*Solutions of a Riccati Matrix Differential Equation as Functions of Initial Values**

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1. Introduction. REDHEFFER [1, 2] has recently studied the character of the solution $u = u(x, y)$ of the scalar Riccati differential equation and initial condition

$$(1.1) \quad u_y = a(y) + 2b(y)u + c(y)u^2, \quad u(x, x) = 0,$$

as a function of the two variables (x, y) . If $v = v(x, y)$ and $w = w(x, y)$ are defined in terms of $u(x, y)$ by the equations

$$(1.2) \quad v_y = b(y) + c(y)u(x, y), \quad v(x, x) = 0,$$

$$(1.3) \quad w_y = c(y) \exp \{2v(x, y)\}, \quad w(x, x) = 0,$$

then he showed that, as functions of x , the triple u, v, w satisfies a system of differential equations similar to the above, and also certain non-linear functional equations. Moreover, REDHEFFER proved that if these functional equations are satisfied by a triple of functions u, v, w which are continuous, are such that $u(x, x) = v(x, x) = w(x, x) = 0$, and the partial derivatives u_y, v_y, w_y exist along the line $y = x$, then there exist functions $a(y), b(y), c(y)$ such that u, v, w are solutions of (1.1), (1.2), (1.3).

It is the purpose of the present paper to extend the results of REDHEFFER to matrix differential equations of Riccati type. The method of proof used here is based on the properties of solutions of a related system of linear differential equations. It is felt that this method of attack is simpler than that employed by REDHEFFER for the scalar problem. Moreover, for the case considered by him the present method shows that the $v(x, y)$ in Theorem I of REDHEFFER [1] is continuous in (x, y) , so that the assumption of continuity stated there is extraneous; also, the continuity assumption of Theorem IV of [1] is shown to be unnecessary.

Matrix notation is used throughout the present paper; in particular, matrices

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of one column are termed vectors, and for a vector (p_α) ($\alpha = 1, \dots, n$) the norm $|p|$ is given by $(|p_1|^2 + \dots + |p_n|^2)^{\frac{1}{2}}$. The symbols E and 0 are used for the identity and zero matrices, respectively, independent of the dimensions of these matrices. For a given matrix M its transpose is denoted by $\tilde{M}$, its conjugate by $\bar{M}$, and its conjugate transpose by M^* . If M is an $n \times n$ matrix the symbol $|M|$ is used for the supremum of $|Mp|$ on the unit sphere $|p| = 1$. Finally, if all elements of a matrix $M(x, y)$ possess a property of continuity or differentiability, for brevity we shall say that $M(x, y)$ possesses this property.

2. Associated differential systems. In the following $\mathfrak{R}$ will denote an open connected region in the (x, y) -plane which is such that if $(x_0, y_0) \in \mathfrak{R}$ then the closed line segment with end-points (x_0, y_0) , (x_0, x_0) belongs to $\mathfrak{R}$; although this type of region is not necessarily of the kind considered by REDHEFFER [1], it is the type of region appropriate for the present treatment. Corresponding to the system (1.1), (1.2), (1.3) of the introduction, we shall be concerned with complex-valued $n \times n$ matrices $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ defined in $\mathfrak{R}$ and possessing throughout this region the following properties:

(H₁) the partial derivatives $U_y(x, y)$, $R_y(x, y)$, $S_y(x, y)$, $W_y(x, y)$ exist (and are finite);

(H₂) $R(x, y)$ and $S(x, y)$ are non-singular;

(H₃) there exist complex-valued $n \times n$ matrices $A(y)$, $B(y)$, $C(y)$, $D(y)$ such that U , R , S , W satisfy throughout $\mathfrak{R}$ the differential system

$$\begin{aligned}
 (2.1) \quad & \text{(a)} \quad U_y = A(y) + D(y)U + UB(y) + UC(y)U, & U(x, x) &= 0, \\
 & \text{(b)} \quad R_y = R[B(y) + C(y)U(x, y)], & R(x, x) &= E, \\
 & \text{(c)} \quad S_y = [D(y) + U(x, y)C(y)]S, & S(x, x) &= E, \\
 & \text{(d)} \quad W_y = R(x, y)C(y)S(x, y), & W(x, x) &= 0.
 \end{aligned}$$

It is to be remarked that if $D(y) \equiv \tilde{B}(y)$ and the matrices $A(y)$, $C(y)$, $U(x, y)$ are symmetric, then equation (2.1c) may be deleted from the above system, since for $R(x, y)$ a solution of (2.1b) it follows that $S(x, y) = \tilde{R}(x, y)$ is a solution of (2.1c). In particular, this simplification occurs for the scalar system considered by REDHEFFER, in which case $S(x, y) = R(x, y) = \exp \{v(x, y)\}$.

