The Solution and Huygens' Principle for a Singular Cauchy Problem

DAVID W. FOX

Communicated by A. ZYGMUND

1. Introduction. The subject of this paper is a study of a singular Cauchy problem for a hyperbolic partial differential equation in $m + 1$ variables. The classical problem in this domain is that for the Euler-Poisson-Darboux (EPD) equation, which in its full generality has been investigated extensively since the appearance of WEINSTEIN's paper in 1952. Our equation [see (1.1)], restricted to two variables, goes back also to EULER and DARBOUX. In 1955, without connection with the earlier studies, STELLMACHER, investigating Huygens' principle, found for (1.1) the fundamental solution and thus, in principle, solved the *regular* Cauchy problem.

The equation which we consider is

$$(1.1) \quad \frac{\partial^2 u}{\partial t^2} + \frac{k}{t} \frac{\partial u}{\partial t} - \sum_{i=1}^m \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\lambda_i}{x_i^2} u \right) \equiv L(u) = 0$$

where k and λ_i are real or sometimes complex parameters. We exclude the so-called exceptional values, $k = -1, -3, -5, \dots$ [1], [14]. The *singular* Cauchy problem corresponds to the initial data

$$(1.2) \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0$$

in which, for convenience, we write x in place of $x_1, x_2, \dots, x_m$ and use subscripts to denote differentiation. The differentiability assumptions on f will be formulated later. The differential equation in the form considered by STELLMACHER is obtained from (1.1) by putting $u = vt^{-\frac{1}{2}k}$. For $\lambda_i = 0$ for every i (1.1) reduces to the EPD equation. Assuming that for each i $|x_i| > |t| \geq 0$, the right hand sum is a regular analytic elliptic differential operator $X(u)$, and therefore (1.1) belongs to the class of singular hyperbolic equations considered by BUREAU in 1955. For such equations BUREAU used new devices to develop a solution for the singular Cauchy problem. His result generalizes WEINSTEIN's for the EPD equation, but it is valid only in the small and has the character of an existence theorem rather than that of an explicit solution. Some points, such as differentiability conditions, uniqueness, and Huygens' principle, are not examined.

In the present paper we give an explicit solution in the large of the singular Cauchy problem (1.1), (1.2) in the domain $|x_i| > |t| \geq 0$. We determine the exact range of the parameter k for which the solution is given by an integral operator. For all other non-exceptional values of k we obtain the solution by means of a recursive procedure which gives the analytic continuation in k of the integral. The question of uniqueness is treated extensively by means of techniques of ZAREMBA and FRIEDRICH & LEWY, the generality achieved being the same as that obtained in 1957 by WALTER for the EPD equation. We show that the equation (1.1) admits several useful transformations associated with WEINSTEIN's correspondence principles. A main part of our investigation is devoted to a formulation of a necessary and sufficient criterion for Huygens' principle and the comparison of it with STELLMACHER's criterion for the regular Cauchy problem and with the necessary and sufficient condition for the EPD equation. Some results of this paper were summarized in a note in the *Comptes Rendus*, Paris, 13 January 1958.

2. The basic solution. Our purpose in this section is to obtain for equation (1.1) a solution which can be used to carry out the integration of our Cauchy problem (1.1), (1.2). This solution will be called the basic solution. To do this we consider first the equation (1.1) in which k has been replaced by $2 - k$; this equation we shall call (1.1)' for convenience. For it we will find a particular solution and then, by a correspondence principle of A. WEINSTEIN [1, 2], obtain the desired solution of (1.1).

In the following we shall denote a point in the $(m + 1)$ -dimensional space with coordinates $(x_1, x_2, \dots, x_m, t)$ by x, t and a point in the singular hyperplane $t = 0$ by x alone.

Let Γ be the square of the Lorentz distance between a point x, t and a point ξ in the singular hyperplane. Γ is given explicitly by the formula

$$(2.1) \quad \Gamma(x, t; \xi) = t^2 - \sum_{i=1}^m (x_i - \xi_i)^2.$$

We now seek a solution of (1.1)' which depends on the coordinates x, t and parametrically on ξ . To facilitate this we introduce, between the $2m + 1$ coordinates x, t, ξ and the new quantities $z_1, z_2, \dots, z_m; \Gamma; y_1, y_2, \dots, y_m$, a transformation defined by equation (2.1) and the equations

$$(2.2) \quad z_i = \frac{\Gamma}{4x_i\xi_i}, \quad y_i = \frac{\xi_i}{x_i} \Gamma \quad (i = 1, 2, \dots, m).$$

