^{\frac{1}{2}y} = \prod_{k=1}^n \frac{\sin(\frac{1}{2}[\varphi - \alpha_k])}{\sin(\frac{1}{2}[\varphi - \beta_k])}.$$

Let

$$q^+(z) = \prod_{k=1}^n (ze^{-\frac{1}{2}i\alpha_k} - e^{\frac{1}{2}i\alpha_k}) - e^{\frac{1}{2}y} \prod_{k=1}^n (ze^{-\frac{1}{2}i\beta_k} - e^{\frac{1}{2}i\beta_k})$$

and

$$q^-(z) = \prod_{k=1}^n (ze^{-\frac{1}{2}i\alpha_k} - e^{\frac{1}{2}i\alpha_k}) + e^{\frac{1}{2}y} \prod_{k=1}^n (ze^{-\frac{1}{2}i\beta_k} - e^{\frac{1}{2}i\beta_k}).$$

The polynomials q^+ and q^- are each of degree n and we see from (4.7) and (4.8) that the n roots of q^- are the elements of S , while the n roots of q^+ are the elements of T .

Since $\exp\{\tilde{\chi}_E(\varphi)\}$ is the restriction to the perimeter of the unit circle of the quotient of the products making up Q (see (4.5)), and is real valued there, we can easily obtain a precise value for $|E_y^+|$ in terms of the roots $\{s_k, t_k\}$, $k = 1, 2, \dots, n$. This can be done in the following manner: The roots φ of the equation $\exp\{\tilde{\chi}_E(\varphi)\} = e^y$ are precisely the arguments of the roots of Q ; furthermore, since $\tilde{\chi}_E(\varphi) \rightarrow +\infty$ as $\varphi \rightarrow \beta_k$ and $\tilde{\chi}_E(\varphi) \rightarrow -\infty$ as $\varphi \rightarrow \alpha_k$, and since Q can have no more roots than the ones in $S \cup T$, we see that $\exp\{\tilde{\chi}_E(\varphi)\} > e^y$ precisely in the intervals (s_k, t_k) , $k = 1, 2, \dots, n$. Thus we have shown that

$$(4.9) \quad \lambda^+(y) = |E_y^+| = \sum_{k=1}^n (t_k - s_k) = \sum_{k=1}^n t_k - \sum_{k=1}^n s_k.$$

We have

$$\begin{aligned} q^+(z) &= \prod_{k=1}^n (ze^{-\frac{1}{2}i\alpha_k} - e^{\frac{1}{2}i\alpha_k}) - e^{\frac{1}{2}y} \prod_{k=1}^n (ze^{-\frac{1}{2}i\beta_k} - e^{\frac{1}{2}i\beta_k}) \\ &= \left\{ \prod_{k=1}^n e^{-\frac{1}{2}i\alpha_k} - e^{\frac{1}{2}y} \prod_{k=1}^n e^{-\frac{1}{2}i\beta_k} \right\} \prod_{k=1}^n (z - e^{it_k}). \end{aligned}$$

Thus, using the fact that the constant coefficient of q^+ is expressible in terms of the product of its roots, we have

$$(-1)^n \prod_{k=1}^n e^{\frac{1}{2}i\alpha_k} - e^{\frac{1}{2}y} (-1)^n \prod_{k=1}^n e^{\frac{1}{2}i\beta_k} = \left\{ \prod_{k=1}^n e^{-\frac{1}{2}i\alpha_k} - e^{\frac{1}{2}y} \prod_{k=1}^n e^{-\frac{1}{2}i\beta_k} \right\} (-1)^n \prod_{k=1}^n e^{it_k}.$$

That is,

$$(4.10) \quad \exp\left\{i \sum_{k=1}^n t_k\right\} = \frac{\exp\left\{\frac{1}{2}i \sum_{k=1}^n \alpha_k\right\} - \exp\left\{\frac{1}{2}y + \frac{1}{2}i \sum_{k=1}^n \beta_k\right\}}{\exp\left\{-\frac{1}{2}i \sum_{k=1}^n \alpha_k\right\} - \exp\left\{\frac{1}{2}y - \frac{1}{2}i \sum_{k=1}^n \beta_k\right\}}.$$