In terms of the solutions U , R , S , W of (2.1) the $n \times n$ matrices $P_i(x, y)$, $Q_i(x, y)$ ($i = 1, 2$) are defined as follows:

$$\begin{aligned}
 (2.2) \quad & \text{(a)} \quad P_1(x, y) = R^{-1}(x, y), \\
 & \text{(b)} \quad Q_1(x, y) = U(x, y)P_1(x, y), \\
 & \text{(c)} \quad P_2(x, y) = -P_1(x, y)W(x, y), \\
 & \text{(d)} \quad Q_2(x, y) = -Q_1(x, y)W(x, y) + S(x, y) = U(x, y)P_2(x, y) + S(x, y).
 \end{aligned}$$

For brevity, let $H(x, y)$ and $G(y)$ denote the $2n \times 2n$ matrices

$$(2.3) \quad H(x, y) = \begin{vmatrix} P_1(x, y) & P_2(x, y) \\ Q_1(x, y) & Q_2(x, y) \end{vmatrix},$$

$$(2.4) \quad G(y) = \begin{vmatrix} -B(y) & -C(y) \\ A(y) & D(y) \end{vmatrix}.$$

With the aid of the differential equations (2.1) one may verify directly the following result, which is basic for the following discussion. It is to be commented that the introduction of the matrices $P_i(x, y)$, $Q_i(x, y)$ is motivated by known results on Riccati matrix differential equations that are associated with variational problems (see REID [3]).

Lemma 2.1. *The matrix $H(x, y)$ is a solution of the linear matrix differential system*

$$(2.5) \quad H_v(x, y) = G(y)H(x, y), \quad H(x, x) = E;$$

moreover, the inverse $H^{-1} = H^{-1}(x, y)$ is given by

$$(2.6) \quad H^{-1} = \begin{vmatrix} -WS^{-1}U + R & WS^{-1} \\ -S^{-1}U & S^{-1} \end{vmatrix}.$$

In the spirit of REDHEFFER [1], for the above discussion there has been no assumption as to the integrability of the coefficient matrices $A(y)$, $B(y)$, $C(y)$, $D(y)$ of (2.1); rather, it has been assumed that these coefficient matrices are finite-valued, and throughout $\mathcal{R}$ the matrices U , R , S , W have finite partial derivatives with respect to y which satisfy the equations (2.1). As the differentiability property (H_1) implies that throughout $\mathcal{R}$ the matrix $H^{-1}(x, y)$ has a finite partial derivative with respect to y , from the relation $H(H^{-1})_v + H_v H^{-1} \equiv 0$ it follows that throughout $\mathcal{R}$ the equation

$$(2.7) \quad [H^{-1}(x, y)]_v = -H^{-1}(x, y)G(y)$$

holds. Moreover, if $H_0 = H_0(x, y)$ is any solution of $H_{0v} = G(y)H_0$ on $\mathcal{R}$ then $(H^{-1}H_0)_v \equiv 0$ on $\mathcal{R}$. In particular, if $H_{0v} = G(y)H_0$ and $H_0(x, x) = E$, then $H^{-1}H_0 \equiv E$, so that $H_0(x, y)$ is the matrix (2.3) with elements expressible in terms of any solution U , R , S , W of (2.1) by the relations (2.2). Consequently, the solution U , R , S , W of the differential system (2.1) is unique. In particular, if $A(y)$, $C(y)$ are symmetric matrices, $D(y) \equiv \tilde{B}(y)$, and $U = U_0(x, y)$, $R = R_0(x, y)$, $S = S_0(x, y)$, $W = W_0(x, y)$ is a solution of (2.1), then $U = \tilde{U}_0(x, y)$, $R = \tilde{S}_0(x, y)$, $S = \tilde{R}_0(x, y)$, $W = \tilde{W}_0(x, y)$ is also a solution of this system, so that $U_0(x, y) \equiv \tilde{U}_0(x, y)$, $R_0(x, y) \equiv \tilde{S}_0(x, y)$.