This transformation clearly is the composition of two transformations, the first defined by (2.1) between x, t, ξ and x, Γ, ξ and the second defined by (2.2) between x, Γ, ξ and z, Γ, y . Elementary computations show that the first is locally non-singular as long as $t \neq 0$ and the second as long as $\Gamma \neq 0, x_i \neq 0, \xi_i \neq 0$. For z_i and y_i given by equations (2.2) it is true that for x, ξ , and t real (as will be assumed) pairwise z_i and y_i always have the same sign. The inverse transforma-

tion is given by the equations

$$(2.3) \quad x_i^2 = \frac{\Gamma^2}{4y_i z_i}, \quad \xi_i^2 = \frac{y_i}{4z_i} \quad (i = 1, 2, \dots, m)$$

and

$$(2.4) \quad t^2 = \Gamma + \sum_{i=1}^m (x_i - \xi_i)^2.$$

We consider as a possible solution of equation (1.1)' a function of the form

$$(2.5) \quad v_\lambda^{2-k} = \Gamma^{\frac{1}{2}(k-m-1)} V(z)$$

where V is to be determined. Here, as in the following, we will indicate the particular equation of type (1.1) which a solution satisfies by a superscript k for the equation's first index and a subscript λ ($= \lambda_1, \lambda_2, \dots, \lambda_m$) for the remaining indices. The insertion of the expression (2.5) into the $2 - k$ equation yields after some computation

$$(2.6) \quad -4\Gamma^{-p-2} \sum_{i=1}^m y_i z_i \Lambda_i(V) = 0,$$

in which we have put $p = \frac{1}{2}(m - k + 1)$ and $\Lambda_i(V)$ is given for each i by

$$(2.7) \quad \Lambda_i(V) = z_i^2 \frac{\partial^2 V}{\partial z_i^2} + 2z_i \frac{\partial V}{\partial z_i} - \sum_{j=1}^m z_j \frac{\partial^2 V}{\partial z_i \partial z_j} + (p - 1) \frac{\partial V}{\partial z_i} + \lambda_i V.$$

Clearly, v_λ^{2-k} will be a solution of (1.1)' if V satisfies the system of equations

$$(2.8) \quad \Lambda_i(V) = 0 \quad (i = 1, 2, \dots, m).$$

These, however, are satisfied by the function F_B of LAURICELLA [3, Ch. VII] which is defined for $|z_i| < 1$ ($i = 1, 2, \dots, m$) by

$$(2.9) \quad \begin{aligned} & F_B(\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_m; \gamma; z_1, z_2, \dots, z_m) \\ &= \sum_{\rho_1, \rho_2, \dots, \rho_m=0}^{\infty} \frac{1}{(\gamma, \rho_1 + \rho_2 + \dots + \rho_m)} \prod_{i=1}^m \frac{(\alpha_i, \rho_i)(\beta_i, \rho_i)}{\rho_i!} z_i^{\rho_i}, \end{aligned}$$

in which $(\alpha, \rho) = \Gamma(\alpha + \rho)/\Gamma(\alpha)$ and the indices α_i, β_i and γ have the special values

$$(2.10) \quad \alpha_i = \frac{1 + \sqrt{1 - 4\lambda_i}}{2}, \quad \beta_i = \frac{1 - \sqrt{1 - 4\lambda_i}}{2},$$

$$\gamma = 1 - p = \frac{1}{2}(k - m + 1).$$

Let us denote our special case of the function F_B by $F_B(\alpha, 1 - \alpha, \frac{1}{2}(k - m + 1), z)$, in which it is understood that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ and $1 - \alpha = (1 - \alpha_1, 1 - \alpha_2, \dots, 1 - \alpha_m)$. While it will be necessary to return to examine some properties of F_B , the essential point to consider is that the expression

$$(2.11) \quad v_\lambda^{2-k} = \Gamma^{\frac{1}{2}(k-m-1)} F_B(\alpha, 1 - \alpha, \frac{1}{2}(k - m + 1), z)$$

is, in fact, for $|z_i| < 1$ ($i = 1, 2, \dots, m$) a solution of equation (1.1)'. We now make use of that correspondence principle of A. WEINSTEIN which states that for any equation of the form $u_{it} + (k/t)u_i = X(u)$, in which X is a differential operator that does not depend on t , the relation

$$(2.12) \quad u^k = t^{1-k} u^{2-k}$$

holds. That is, for any k a solution of the k -equation is obtained by multiplying by t^{1-k} any solution of the same equation in which k has been replaced by $2 - k$. In our case this implies that v_λ^k , defined by

$$(2.13) \quad v_\lambda^k = |t|^{1-k} \Gamma^{\frac{1}{2}(k-m-1)} F_B(\alpha, 1 - \alpha, \frac{1}{2}(k - m + 1), z),$$

is, in fact, a solution of equation (1.1). This solution will lead directly to the integration procedure for our Cauchy problem.