Similarly, using q^- in place of q^+ , we have

$$(4.11) \quad \exp \left\{ i \sum_{k=1}^n s_k \right\} = \frac{\exp \left\{ \frac{1}{2} i \sum_{k=1}^n \alpha_k \right\} + \exp \left\{ \frac{1}{2} y + \frac{1}{2} i \sum_{k=1}^n \beta_k \right\}}{\exp \left\{ -\frac{1}{2} i \sum_{k=1}^n \alpha_k \right\} + \exp \left\{ \frac{1}{2} y - \frac{1}{2} i \sum_{k=1}^n \beta_k \right\}}.$$

Thus, by (4.9), (4.10) and (4.11), we have

$$(4.12) \quad \exp \{ i \lambda^+(y) \} = \exp \left\{ i \sum_{k=1}^n (t_k - s_k) \right\} = \frac{\sinh \frac{1}{2} y + i \sin \frac{1}{2} |E|}{\sinh \frac{1}{2} y - i \sin \frac{1}{2} |E|}.$$

A completely analogous argument yields

$$(4.13) \quad \exp \{ i \lambda^-(y) \} = \frac{\sinh \frac{1}{2} y + i \sin \frac{1}{2} |E|}{\sinh \frac{1}{2} y - i \sin \frac{1}{2} |E|}.$$

From (4.12) and (4.13) we obtain

$$(4.14) \quad \exp \{ i \lambda(y) \} = \exp \{ i \lambda^+(y) \} \exp \{ i \lambda^-(y) \} = \left[\frac{\sinh \frac{1}{2} y + i \sin \frac{1}{2} |E|}{\sinh \frac{1}{2} y - i \sin \frac{1}{2} |E|} \right]^2$$

and (4.3) is proved.

The last part of the lemma can now be proved in the following manner. From (4.3) we have

$$(\cos \tfrac{1}{2} \lambda(y) + i \sin \tfrac{1}{2} \lambda(y)) (\sinh \tfrac{1}{2} y - i \sin \tfrac{1}{2} |E|) = \sinh \tfrac{1}{2} y + i \sin \tfrac{1}{2} |E|.$$

Thus, taking real parts,

$$\cos \tfrac{1}{2} \lambda(y) \cdot \sinh \tfrac{1}{2} y + \sin \tfrac{1}{2} \lambda(y) \cdot \sin \tfrac{1}{2} |E| = \sinh \tfrac{1}{2} y.$$

Therefore

$$\sinh \tfrac{1}{2} y = \frac{\sin \tfrac{1}{2} \lambda(y) \sin \tfrac{1}{2} |E|}{1 - \cos \tfrac{1}{2} \lambda(y)} = \left[\frac{\sin \tfrac{1}{2} \lambda(y)}{2 \sin^2 \tfrac{1}{4} \lambda(y)} \right] \cdot \sin \tfrac{1}{2} |E|.$$

Since $0 \leq \lambda(y) \leq 2\pi$, there exists an $A > 0$ such that

$$\frac{1}{A} \lambda(y) \leq \frac{2 \sin^2 \tfrac{1}{4} \lambda(y)}{\sin \tfrac{1}{2} \lambda(y)}.$$

Thus

$$\frac{1}{A} \lambda(y) \leq \frac{2 \sin^2 \tfrac{1}{4} \lambda(y)}{\sin \tfrac{1}{2} \lambda(y)} = \left(\frac{1}{\sinh \tfrac{1}{2} y} \right) \sin \tfrac{1}{2} |E| \leq \frac{|E|}{\sinh \tfrac{1}{2} y},$$

from which the lemma follows.

5. Theorems of Paley. If $c_1, c_2, \dots, c_n, \dots$ is a sequence of complex numbers, we let $c_1^*, c_2^*, \dots, c_n^*, \dots$ denote a rearrangement of the moduli of the first sequence such that $c_1^* \geq c_2^* \geq \dots \geq c_n^* \geq \dots$.