Theorem 2.1. *If $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ satisfy hypotheses (H_1) , (H_2) , (H_3) on $\mathcal{R}$, then throughout this region these matrices are continuous in (x, y) and have (finite) partial derivatives with respect to x which satisfy the differ-*

ential equations

$$\begin{aligned}
 (2.8) \quad & (a) \quad W_x = -C(x) - B(x)W - WD(x) - WA(x)W, \\
 & (b) \quad S_x = -S[D(x) + A(x)W(x, y)], \\
 & (c) \quad R_x = -[B(x) + W(x, y)A(x)]R, \\
 & (d) \quad U_x = -S(x, y)A(x)R(x, y).
 \end{aligned}$$

For a given point $(x_0, y_0) \in \mathcal{R}$, let y_1, y_2 be values such that the line segment $x = x_0, y_1 < y < y_2$ is in $\mathcal{R}$ and contains the closed segment with end-points $(x_0, x_0), (x_0, y_0)$, and choose a $\delta > 0$ such that $\delta < \min(y_2 - x_0, x_0 - y_1)$ and the oriented rectangular region $\mathcal{R}_0$ with opposite vertices $(x_0 - \delta, y_1), (x_0 + \delta, y_2)$ lies in $\mathcal{R}$. The $\mathcal{R}_0$ thus defined is a subregion of $\mathcal{R}$ which satisfies the properties specified for $\mathcal{R}$ at the beginning of this section. Moreover, on $\mathcal{R}_0$ the matrix $H_0(x, y) = H(x_0, y)H^{-1}(x_0, x)$ is a solution of

$$H_{0y} = G(y)H_0, \quad H_0(x, x) = E,$$

and consequently by the results established above the relation $H_0(x, y) = H(x, y)$ holds on $\mathcal{R}_0$; that is,

$$(2.9) \quad H(x, y) = H(x_0, y)H^{-1}(x_0, x) \quad \text{for} \quad |x - x_0| < \delta, \quad y_1 < y < y_2.$$

As $H(x_0, y)$ has a finite partial derivative with respect to y on $y_1 < y < y_2$ and $H^{-1}(x_0, x)$ has a finite partial derivative with respect to x on $|x - x_0| < \delta$, it follows from (2.9) that on $\mathcal{R}_0$ the matrix $H(x, y)$ is continuous in (x, y) and has a finite partial derivative with respect to x ; indeed, since

$$[H^{-1}(x_0, x)]_x = -H^{-1}(x_0, x)[H(x_0, x)]_x H^{-1}(x_0, x) = -H^{-1}(x_0, x)G(x)$$

on $|x - x_0| < \delta$, it follows that

$$(2.10) \quad H_x(x, y) = -H(x, y)G(x)$$

holds on $\mathcal{R}_0$. In view of the arbitrariness of the point (x_0, y_0) in $\mathcal{R}$, it follows that $H(x, y)$ is continuous in (x, y) on $\mathcal{R}$, and on this region has a finite partial derivative with respect to x satisfying (2.10). In terms of the component matrices $P_i(x, y), Q_i(x, y)$ the equation (2.10) becomes

$$\begin{aligned}
 (2.11) \quad & P_{1x} = P_1B(x) - P_2A(x), \quad P_{2x} = P_1C(x) - P_2D(x), \\
 & Q_{1x} = Q_1B(x) - Q_2A(x), \quad Q_{2x} = Q_1C(x) - Q_2D(x).
 \end{aligned}$$

As (2.2) is equivalent to

$$\begin{aligned}
 (2.12) \quad & (a) \quad R(x, y) = P_1^{-1}(x, y), \\
 & (b) \quad U(x, y) = Q_1(x, y)P_1^{-1}(x, y), \\
 & (c) \quad W(x, y) = -P_1^{-1}(x, y)P_2(x, y), \\
 & (d) \quad S(x, y) = Q_2(x, y) - Q_1(x, y)P_1^{-1}(x, y)P_2(x, y),
 \end{aligned}$$

the functions U , R , S , W clearly possess the continuity and differentiability properties specified in Theorem 2.1, and the differential equations of (2.8) may be derived directly from (2.12), (2.11).