Before that is carried out it is essential to examine the function F_B in some detail. Here we will regard the parameters α_i , β_i , and γ as complex quantities and the z_i as real. Since the series defining $F_B(\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_m; \gamma; z_1, z_2, \dots, z_m)$ converges for $|z_i| < 1$, we may conclude immediately that the convergence is uniform with respect to α, β, γ, z for all α, β in a bounded region, γ bounded ($\neq -1, -2, -3, \dots$), and z in any closed subset of the cube $|z_i| < 1$. Clearly then for each i , F_B is an entire function of each parameter α_i , all other quantities being regarded as fixed. The same is true for the quantities β_i , while it is clear that, regarded as a function of γ alone, F_B is a meromorphic function with simple poles at $\gamma = -1, -2, \dots$.

In our particular case it is pertinent to notice that, because for each real λ_i the α_i and β_i defined by equation (2.10) are complex conjugates, all products of the form $(\alpha_i, \rho_i) \cdot (\beta_i, \rho_i)$ are purely real and thus for convenience may be written as

$$(2.14) \quad [\lambda_i, \rho_i] \equiv 1 \cdot \lambda_i \cdot (\lambda_i + 1 \cdot 2) \cdots (\lambda_i + (\rho_i - 1) \cdot \rho_i).$$

We note that $[\lambda_i, 0] \equiv 1$ for each j . As a consequence of this, $F_B(\alpha, 1 - \alpha, \frac{1}{2}(k - m + 1), z)$ is purely real as long as λ and k are real. For complex k the poles of F_B (considered now as a meromorphic function of k) are $k = m - 2n - 1$ ($n = 0, 1, 2, \dots$). One further observation is important: If $\lambda_i = 0$ for some i then F_B does not depend on z_i and hence v_λ^k depends on ξ_i only through Γ and not through F_B .

3. Solution of the Cauchy problem. In this section we carry out the integration of our singular Cauchy problem. Our program is to establish a form for the solution when k is complex and has a sufficiently large real part and then to extend this solution by analytic continuation *with respect to k* to all k except the values $k = -1, -3, -5, \dots$ * We will establish sufficient differentiability conditions on f and verify that the initial conditions are satisfied.

*The exceptional values were found by WEINSTEIN [2] and, as BUREAU [4] states, have no counterpart in the theory of equations with regular coefficients.

Assume that $\operatorname{Re}(k)$ ($=$ real part of k) satisfies $\operatorname{Re}(k) > m - 1$ and $|x_i| > |t| > 0$ ($i = 1, 2, \dots, m$). Following BUREAU [4] we consider the expression

$$(3.1) \quad u_{\lambda}^k(x, t, f) = \frac{\Gamma[\frac{1}{2}(k+1)]}{\pi^{\frac{1}{2}m} \Gamma[\frac{1}{2}(k-m+1)]} \int_{S(x,t)} f(\xi) v_{\lambda}^k(x, t, \xi) d\xi,$$

in which $d\xi$ is written for $d\xi_1 d\xi_2 \cdots d\xi_m$ and $S(x, t)$ is the domain of the singular hyperplane $t = 0$ cut out by the characteristic cone with vertex at x, t . $S(x, t)$ is, in fact, a closed m -dimensional sphere of radius $|t|$ and center x in the initial hyperplane and is given by $\Gamma(x, t, \xi) \geq 0$.

We now establish, as is essential for (3.1) to have meaning, that for any fixed point x, t the series defining F_B is convergent for ξ in $S(x, t)$. Clearly the restriction $|x_i| > |t| \geq 0$ is necessary and sufficient to insure that the sphere $S(x, t)$ does not intersect any of the hyperplanes $x_i = 0$. This means that $S(x, t)$ is confined to the interior of an "octant" of the hyperplane $t = 0$, and, as a consequence, for each i the quantities x_i and ξ_i have the same sign and z_i is thus always non-negative in S . (See Figure 1). For i fixed z_i vanishes on the boundary of S and takes only positive values in the interior. Since, for x, t fixed, Γ depends only on the radial distance r ($= [\sum_{i=1}^m (x_i - \xi_i)^2]^{\frac{1}{2}}$) from the center x to the point ξ , it is clear that the maximum of z_i on each m -sphere of radius $r \leq |t|$ and center x is assumed at the point given by $\xi_k = x_k, k \neq i$; $|\xi_i| = |x_i| - r$. Consequently z_i takes its maximum in S along the line l_i defined by $\xi_k = x_k, k \neq i$. Now z_i , which is given by $z_i = [t^2 - (x_i - \xi_i)^2]/4x_i\xi_i$ on the line l_i , is a function of the single variable ξ_i . Elementary computation shows that z_i takes its maximum on l_i at the point given by $|\xi_i| = \sqrt{x_i^2 - t^2}$, at which z_i has the value

$$(3.2) \quad \operatorname{Max}_{\xi \in S(x,t)} z_i = \frac{1}{2} \left(1 - \frac{\sqrt{x_i^2 - t^2}}{|x_i|} \right) \quad (i = 1, 2, \dots, m).$$