Let $\{\varphi_n\}$ be an orthonormal system of functions defined on $[0, 1]$, satisfying $|\varphi_n(x)| \leq M$ for all n and x . We will prove the following theorems of PALEY (see [3]):

Theorem V. Let $\{c_n\}$ be a sequence of complex numbers for which $\sum_{n=1}^{\infty} (c_n^*)^q n^{q-2}$ is finite, $2 < q < \infty$. Let $S^*(x) = \sup_{n \geq 1} |\sum_{\nu=1}^n c_\nu \varphi_\nu(x)|$. Then

$$\|S^*(x)\|_q \leq A_q M^{(q-2)/q} \left(\sum_{n=1}^{\infty} (c_n^*)^q n^{q-2} \right)^{1/q}.$$

Theorem VI. Let $\{c_n\}$ be a sequence of complex numbers for which $\sum_{n=1}^{\infty} |c_n|^p$ is finite, $1 \leq p < 2$. If $(1/q) + (1/p) = 1$, then

$$\|S^*(x)\|_q \leq A_q M^{(q-2)/q} \left(\sum_{n=1}^{\infty} |c_n|^p \right)^{1/p}.$$

Remarks. Except for the trivial case $p = 1$, Theorem VI follows from Theorem V immediately, because of the well known fact (see [6], p. 206)

$$\left(\sum_{n=1}^{\infty} (c_n^*)^q n^{q-2} \right)^{1/q} \leq A_q \left(\sum_{n=1}^{\infty} |c_n|^p \right)^{1/p}.$$

We shall not need to know this, however. As a matter of fact, we shall first prove Theorem VI by our interpolation theorem, and also by a somewhat complicated preliminary argument. Then by a further interpolation argument we shall deduce Theorem V—the general case—after a rather simple reduction.

The importance of these theorems is that by a well known argument one obtains from them the conclusion that the orthonormal series in question converges almost everywhere.

We shall first need a lemma which follows familiar lines in the theory of orthogonal expansions. This is a special case of a lemma also used by PALEY.

Lemma 6. Let $\{\epsilon_\nu\}$ be a finite collection of numbers, N of which are equal to 1 and the remainder equal to zero. Let $F(x) = \sum \epsilon_\nu \varphi_\nu(x)$ and $F^*(x) = \sup_{n \geq 1} |\sum_{\nu=1}^n \epsilon_\nu \varphi_\nu(x)|$. Then if $2 < q \leq \infty$, $(1/p) + (1/q) = 1$,

$$\|F^*(x)\|_q \leq A_q M^{(q-2)/q} N^{1/p}.$$

Proof. By excluding certain of the φ_ν from our given set $\{\varphi_\nu\}$, if necessary, we may assume that $\epsilon_1 = \epsilon_2 = \dots = \epsilon_N = 1$, the rest being zero. Hence

$$F^*(x) = \sup_{n \leq N} \left| \sum_{\nu=1}^n \varphi_\nu(x) \right|.$$

Let m be the integer satisfying $2^{m-1} \leq N < 2^m$. Corresponding to each fixed integer l , $0 \leq l \leq m$, we split $F(x)$ into 2^l blocks, each block containing 2^{m-l} terms. For k given, we denote the k^{th} such block by

$$(5.1) \quad F_{k,l}(x) = \sum_{(k-1)2^{m-l}+1}^{k2^{m-l}} \varphi_\nu(x).$$

Also

$$(5.2) \quad F_l^*(x) = \sup_k |F_{k,l}(x)|.$$

Now given an integer n , $n \leq 2^m$, consider its binary expansion; it can be written uniquely as a sum of l^{th} powers of 2, for appropriate l , $l \leq m$. A moment's reflection reveals that, therefore,

$$F^*(x) \leq \sum_{l=0}^m F_l^*(x).$$

Hence

$$(5.3) \quad \|F^*(x)\|_a \leq \sum_{l=0}^m \|F_l^*(x)\|_a.$$

We now estimate $\|F_l^*\|_a$. By (5.2),

$$(5.4) \quad \int |F_l^*(x)|^a dx \leq \sum_{k=1}^{2^l} \int |F_{k,l}(x)|^a dx.$$

But

$$\int |F_{k,l}(x)|^a dx \leq \sup_x |F_{k,l}(x)|^{a-2} \int |F_{k,l}(x)|^2 dx.$$