3. Functional equations. Corresponding to Theorem II of REDHEFFER [1], the following result holds for the matrix differential system (2.1).

Theorem 3.1. *If $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ satisfy hypotheses (H_1) , (H_2) , (H_3) on $\mathfrak{R}$, then for arbitrary points (x, y) , (y, z) , (x, z) of this region*

$$(3.1) \quad \begin{aligned} (a) \quad & U(x, z) - U(y, z) = S(y, z)U(x, y)R^{-1}(x, y)R(x, z), \\ (b) \quad & E - W(y, z)U(x, y) = R(y, z)R^{-1}(x, z)R(x, y), \\ (c) \quad & E + U(x, y)R^{-1}(x, y)R(x, z)R^{-1}(y, z)W(y, z) = S^{-1}(y, z)S(x, z)S^{-1}(x, y). \\ (d) \quad & W(x, z) - W(x, y) = R(x, z)R^{-1}(y, z)W(y, z)S(x, y). \end{aligned}$$

If (x, y) , (y, z) , and (x, z) are points of $\mathfrak{R}$ then the vertical segments with end-points (x, y) , (x, z) and (y, y) , (y, z) belong to this region. In order to establish (3.1a), consider the matrix function

$$F_1(x, y, t) = S^{-1}(y, t)[U(x, t) - U(y, t)]R^{-1}(x, t),$$

for t on the closed interval of the real line with end-points y and z . Then $F_1(x, y, y) = U(x, y)R^{-1}(x, y)$, while use of equations (a), (b), (c) of (2.1) to compute the partial derivative of $F_1(x, y, t)$ with respect to t leads to the result that $F_{1t}(x, y, t) \equiv 0$. Consequently

$$\begin{aligned} S^{-1}(y, z)[U(x, z) - U(y, z)]R^{-1}(x, z) \\ = F_1(x, y, z) = F_1(x, y, y) = U(x, y)R^{-1}(x, y), \end{aligned}$$

which is clearly equivalent to (3.1a).

In a similar fashion (3.1b) is obtained by the consideration of the function

$$F_2(x, y, t) = E - W(y, t)U(x, y) - R(y, t)R^{-1}(x, t)R(x, y).$$

Use of (b), (d) of (2.1) to compute $F_{2t}(x, y, t)$ shows that

$$F_{2t}(x, y, t) = R(y, t)C(t)\{[U(x, t) - U(y, t)]R^{-1}(x, t)R(x, y) - S(y, t)U(x, y)\},$$

and in view of (3.1a) we have $F_{2t}(x, y, t) \equiv 0$. Therefore we have the relation $F_2(x, y, z) = F_2(x, y, y) = 0$, so that (3.1b) holds.

Correspondingly, if

$$F_3(x, y, t) = S^{-1}(x, t)S(y, t)[E + U(x, y)R^{-1}(x, y)R(x, t)R^{-1}(y, t)W(y, t)],$$

with the aid of (b), (c), (d) of (2.1) and a subsequent use of (3.1a) it follows that $F_{3t}(x, y, t) \equiv 0$, and hence $F_3(x, y, z) = F_3(x, y, y) = S^{-1}(x, y)$, which is equivalent to (3.1c).

Finally, if

$$F_4(x, y, t) = W(x, t) - W(x, y) - R(x, t)R^{-1}(y, t)W(y, t)S(x, y),$$

use of (b), (d) of (2.1) and (3.1a) leads to the relation

$$F_{4t}(x, y, t) = R(x, t)C(t)\{S(x, t) - S(y, t)[E + U(x, y)R^{-1}(x, y)R(x, t)R^{-1}(y, t)W(y, t)]S(x, y)\}.$$

From (3.1c) it then follows that $F_{4t}(x, y, t) \equiv 0$ and $F_4(x, y, z) = F_4(x, y, y) = 0$, thus establishing (3.1d).