Thus $|z_i| \leq \frac{1}{2}$ in $S(x, t)$ for each i , and consequently the convergence of F_B is uniform with respect to z and hence with respect to ξ in $S(x, t)$.

For $\operatorname{Re}(k) > m + 3$ it is clear that u_{λ}^k , given by (3.1), is a solution of (1.1) since, when the derivatives of L are performed on the integral, all boundary integrals which arise vanish leaving only

$$(3.3) \quad L(u_{\lambda}^k) = \frac{\Gamma[\frac{1}{2}(k+1)]}{\pi^{\frac{1}{2}m} \Gamma[\frac{1}{2}(k-m+1)]} \int_{S(x,t)} f(\xi) L(v_{\lambda}^k) d\xi = 0.$$

For this it is sufficient that f be continuous.

We will now show that (3.1) gives a solution of equation (1.1) for $\operatorname{Re}(k) > m - 1$. We observe that for $\operatorname{Re}(k)$ in this range the integral in (3.1) converges and the convergence is uniform with respect to k as long as $\operatorname{Re}(k) \geq \delta > m - 1$. Since $F_B(\alpha, 1 - \alpha, \frac{1}{2}(k - m + 1), z)$ is analytic in k for $\operatorname{Re}(k) > m - 1$, the solution (3.1) is an analytic function of k in the right half plane $\operatorname{Re}(k) > m - 1$. Further, if f has at least two continuous derivatives, $L(u_{\lambda}^k)$ is also an analytic

FIGURE 1

function of k in the same domain. This last statement is easily verified by writing u_λ^k as an integral over a fixed domain.

Let $\xi_i = x_i + \eta_i |t|$ ($i = 1, 2, \dots, m$) define the new coordinates η_i . Then u_λ^k can be written as

$$(3.4) \quad u_\lambda^k(x, t, f) = \frac{\Gamma[\frac{1}{2}(k+1)]}{\pi^{\frac{1}{2}m} \Gamma[\frac{1}{2}(k-m+1)]} \cdot \int_{\sum \eta_i^2 \leq 1} f(x + \eta |t|) \left(1 - \sum_{i=1}^m \eta_i^2\right)^{\frac{1}{2}(k-m-1)} F_B(\alpha, 1-\alpha, \frac{1}{2}(k-m+1), z) d\eta$$

in which the coordinates z_i are now given by $z_i = t^2(1 - \sum \eta_i^2)/4x_i(x_i + \eta_i |t|)$. The operator L applied to (3.4) yields

$$(3.5) \quad L(u_\lambda^k) = \frac{\Gamma[\frac{1}{2}(k+1)]}{\pi^{\frac{1}{2}m} \Gamma[\frac{1}{2}(k-m+1)]} \int_{\sum \eta_i^2 \leq 1} L(f F_B) \left(1 - \sum_{i=1}^m \eta_i^2\right)^{\frac{1}{2}(k-m-1)} d\eta.$$

Certainly for $\text{Re}(k) > m+3$, (3.5) must coincide with (3.3), while for $\text{Re}(k) > m-1$ the differentiation under the integral sign in (3.5) disturbs neither the convergence of the integral nor the analyticity of the integrand

as a function of k . Thus for $\operatorname{Re}(k) > m - 1$ and x, t fixed, $L(u_\lambda^k)$ is an analytic function of k , vanishes for $\operatorname{Re}(k) > m + 3$, and thus is identically zero. It is clear from (3.4) that the integral that appears there is divergent for $\operatorname{Re}(k) \leq m - 1$, and some other form must be found to give meaning to the solution for k with smaller real part. We shall assume in all that follows that k is non-exceptional.