Using the bound M on the system $\{\varphi_\nu\}$, the definition (5.1), and Bessel's inequality, we obtain

$$\int |F_{k,l}(x)|^a dx \leq M^{a-2} 2^{(a-2)(m-l)} 2^{m-l}.$$

Thus, by (5.4),

$$\int |F_l^*(x)|^a dx \leq M^{a-2} 2^{(a-2)(m-l)} 2^m.$$

Hence

$$(5.5) \quad \|F_l^*\|_a \leq M^{(a-2)/a} 2^{(a-2)(m-l)/a} 2^{m/a}.$$

Therefore, by (5.3),

$$\begin{aligned} \|F^*\|_a &\leq \left(\sum_{l=0}^m 2^{(a-2)(m-l)/a} \right) M^{(a-2)/a} 2^{m/a} \\ &\leq A_q 2^{(a-2)m/a} M^{(a-2)/a} 2^{m/a} \leq A_q M^{(a-2)/a} 2^{m(a-1)/a}. \end{aligned}$$

However, $2^{m-1} \leq N$, and $(q-1)/q = 1/p$; we therefore obtain

$$\|F^*\|_a \leq A_q M^{(a-2)/a} N^{1/p},$$

and the lemma is proved.

Proof of Theorem VI. We should like to use our interpolation theorems, but the operators in question are not linear. We therefore proceed as follows: Choose an arbitrary positive-integer-valued function $n(x)$. Let

$$(5.6) \quad S(x) = \sum_{\nu=1}^{n(x)} c_{\nu} \varphi_{\nu}.$$

We then have $|S(x)| \leq S^*(x)$, and

$$\sup_{n(x)} \|S\|_q = \|S^*\|_q.$$

For each $n(x)$, (5.6) defines a linear operator acting on the space of sequences $\{c_n\}$ having at most a finite number of non-zero values, and taking values that are Lebesgue measurable functions. Lemma 6 tells us exactly that this operator is of restricted type (p, q) , $1 \leq p < 2$, with $(1/p) + (1/q) = 1$, and with bounds independent of $n(x)$. Our interpolation Theorem II then implies that $S(x)$ is of type (p, q) , $1 < p < 2$, with bounds independent of $n(x)$. Choosing appropriate $n(x)$ proves Theorem VI, when $1 < p < 2$. Of course, the theorem is trivial when $p = 1$. The extension to infinite sequences satisfying $\sum |c_n|^p < \infty$ is immediate.

Proof of Theorem V. Let $m_1, m_2, \dots, m_{\nu}, \dots$ be an arbitrary rearrangement of the integers $1, 2, \dots, \nu, \dots$, and let $n(x)$ be an arbitrary positive-integer-valued function. Define the function $S(x)$ as in (5.6) before. Then it is sufficient to prove

$$(5.7) \quad \|S(x)\|_q \leq A_q M^{\frac{1}{2}(q-2)} \left(\sum_{\nu=1}^{\infty} |c_{\nu}|^q (m_{\nu})^{q-2} \right)^{1/q},$$

$2 < q < \infty$, with A_q independent of $n(x)$ and the rearrangement $\{m_{\nu}\}$. Consider the operator T which assigns to each sequence $\{a_{\nu}\}$ the function $S(x)$, with $c_{\nu} = a_{\nu}/m_{\nu}$. Assign to the positive integers the following measure: The integer ν is given the measure $1/m_{\nu}^2$. With this understood, the inequality (5.7), which is to be proved, reduces to the statement that T is of type (q, q) , $2 < q < \infty$, and with bounds independent of $n(x)$ and $\{m_{\nu}\}$.

In order to reach this conclusion, we shall show that T is of restricted type (q, q) , $2 < q < \infty$ (and with bounds independent of $n(x)$ and $\{m_{\nu}\}$).

Let I be any set of integers. Then Theorem VI shows

$$(5.8) \quad \|Ta\|_q \leq A_q M^{(q-2)/q} \left(\sum_I \frac{1}{m_{\nu}^2} \right)^{1/p},$$

where a is the sequence $\{a_{\nu}\}$ with $a_{\nu} = 1$ if ν is in I and $a_{\nu} = 0$ otherwise. This is clearly seen by remembering that $c_{\nu} = 1/m_{\nu}$ ($= a_{\nu}/m_{\nu}$) if ν is in I and $c_{\nu} = 0$ otherwise, and comparing with (5.6).