Corresponding to Theorem IV of REDHEFFER [1], we have the following theorem.

Theorem 3.2. *Suppose that $\mathcal{R}$ is a region of the type specified at the beginning of Section 2, and that on $\mathcal{R}$ the $n \times n$ matrices $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ are such that the two matrices $R(x, y)$, $S(x, y)$ are non-singular; moreover, the functional equations (3.1) hold for all points (x, y) , (y, z) , (x, z) of this region,*

$$(3.2) \quad U(x, x) \equiv 0 \equiv W(x, x), \quad R(x, x) \equiv E \equiv S(x, x),$$

and the partial derivatives U_y , R_y , S_y , W_y exist and are finite at points of the line $y = x$ in $\mathcal{R}$. Then throughout $\mathcal{R}$ the functions U , R , S , W have finite partial derivatives with respect to x and y , and there exist matrices $A(y)$, $B(y)$, $C(y)$, $D(y)$ such that the differential systems (2.1) and (2.8) are valid in $\mathcal{R}$.

The above theorem is devoid of any specific continuity hypothesis as in Theorem IV of REDHEFFER [1]. However, the existence of the finite partial derivatives U_y , R_y , S_y , W_y at points of the line $y = x$ in $\mathcal{R}$ implies the continuity of $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ as functions of y at such points, and relations (3.1) require that throughout $\mathcal{R}$ these matrices are continuous in y for fixed x . Indeed, since $W(y, z) \rightarrow W(y, y) = 0$ and $R(y, z) \rightarrow R(y, y) = E$ as $z \rightarrow y$, it follows from (3.1b) that $R(x, z) \rightarrow R(x, y)$ as $z \rightarrow y$, for arbitrary $(x, y) \in \mathcal{R}$. Using this continuity property of $R(x, y)$ and similar arguments, the stated continuity property of U , S and W follows from the respective equations (3.1a), (3.1c) and (3.1d).

It will be shown that the result of this theorem holds with

$$(3.3) \quad A(x) = U_x(x, x), \quad B(x) = R_x(x, x), \quad C(x) = W_x(x, x), \quad D(x) = S_x(x, x).$$

If $F(x, y)$ is a matrix defined on $\mathcal{R}$ the symbol $F_\Delta(x, y, z)$ will be employed for the difference quotient $[F(x, z) - F(x, y)]/(z - y)$, it being understood that (x, y) , (x, z) are points of $\mathcal{R}$ with $z \neq y$. Also, for brevity we write $T(x, y, z)$ for the matrix

$$R^{-1}(x, y)R(x, z)R^{-1}(y, z) = [E - W(y, z)U(x, y)]^{-1}.$$

The equation (3.1a) may be written as

$$U(x, z) = U(y, z) + S(y, z)U(x, y)T(x, y, z)R(y, z).$$

In view of the initial values (3.2) and the identity $T(x, y, z) - E = W(y, z)U(x, y)T(x, y, z)$, algebraic manipulation yields the relation

$$U_{\Delta}(x, y, z) = U_{\Delta}(y, y, z) + S_{\Delta}(y, y, z)U(x, y)T(x, y, z) \\ + S(y, z)U(x, y)T(x, y, z)R_{\Delta}(y, y, z) + U(x, y)W_{\Delta}(y, y, z)U(x, y)T(x, y, z),$$

and when $z \rightarrow y$ equation (2.1a) is obtained.

As (3.1b) is equivalent to

$$R(x, y) = R(x, z)R^{-1}(y, z)[E - W(y, z)U(x, y)],$$

one is lead readily to the relation

$$R_{\Delta}(x, y, z) = R(x, z)R^{-1}(y, z)[R_{\Delta}(y, y, z) + W_{\Delta}(y, y, z)U(x, y)],$$

and this latter equation for $z \rightarrow y$ yields (2.1b). By a similar argument, (2.1c) results from writing (3.1c) as

$$S(x, z) = S(y, z)[E + U(x, y)T(x, y, z)W(y, z)]S(x, y),$$

and, noting that, consequently

$$S_{\Delta}(x, y, z) = [S_{\Delta}(y, y, z) + S(y, z)U(x, y)T(x, y, z)W_{\Delta}(y, y, z)]S(x, y).$$

In turn, (2.1d) follows from the relation

$$W_{\Delta}(x, y, z) = R(x, z)R^{-1}(y, z)W_{\Delta}(y, y, z)S(x, y),$$

which is an immediate consequence of (3.1d). Finally, it is to be remarked that in view of Theorem 2.1 the functions U, R, S, W have finite partial derivatives with respect to x and the differential system (2.8) holds throughout $\mathfrak{R}$.