We first remark that the series for F_B can be written in a form useful for analytic continuation by grouping the terms into homogenous polynomials. This done, F_B has the form

$$(3.6) \quad F_B(\alpha, 1 - \alpha, \tfrac{1}{2}(k - m + 1), z) = \sum_{\nu=0}^{\infty} \left(\frac{\Gamma}{4}\right)^{\nu} \frac{1}{(\tfrac{1}{2}(k - m + 1), \nu)} \sum_{\substack{\rho_1, \rho_2, \dots, \rho_m=0 \\ \sum \rho_i = \nu}}^{\nu} \left(\prod_{i=1}^m \frac{[\lambda_i, \rho_i]}{\rho_i! (x_i \xi_i)^{\rho_i}} \right).$$

Inserting this in the formula (3.4) the following expression for u_λ^k is obtained:

$$(3.7) \quad u_\lambda^k = \frac{\Gamma[\tfrac{1}{2}(k + 1)]}{\pi^{\frac{1}{2}m} \Gamma[\tfrac{1}{2}(k - m + 1)]} \int_{\sum \eta_i^2 \leq 1} f(x + \eta |t|) (1 - \sum \eta_i^2)^{\frac{1}{2}(k-m-1)} \cdot \sum_{\nu=0}^{\infty} \left(\frac{t}{2}\right)^{2\nu} \frac{(1 - \sum \eta_i^2)^{\nu}}{(\tfrac{1}{2}(k - m + 1), \nu)} \sum_{\substack{\rho_1, \rho_2, \dots, \rho_m=0 \\ \sum \rho_i = \nu}}^{\nu} \left(\prod_{i=1}^m \frac{[\lambda_i, \rho_i]}{\rho_i! [x_i(x_i + \eta_i |t|)]^{\rho_i}} \right) d\eta.$$

First, for compactness of notation, let

$$(3.8) \quad C(\lambda, \rho, x) = \prod_{i=1}^m \frac{[\lambda_i, \rho_i]}{\rho_i! x_i^{\rho_i}}$$

and

$$(3.9) \quad g_\rho(x) = \frac{f(x_1, x_2, \dots, x_m)}{x_1^{\rho_1} x_2^{\rho_2} \dots x_m^{\rho_m}}.$$

Then, using the definition of $(\tfrac{1}{2}(k - m + 1), \nu)$, (3.7) can be written by integrating term by term as

$$(3.10) \quad u_\lambda^k = \sum_{\nu=0}^{\infty} \left(\frac{t}{2}\right)^{2\nu} \frac{\Gamma[\tfrac{1}{2}(k + 1)]}{\Gamma[\tfrac{1}{2}(k + 1) + \nu]} \cdot \sum_{\substack{\rho_1, \rho_2, \dots, \rho_m=0 \\ \sum \rho_i = \nu}}^{\nu} C(\lambda, \rho, x) \frac{\Gamma[\tfrac{1}{2}(k + 1) + \nu]}{\pi^{\frac{1}{2}m} \Gamma[\tfrac{1}{2}(k - m + 1) + \nu]} \cdot \int_{\sum \eta_i^2 \leq 1} g_\rho(x + \eta |t|) (1 + \sum \eta_i^2)^{\frac{1}{2}(k-m-1)+\nu} d\eta.$$

Each integral of the form

$$(3.11) \quad \frac{\Gamma[\tfrac{1}{2}(k + 1) + \nu]}{\pi^{\frac{1}{2}m} \Gamma[\tfrac{1}{2}(k - m + 1) + \nu]} \int_{\sum \eta_i^2 \leq 1} g_\rho(x + \eta |t|) (1 - \sum \eta_i^2)^{\frac{1}{2}(k-m-1)+\nu} d\eta$$

which occurs in (3.10) is for $\operatorname{Re}(k) > m - 2\nu - 1$ exactly a solution of a singular Cauchy problem for an EPD equation of index $k + 2\nu$ with the initial datum g_ρ . Here we will denote such functions by $u_0^{k+2\nu}(x, t, g_\rho)$. Using this, (3.10) can be written in a form suitable for analytic continuation as follows:

$$(3.12) \quad u_\lambda^k(x, t, f) = \sum_{\nu=0}^{\infty} \left(\frac{t}{2}\right)^{2\nu} \frac{\Gamma[\frac{1}{2}(k+1)]}{\Gamma[\frac{1}{2}(k+1) + \nu]} \cdot \sum_{\substack{\rho_1, \rho_2, \dots, \rho_m=0 \\ \sum \rho_i = \nu}} C(\lambda, \rho, x) u_0^{k+2\nu}(x, t, g_\rho).$$

The next step is to give meaning to (3.12) when $\operatorname{Re}(k) \leq m - 1$. Let us assume that k is restricted to a right half plane given by $\operatorname{Re}(k) \geq m - 2n - 1$, in which n is a non-negative integer. Then, in general, the integrals (3.11) which define the functions $u_0^{k+2\nu}(x, t, g_\rho)$ in (3.12) are always convergent for $\nu > n$, while for $\nu \leq n$ they may be divergent. Thus the analytic continuation of $u_\lambda^k(x, t, f)$ involves at most the analytic continuation of the terms of index $\nu \leq n$ in the ν -summation of (3.12). The continuation of these terms is easily accomplished by employing the continuations with respect to k that already have been developed for the solutions of singular Cauchy problems for the EPD equation.