We now claim the following fact:

$$(5.9) \quad \left(\sum_J \frac{1}{m_{\nu}^2} \right)^{1/p} \leq A_p \left(\sum_J \frac{1}{m_{\nu}^2} \right)^{1/q},$$

where $1 < p \leq 2$, $(1/p) + (1/q) = 1$, J is any set of integers and A_p is independent of J . In fact, let $\sum_J 1/m^2 = \sigma$. We then write

$$\sum_J \frac{1}{m^p} = \sum_{(1)} \frac{1}{m^p} + \sum_{(2)} \frac{1}{m^p},$$

where in $\sum_{(1)}$ the summation is extended over the integers in J which are smaller than $1/\sigma$, and $\sum_{(2)}$ includes the remaining integers. Thus

$$\sum_{(1)} \frac{1}{m^p} = \sum_{(1)} \frac{m^{2-p}}{m^2} \leq \left(\frac{1}{\sigma}\right)^{2-p} \sum_J \frac{1}{m^2} = \sigma^{p-2} \cdot \sigma = \sigma^{p-1}.$$

Also

$$\sum_{(2)} \frac{1}{m^p} \leq \sum_{m \geq 1/\sigma} \frac{1}{m^p} \leq A_p \sigma^{p-1}.$$

Hence

$$\sum_J \frac{1}{m^p} \leq A_p \sigma^{p-1} = A_p \left(\sum_J \frac{1}{m^2} \right)^{p-1}.$$

This proves (5.9).⁹ Combining (5.8) and (5.9) we obtain

$$\|Ta\|_q \leq A_q M^{(q-2)/q} \left(\sum_I \frac{1}{m_i^2} \right)^{1/q},$$

for $2 < q < \infty$.

This proves that the operator T , defined above, is of restricted type (q, q) , $2 < q < \infty$, and that the bounds are independent of $n(x)$ and $\{m_\nu\}$. Our interpolation theorem then proves (5.7). This concludes the proof of Theorem V.

6. Fractional integrals. As a preliminary matter we need a variant of Theorem II. Let T be a linear operator mapping $\mathfrak{S}(M)$ into $\mathfrak{M}(N)$. We then consider the bilinear form B that it defines, that is,

$$B(f, g) = \int g T f \, d\nu.$$

We then say that B is of *restricted type* (p, q) if

$$|B(f, g)| \leq A \|f\|_p \|g\|_q, \quad (p, q \geq 1)$$

whenever f and g are characteristic functions of measurable sets of finite measure. We then have the following

Theorem VII. *If $B(f, g)$ is of restricted types (p_0, q_0) and (p_1, q_1) then T is of type (p_t, q_t) , $0 < t < 1$.*

⁹ We are indebted to D. J. NEWMAN for the proof of (5.9).

The proof of this theorem is an immediate application of Theorem II, once we observe that if B is of restricted type (p, q) then T is of restricted weak type (p, q) . The proof of this last fact, however, is so similar to the proof of Lemma 1 that we omit it (in fact, it is much simpler than that argument).

We now proceed to derive a fundamental result of HARDY & LITTLEWOOD (see [1]) on fractional integrals: Let T_λ be the operator defined by the integral

$$T_\lambda(f) = \int_{-\infty}^{\infty} \frac{f(y)}{|x - y|^\lambda} dy, \quad 0 < \lambda < 1.$$

We then have the

Theorem VIII. T_λ is of type (p, q) if $1 < p < \infty$, $1 < q < \infty$ and $1/p = (1/q) + 1 - \lambda$.

Proof. Let

$$B_\lambda(f, g) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(y)g(x)}{|x - y|^\lambda} dy dx.$$

Theorem VIII is then an immediate consequence of Theorem VII if we establish

Lemma 7. B_λ is of restricted type (p, q) if $1 \leq p < \infty$, $1 < q \leq \infty$ and $1/p = (1/q) + 1 - \lambda$.

We thus devote the remainder of this section to the proof of this lemma.

Fix p and q . We must show

$$(6.1) \quad |B_\lambda(f, g)| \leq A \|f\|_p \|g\|_{q'},$$

whenever f and g are characteristic functions of measurable sets of finite measure; i.e. if E and F are two such sets, then

$$(6.1^*) \quad B_\lambda(\chi_E, \chi_F) \leq A |E|^{1/p} |F|^{1-(1/q)}.$$

We claim that it is sufficient to prove (6.1*) when E and F are finite unions of intervals of the form $(k, k + 1)$, where k is an integer. For suppose that we have established this. Making the transformation $x \rightarrow \mu x$, $y \rightarrow \mu y$, $\mu > 0$, we then know (due to the homogeneity of the kernel $|x - y|^{-\lambda}$) (6.1*) when E and F are finite unions of intervals of the form $(\mu k, \mu k + \mu)$. Then by a simple limiting process we know (6.1*) for all sets E and F of finite measure.