4. General solutions of (2.1a). If K is a given $n \times n$ constant matrix and

$$P(x, y; K) = P_1(x, y) + P_2(x, y)K, \quad Q(x, y; K) = Q_1(x, y) + Q_2(x, y)K,$$

then $P = P(x, y; K)$, $Q = Q(x, y; K)$ is a solution of the differential system

$$(4.1) \quad P_v = -B(y)P - C(y)Q, \quad Q_v = A(y)P + D(y)Q,$$

satisfying

$$P(x, x; K) = E, \quad Q(x, x; K) = K.$$

The corresponding matrix $U(x, y; K) = Q(x, y; K)P^{-1}(x, y; K)$ is then a solution of (2.1a) on any subinterval of I throughout which $P(x, y; K)$ is non-singular. Moreover, in view of (2.2c), the non-singularity of $P(x, y; K)$ is equivalent to the non-singularity of the matrix $E - W(x, y)K$, and from (2.2c), (2.2d) it follows that

$$U(x, y; K) = U(x, y) + S(x, y)K[E - W(x, y)K]^{-1}R(x, y)$$

on an interval containing $y = x$ throughout which $P(x, y; K)$ is non-singular.

For simplicity, it will now be assumed that $\mathfrak{R} = I \times I$, where I is an open interval of the real line, and that $A(y), B(y), C(y), D(y)$ are piecewise continuous on I .

In place of (H_1) , (H_2) , (H_3) we now make the following assumptions:

(H'_1) for $x \in I$ the $n \times n$ matrices $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ are absolutely continuous in y on I ;

(H'_3) $U(x, x) \equiv 0 \equiv W(x, x)$, $R(x, x) \equiv E \equiv S(x, x)$ for $x \in I$, and the differential equations (2.1) hold at each $y \in I$ that is a point of continuity of $A(y)$, $B(y)$, $C(y)$, $D(y)$.

Clearly the significant condition of (H'_1) , (H'_3) is that for $x \in I$ the matrix $U(x, y)$ is absolutely continuous in y , with $U(x, x) \equiv 0$ and (2.1a) holding almost everywhere on I , since under this condition the matrix $U(x, y)$ is unique, and the corresponding matrices R , S , W are determined uniquely by the initial value problems (2.1b), (2.1c), (2.1d). In particular, from well-known existence theorems it follows that $R(x, y)$ and $S(x, y)$ are non-singular on $\mathcal{R}$, that is, hypothesis (H_2) is satisfied by these matrices. Moreover, for $x \in I$ the matrices $P_i(x, y)$, $Q_i(x, y)$ ($i = 1, 2$), $P(x, y; K)$, $Q(x, y; K)$, $U(x, y)$, $U(x, y; K)$ are absolutely continuous in y on I , and are solutions of the above encountered differential equations in the sense that these equations hold at each point of continuity of the matrices $A(y)$, $B(y)$, $C(y)$, $D(y)$. Corresponding to part of the results of Theorem IV of REDHEFFER [2], we shall determine the condition under which the solutions $U(x, y; K)$ of (2.1a) possess the following property:

$\mathcal{P}$: if $x \in I$ and $|K| < 1$, then $|U(x, y; K)| < 1$ on $y \geq x$, $y \in I$.