We recall that for the solution $u_0^{\bar{k}}(x, t, h)$ of an EPD equation of index $\bar{k}$ ($\neq -1, -3, \dots$) with initial datum h on the singular hyperplane $t = 0$ the analytic continuation for $\operatorname{Re}(\bar{k}) \leq m - 1$ is given by

$$(3.13) \quad u_0^{\bar{k}}(x, t, h) = \frac{\Gamma[\frac{1}{2}(\bar{k}+1)]}{\Gamma[\frac{1}{2}(\bar{k}+p+1)]} \tau^{\frac{1}{2}(1-\bar{k})} \left(\frac{\partial}{\partial \tau}\right)^p \{ \tau^{\frac{1}{2}(\bar{k}+2p-1)} u_0^{\bar{k}+2p}(x, \sqrt{\tau}, h) \},$$

in which $\tau = t^2$ and p is a non-negative integer such that $\bar{k} + 2p \geq m - 1$. Here it is to be observed that $u_0^{\bar{k}+2p}(x, t, h)$ which appears in (3.13) is given either by an integral of the type (3.11) or by the mean value of h over the surface of an m -dimensional sphere of radius $|t|$ ($= \sqrt{\tau}$) with center at x . For (3.13) to be valid it is sufficient to assume that h has at least $\bar{p} + 2$ continuous derivatives where $\bar{p}$ is the smallest non-negative integer which satisfies the inequality placed on p . Let us note that DIAZ & WEINBERGER [5] in 1953 were the first to point out that the solution of a singular problem for the EPD equation for $k \leq m - 1$ can be obtained by analytic continuation with respect to k . It should be emphasized, however, that the formula (3.13) of A. WEINSTEIN [2] (also published in 1953), which we have used here, is an expression for the analytic continuation, as can be seen from the representation of $u_0^{\bar{k}}(x, t, h)$ as a Riemann-Liouville integral [2, formula (36)].

In order to apply (3.13) in the continuation of (3.12) it is sufficient to assume, under the restriction we have placed on $\operatorname{Re}(k)$, that $f(x)$ (and thus g_ρ) has at least $n + 2$ continuous derivatives. Then for each ν less than or equal to n we replace in (3.12) the integral defining $u_0^{k+2\nu}(x, t, g_\rho)$ by the analytic continuation given by (3.13), in which we set $\bar{k} = k + 2\nu$, $p = n - \nu$, and $h = g_\rho$. For

convenience, the notation $u_0^{k+2\nu}(x, t, g_\nu)$ will be retained when the analytic continuation is intended. With this understood the expression (3.12) is the analytic continuation we have sought.

It is clear that $L(u_\lambda^k)$, in which u_λ^k is given by (3.12) in its continued form, is an analytic function of k for x, t fixed, $\operatorname{Re}(k) \geq m - 2n - 1$, $k \neq -1, -3, -5, \dots$. For $\operatorname{Re}(k) > m - 1$ this expression for $L(u_\lambda^k)$ must be equal to that given by (3.5), which, as we have established, has the value zero. Thus for the non-exceptional values of k that satisfy $\operatorname{Re}(k) \geq m - 2n - 1$ we have $L(u_\lambda^k) = 0$.

Since there have been no restrictions placed on the magnitude of n it is clear that by the above process $u_\lambda^k(x, t, f)$ can be defined by analytic continuation in k for any non-exceptional value of k provided that f is assumed to be sufficiently differentiable.

So far we have noted that in order for the various expressions we have considered to exist and satisfy (1.1), it is sufficient that f be continuous for $\operatorname{Re}(k) > m + 3$ and that f have continuous derivatives of order $n + 2$ for $\operatorname{Re}(k) \geq m - 2n - 1$ (n a non-negative integer), $k \neq -1, -3, -5, \dots$. Returning to our expression (3.1) examination shows that it is sufficient that f be continuous for $\operatorname{Re}(k) \geq m + 3$ and have continuous derivatives of the first order for $\operatorname{Re}(k) \geq m + 1$. All of these conditions on the differentiability of f can be conveniently summarized in the statement

$$(3.14) \quad f \in C^{(2+p)}$$

where p is the smallest integer, positive, negative, or zero, such that $\operatorname{Re}(k) \geq m - 2p - 1$, $\{2 + p\}$ equals $2 + p$ if $2 + p$ is non-negative and equals zero otherwise, and $C^{(2+p)}$ is the class of all continuous real valued functions of m real variables that possess continuous derivatives through the order $\{2 + p\}$.