Hence it remains for us to prove (6.1*) when E and F are finite unions of intervals of the form $(k, k + 1)$. Let I and J be two finite sets of integers, with n and m elements respectively. We must then show

$$(6.2) \quad \sum_{i \in I} \sum_{k \in J} \int_i^{i+1} \int_k^{k+1} \frac{dx dy}{|x - y|^\lambda} \leq A n^{1/p} m^{1-(1/q)}.$$

Let us first make the obvious observation

$$(6.3) \quad \int_i^{i+1} \int_k^{k+1} \frac{dx dy}{|x - y|^\lambda} \leq A_\lambda (1 + |l - k|)^{-\lambda}, \quad 0 < \lambda < 1.$$

Now consider the pairs (l, k) where l is in J and k is in I . Let N_r be the number of such pairs satisfying $|l - k| = r$. Since the totality of all our pairs is $n \cdot m$, we have

$$(6.4) \quad \sum_{r=0}^{\infty} N_r = n \cdot m \quad (\text{actually a finite sum}).$$

Because of (6.3), then (6.2) follows if we know

$$(6.5) \quad \sum_{r=0}^{\infty} N_r (r+1)^{-\lambda} \leq A_{\lambda} n^{1/p} m^{1-(1/q)}.$$

Let us make one last remark about N_r . Given any fixed l , there are at most two integers k such that $|l - k| = r$. Hence

$$(6.6) \quad N_r \leq 2 \min(n, m), \quad r = 0, 1, 2, \dots$$

Let us now evaluate the left hand side of (6.5):

$$\begin{aligned} \sum_{r=0}^{\infty} N_r (r+1)^{-\lambda} &= \sum_{r \leq w} N_r (r+1)^{-\lambda} + \sum_{r > w} N_r (r+1)^{-\lambda} \\ &\leq 2 \min(n, m) \sum_{r \leq w} (r+1)^{-\lambda} + (w)^{-\lambda} \sum_{r=0}^{\infty} N_r \\ &\leq A_{\lambda} \min(n, m) w^{1-\lambda} + w^{-\lambda} n \cdot m. \end{aligned}$$

We now choose w satisfying $w^{-\lambda} = n^{(1/p)-1} \cdot m^{-(1/q)}$, thus obtaining (6.5).

Appendix. In this paper we have dealt with four conditions on a linear transformation T mapping $\mathcal{S}(M)$ into $\mathfrak{M}(N)$. “Type (p, q) ” and “weak type (p, q) ” have already been discussed by several authors; however, “restricted type (p, q) ” and “restricted weak type (p, q) ” are new concepts. In order to gain a better understanding of these notions we now investigate various relations between them.

Throughout this appendix p and q are real numbers satisfying $1 \leq p, q < \infty$. If f is in $\mathcal{S}(M)$ we write $h = Tf$. λ will always denote the distribution function of $|h|$ —that is, $\lambda(y)$, $y > 0$, is the measure of the set $E_y = \{x \text{ in } N: |h(x)| > y\}$.

(a) **Type (p, q) implies weak type (p, q) , but the converse is false.** The first part of (a) is proved in several places (see, for example, [6]). For the sake of completeness we include the proof. Thus we assume the existence of an absolute constant $A > 0$ such that

$$\|h\|_q \leq A \|f\|_p$$

for all f in $\mathcal{S}(M)$. Using this fact, therefore, we have, for each $y > 0$,

$$(A \|f\|_p)^q \geq \|h\|_q^q = \int_N |h|^q d\nu \geq \int_{E_y} |h|^q d\nu \geq y^q \int_{E_y} d\nu = y^q \nu(E_y) = y^q \lambda(y).$$

Thus

$$\lambda(y) \leq \left[\frac{A}{y} \|f\|_p \right]^q,$$

which is the desired “weak type (p, q) ” condition (see (1.2)).