Theorem 4.1. Suppose that $\mathcal{R} = I \times I$, the matrices $A(y)$, $B(y)$, $C(y)$, $D(y)$ are piecewise continuous on I , and $U(x, y)$, $R(x, y)$, $S(x, y)$, $W(x, y)$ satisfy hypotheses (H'_1) , (H'_3) on $\mathcal{R}$. Then a necessary condition for solutions $U(x, y; K)$ of (2.1a) to possess property $\mathcal{P}$ is that for y a point of continuity of $A(y)$, $B(y)$, $C(y)$, $D(y)$ the hermitian form

$$(4.2) \quad \begin{aligned} \Omega(y; p, q) \equiv & q^*[D(y) + D^*(y)]q + q^*[A(y) + C^*(y)]p \\ & + p^*[A^*(y) + C(y)]q + p^*[B(y) + B^*(y)]p \end{aligned}$$

be non-positive in the class of n -dimensional vectors $p \equiv (p_\alpha)$, $q \equiv (q_\alpha)$ ($\alpha = 1, \dots, n$) satisfying $|p|^2 - |q|^2 = 0$. A sufficient condition for property $\mathcal{P}$ to hold is that for y a point of continuity of $A(y)$, $B(y)$, $C(y)$, $D(y)$ the hermitian form $\Omega(y; p, q)$ be non-positive for all p, q .

Suppose that there exists a point $x_0 \in I$ which is a point of continuity of the matrices A , B , C , D and vectors p_0, q_0 such that $|p_0|^2 - |q_0|^2 = 0$ and $\Omega(x_0; p_0, q_0) > 0$. The vectors p_0, q_0 may be chosen so that $|p_0| = 1 = |q_0|$ and there exist $n \times n$ unitary matrices $P_0 \equiv ||P_{0\alpha\beta}||$, $Q_0 \equiv ||Q_{0\alpha\beta}||$ with $P_{0\alpha 1} = p_{0\alpha}$, $Q_{0\alpha 1} = q_{0\alpha}$ ($\alpha = 1, \dots, n$). Let K_0 denote the unitary matrix $Q_0 P_0^{-1} = Q_0 P_0^*$ and $\xi = P_0 e^{(1)}$, where $e^{(1)} = (e_\alpha^{(1)})$ with $e_1^{(1)} = 1$, $e_\alpha^{(1)} = 0$ ($\alpha = 2, \dots, n$). Then $p(y) = P(x_0, y; K_0)\xi$, $q(y) = Q(x_0, y; K_0)\xi$ are such that $p(x_0) = p_0$, $q(x_0) = q_0$, and with the aid of equations (4.1) it follows readily that $\phi(y) =$

$|q(y)|^2 - |p(y)|^2$ satisfies the conditions

$$(4.3) \quad \phi'(y) = \Omega(y; p(y), q(y)), \quad \phi(x_0) = 0.$$

Consequently, $\phi(x_0) = 0$, $\phi'(x_0) > 0$, and for $y_1 > x_0$ and sufficiently close to x_0 we have

$$0 < \phi(y_1) = \xi^*[Q^*(x_0, y_1; K_0)Q(x_0, y_1; K_0) - P^*(x_0, y_1; K_0)P(x_0, y_1; K_0)]\xi,$$

so that $|U(x_0, y_1; K_0)| > 1$ for $y_1 > x_0$ and sufficiently close to x_0 . By continuity it then follows that if $0 < r < 1$ and r is sufficiently close to one, then $|U(x_0, y_1; rK_0)| > 1$ and $|U(x_0, x_0; rK_0)| = r|K_0| = r < 1$, so that property $\mathcal{O}$ does not hold. Consequently, for property $\mathcal{O}$ to hold it is necessary that at each $y \in I$ which is a point of continuity of $A(y)$, $B(y)$, $C(y)$, $D(y)$ the hermitian form (4.2) is non-positive in the class of vectors p, q satisfying $|p|^2 - |q|^2 = 0$.

On the other hand, suppose $\Omega(y; p, q) \leq 0$ for y a point of continuity of $A(y)$, $B(y)$, $C(y)$, $D(y)$ and arbitrary n -dimensional vectors p, q . If K is a constant $n \times n$ matrix with $|K| < 1$ and

$$V(x, y; K) = Q^*(x, y; K)Q(x, y; K) - P^*(x, y; K)P(x, y; K),$$

then for an arbitrary non-null constant n -dimensional vector ξ we have at each y which is a point of continuity of A, B, C, D the relation