To show that $u_\lambda^k(x, t, f)$ does, in fact, satisfy the initial conditions (1.2) we use the properties of the solutions of singular Cauchy problems for the EPD equation [2, 6] and the solution formula (3.12). We assume that f satisfies the differentiability requirements given by (3.14) and, in addition, that f is always at least once continuously differentiable. From (3.12) we have that $u_\lambda^k(x, 0, f)$ is given by

$$u_\lambda^k(x, 0, f) = u_0^k(x, 0, f) = f(x)$$

and that $(\partial u_\lambda^k / \partial t)(x, 0, f)$ is given by

$$\frac{\partial u_\lambda^k}{\partial t}(x, 0, f) = \frac{\partial u_0^k}{\partial t}(x, 0, f) = 0.$$

That continuity of f is not sufficient for the second initial condition to be satisfied, even in the limiting sense, for $k \geq m + 3$ can be seen from the following example: Consider the EPD equation with $m = 1$ and let $f(x) = f_0(x) \equiv |x - x_0|^\theta$, θ fixed, $0 < \theta < 1$. Along the line $x = x_0$ the solution is given by

$$(3.15) \quad u_0^k(x_0, t, f_0) = |t|^\theta \frac{\Gamma[\frac{1}{2}(k+1)]}{\Gamma(\frac{1}{2})} \frac{\Gamma[\frac{1}{2}(\theta+1)]}{\Gamma[\frac{1}{2}(\theta+k+1)]},$$

and along the same line we have

$$(3.16) \quad \frac{\partial u_0^k}{\partial t}(x_0, t, f_0) = \frac{\theta |t|^\theta}{t} \frac{\Gamma[\frac{1}{2}(k+1)]}{\Gamma(\frac{1}{2})} \frac{\Gamma[\frac{1}{2}(\theta+1)]}{\Gamma[\frac{1}{2}(\theta+k+1)]}.$$

But (3.16) shows that $(\partial u_0^k / \partial t)(x_0, t, f_0)$ approaches infinity as t approaches zero.

Remarks. (1) It should be noted that, in generalizing WEINSTEIN's solution of 1952, BUREAU obtains his formula as a multiple integral which reappears in our formula (3.1). A reduction of this multiple integral to a simple integral, as given by DIAZ & WEINBERGER [5], seems to be of interest only in the case of the EPD equation.

(2) We observe that the solutions given by integrals over fixed domains or their analytic continuations remain even in t when $|t|$ is replaced by t , so that the solutions of this form obtained for t positive without the use of the absolute value are also valid for t negative.

(3) As in the case of the EPD equation (see [13], page 79), it can be shown that if f satisfies (3.14) and is always in the class $C^{(2)}$ then the limiting form of equation (1),

$$(k+1) \frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^m \left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\lambda_i}{x_i^2} u \right),$$

is satisfied on the hyperplane $t = 0$.

4. Uniqueness. In this section we will demonstrate that the solution $u_\lambda^k(x, t, f)$ of our singular Cauchy problem is unique when k is real and non-negative. To do this we will extend the uniqueness theorem of ZAREMBA and ASGEIRSSON [7, 8], as improved by WALTER [6], to the equation

$$(4.1) \quad L(u) \equiv \frac{\partial^2 u}{\partial t^2} + \frac{k}{t} \frac{\partial u}{\partial t} - \sum_{i=1}^m \frac{\partial^2 u}{\partial x_i^2} - c(x)u = 0,$$

in which $c(x)$ is a continuous real-valued function of the x -coordinates alone.

In the course of the development leading to the uniqueness theorem we will show that the second initial condition, $u_t(x, 0) = 0$, is essential if a solution u is smooth enough at $t = 0$. (In this section we will give the variable t only non-negative values.)