An example of a linear transformation T of weak type (p, q) but not of type (p, q) can be constructed in the following manner: We let $M = N = (0, \infty)$ and $\mu = \nu =$ Lebesgue measure. Suppose Ψ is a function in $L^{p'}(0, \infty)$ such that $\|\Psi\|_{p'} > 0$ and Φ the function satisfying $\Phi(x) = x^{-(1/q)}$, for $0 < x < \infty$. We then define Tf , for f in $\mathcal{S}(M)$, to be the function $h = (\int_0^\infty \Psi f) \Phi$. We then have

$$\begin{aligned} E_y &= \{x : |h(x)| > y\} = \left\{x : \left| \int_0^\infty \Psi f \right| x^{-(1/q)} > y\right\} \\ &\subset \{x : \|f\|_p \cdot \|\Psi\|_{p'} x^{-(1/q)} > y\} = \left\{x : \left(\frac{\|\Psi\|_{p'}}{y} \|f\|_p \right)^q > x\right\}. \end{aligned}$$

But the measure of the last set is $[(\|\Psi\|_{p'}/y)\|f\|_p]^q$. Thus

$$\lambda(y) = |E_y| \leq \left(\frac{\|\Psi\|_{p'}}{y} \|f\|_p \right)^q,$$

which shows that T is of weak type (p, q) . On the other hand, T is clearly not of type (p, q) —in fact, h is not even in $L^q(0, \infty)$ whenever $\int_0^\infty \Psi f \neq 0$.

(b) Type (p, q) implies restricted type (p, q) . The converse is true if and only if $p = 1$. The first part of (b) is trivial. Furthermore, if $p = 1$ and T is of restricted type (p, q) , then it is easily seen, by an application of Minkowski's inequality, that T is of type (p, q) . Thus we only have to give an example, in case $p \neq 1$, of a linear transformation T of restricted type (p, q) but not of type (p, q) . Toward this end, we construct an unbounded linear functional $\mathcal{L}: \mathcal{S}(M) \rightarrow R$ ($=$ real numbers) for which there exists $A > 0$ such that

$$|\mathcal{L}(\chi_E)| \leq A \|\chi_E\|_p,$$

for all characteristic functions χ_E of measurable sets $E \subset M$ with $\mu(M) < \infty$. Once this is done, if (N, ν) is an arbitrary measure space and Φ a fixed function in $L^q(N)$ with $\|\Phi\|_q > 0$, then, if we define $Tf = \mathcal{L}(f) \cdot \Phi$, we have a linear transformation $T: \mathcal{S}(M) \rightarrow L^q(N) \subset \mathfrak{M}(N)$ which is of restricted type (p, q) but not of type (p, q) .

Let $M = (0, 1)$ with Lebesgue measure and, for f in $\mathcal{S}(M)$, let

$$\mathcal{L}(f) = \int_0^1 x^{-1/p'} f(x) dx.$$

If $E \subset (0, 1)$ is a measurable set, we then have

$$|\mathcal{L}(\chi_E)| = \int_E x^{-1/p'} dx \leq \int_0^{1/|E|} x^{-1/p'} dx \leq A_{p'} |E|^{1/p}.$$

Hence $\mathfrak{L}$ is of restricted type (p, q) for any $q \geq 1$. But there exists f in L^p such that $\mathfrak{L}(f) = \infty$, since $x^{-1/p'}$ is not in $L^{p'}$; e.g. choose $f(x) = x^{-1/p} (\log [1/x])^{-1}$, if $0 \leq x \leq \frac{1}{2}$, and $f(x) = 0$ otherwise.

(c) **Restricted type (p, q) implies restricted weak type (p, q) , but the converse is false.** The first part of this assertion is proved in exactly the same way as the first part of (a). In order to construct an example for the last part, we proceed as follows: Let M and N be the interval $(0, 1)$ with Lebesgue measure, and let T be defined by letting

$$(*) \quad (Tf)(x) = x^{-1/q} \int_0^1 y^{-1/p'} f(y) dy.$$

Then, by combining the last arguments in both parts (a) and (b) we easily see that T is of restricted weak type (p, q) but not of restricted type (p, q) .

(d) **Weak type (p, q) implies restricted weak type (p, q) . In case $p \neq 1$ the converse is false.** The first part of (d) is immediate. On the other hand, if T is given by (*) we can easily deduce that, for $p \neq 1$, T is of restricted weak type but not of weak type (p, q) .

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