$$[\xi^*V(x, y; K)\xi]_v = \Omega(y; P(x, y; K)\xi, Q(x, y; K)\xi) \leq 0,$$

and since $\xi^*V(x, x; K)\xi = |K\xi|^2 - |\xi|^2 < 0$, it follows that $\xi^*V(x, y; K)\xi < 0$ for $x, y \in I$, $y \geq x$. As $|P(x, y; K)\xi|^2 + |Q(x, y; K)\xi|^2 > 0$ for arbitrary x, y, K , this latter relation implies that $P(x, y; K)$ is non-singular for $y \geq x$ and $U(x, y; K) = Q(x, y; K)P^{-1}(x, y; K)$ is finite valued for $x, y \in I$, $y \geq x$. Moreover, $\xi^*V(x, y; K)\xi < 0$ is readily seen to be equivalent to $|U(x, y; K)| < 1$, thus establishing the sufficiency statement of the theorem.

For the general matrix differential equation (2.1a) the author does not know a condition that is both necessary and sufficient for solutions $U(x, y; K)$ to possess property $\mathcal{O}$. One does have the following theorem, however, which generalizes the result of Theorem IV of REDHEFFER [2].

Theorem 4.2. *Suppose that the hypotheses of Theorem 4.1 hold, and in addition $D(y) \equiv \tilde{B}(y)$ and $A(y), C(y)$ are symmetric on I . Then a necessary and sufficient condition for solutions $U(x, y; K)$ of (2.1a) to possess property $\mathcal{O}$ is that for y a point of continuity of $A(y), B(y), C(y), D(y)$ the hermitian form $\Omega(y; p, q)$ is non-positive for all n -dimensional vectors p, q .*

Under the hypotheses of Theorem 4.2 we have

$$\begin{aligned} \Omega(y; p, q) &= q^*[\tilde{B}(y) + \bar{B}(y)]q + q^*[A(y) + \bar{C}(y)]p \\ &\quad + p^*[\bar{A}(y) + C(y)]q + p^*[B(y) + B^*(y)]p, \end{aligned}$$

and in view of Theorem 4.1 it follows that in order to establish the result of

the theorem it suffices to show that if $\Omega(y; p, q)$ is non-positive in the class of vectors p, q satisfying $|p|^2 - |q|^2 = 0$ then $\Omega(y; p, q)$ is non-positive for all p, q . Now if there exists a pair of vectors p, q such that $\Omega(y; p, q) > 0$, then the greatest proper value λ of the hermitian form $\Omega(y; p, q)$ is positive. In general, if λ is a proper value of this form and p_0, q_0 is a corresponding proper vector, then $p = p_0, q = q_0$ satisfies

$$(4.4) \quad \begin{aligned} [\tilde{B}(y) + \bar{B}(y)]q + [A(y) + \bar{C}(y)]p &= \lambda q, \\ [\bar{A}(y) + C(y)]q + [B(y) + B^*(y)]p &= \lambda p, \end{aligned}$$

and $\Omega(y; p, q) = \lambda[|p|^2 + |q|^2]$. Now if $p = p_0, q = q_0$ is a non-trivial solution of (4.4), in view of the reality of λ the pair $p = \bar{q}_0, q = \bar{p}_0$ is a solution of (4.4) for the same λ , and so also are the pairs $p = p_1 = p_0 + \bar{q}_0, q = q_1 = q_0 + \bar{p}_0$ and $p = p_2 = p_0 - \bar{q}_0, q = q_2 = q_0 - \bar{p}_0$. As not both the pairs p_1, q_1 and p_2, q_2 can be the pair 0, 0 and since $\bar{p}_1 = q_1, \bar{p}_2 = -q_2$, it follows that for any proper value λ of $\Omega(y; p, q)$ there is a corresponding proper vector p, q with $|p|^2 = |q|^2$. Consequently, if $\Omega(y; p, q)$ is non-positive in the class of vectors p, q satisfying $|p|^2 - |q|^2 = 0$, then $\Omega(y; p, q)$ is non-positive for all p, q .

If the matrices $A(y), B(y), C(y), D(y)$ are merely assumed to be Lebesgue integrable on I , then by suitable modification of the above argument one may establish the results of Theorems 4.1 and 4.2 with the phrase "for y a point of continuity of $A(y), B(y), C(y), D(y)$ " replaced by "for y almost everywhere on I ".

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