Our first step is to write two identities which are valid for $t > 0$, provided u is twice continuously differentiable. These are given by

$$(4.2) \quad \begin{aligned} 2t^\alpha u_t L(u) = & \left[t^\alpha \left(u_t^2 + \sum_{i=1}^m u_i^2 - cu^2 \right) \right]_t - 2 \sum_{i=1}^m (u_i u_t t^\alpha)_i \\ & + t^{\alpha-1} \left[(2k - \alpha) u_t^2 - \alpha \sum_{i=1}^m u_i^2 + \alpha c u^2 \right], \end{aligned}$$

in which α is a non-negative real constant, and by

$$(4.3) \quad 2u_t L(u) = \left(u_t^2 + \sum_{i=1}^m u_i^2 \right)_t - 2 \sum_{i=1}^m (u_i u_t)_i + \frac{2k}{t} u_t^2 - 2cu u_t.$$

Using the identity (4.2) we now deduce

Theorem 1. *If $u^k \in C^{(2)}$ and $L(u^k) = 0$ for $t > 0$ and if $u^k \in C^{(1)}$ for $t \geq 0$, then for any non-zero k $u_i^k(x, 0) = 0$.*

To prove Theorem 1 we let x, t be the arbitrary but fixed point x_0, t_0 and then integrate the identity (4.2) over the domain enclosed by the retrograde cone with vertex x_0, t_0 between the two parallel hyperplanes given by $t = t_1$ and $t = t_2, t_0 \geq t_2 \geq t_1 > 0$. This region we will designate by G_{12} (see Figure 2). We observe that the integrals of the first two terms on the right in (4.2) can be transformed to surface integrals by the theorem of Gauss, so that the result of integrating over G_{12} is expressed by

$$\begin{aligned}
 (4.4) \quad & t_2^\alpha \int_{S(t_2)} (u_i^2 + \sum u_i^2 - cu^2) dS - t_1^\alpha \int_{S(t_1)} (u_i^2 + \sum u_i^2 - cu^2) dS \\
 & + \int_{S_1^2} t^\alpha \left(\frac{\sum (u_i \eta_i - u_i \eta_i)^2}{\eta_i} - cu^2 \eta_i \right) dS \\
 & + \int_{G_{12}} t^{\alpha-1} [(2k - \alpha)u_i^2 - \alpha \sum u_i^2 + \alpha cu^2] dv = 0.
 \end{aligned}$$

In writing (4.4) we have used the following notations: $S(t_1)$ and $S(t_2)$ denote those portions of the hyperplanes $t = t_1$ and $t = t_2$ respectively which bound G_{12} ; S_1^2 is that part of the mantle of the cone that encloses G_{12} ; and η_i and η_t are the components of the outward unit normal to S_1^2 in the x_i and t directions respectively. In order to write the second term in the form in which it appears, we have used the fact that on S_1^2 the components of the normal satisfy the characteristic condition $\eta_i^2 - \sum_{i=1}^m \eta_i^2 = 0$. If in (4.4) we write the integral

FIGURE 2

over G_{12} as an iterated integral and then allow t_1 to approach zero, in the limit we have the equation

$$(4.5) \quad t_2^\alpha \int_{S(t_2)} (u_i^2 + \sum u_i^2 - cu^2) dS + \int_{S_1} t^\alpha \left(\frac{\sum (u_i \eta_i - u_i \eta_i)^2}{\eta_i} - cu^2 \eta_i \right) dS \\ + \int_0^{t_2} t^{\alpha-1} \int_{S(t)} \{ (2k - \alpha) u_i^2 - \alpha \sum u_i^2 + \alpha cu^2 \} dS dt = 0.$$

Now divide both sides of (4.5) by t_2^α and let t_2 approach zero. As the limit is taken the second term of (4.5) vanishes since $(t/t_2)^\alpha$ remains less than one, the quantity in the braces is bounded, and the content of the region of integration vanishes. The limit of the third term, on the other hand, is exactly a derivative at $t_2 = 0$, as may be observed by writing it as

$$\frac{1}{\alpha t_2^\alpha} \int_0^{t_2^\alpha} \int_{S(t^\alpha)} [(2k - \alpha) u_i^2 - \alpha \sum u_i^2 + \alpha cu^2] dS d(t^\alpha).$$

The limit performed, the result is given by the equation

$$(4.6) \quad \int_{S(0)} (u_i^2 + \sum u_i^2 - cu^2) dS \\ + \frac{1}{\alpha} \int_{S(0)} [(2k - \alpha) u_i^2 - \alpha \sum u_i^2 + \alpha cu^2] dS = 0.$$

When (4.6) is condensed we have

$$(4.7) \quad \frac{2k}{\alpha} \int_{S(0)} u_i^2 dS = 0,$$

from which the conclusion of the theorem follows.

A comparable result obtained under less restrictive hypotheses is given by

Theorem 2. Assume $k \geq 0$. If $u^k \in C^{(2)}$ and $L(u^k) = 0$ for $t > 0$, $u^k \in C^{(0)}$ for $t \geq 0$, and the limits $\lim_{t \rightarrow 0+} u_i^k(x, t)$, $\lim_{t \rightarrow 0+} t^k u_i^k$ converge uniformly with respect to x in each bounded set and the last has the value zero, then

$$\lim_{t \rightarrow 0+} \int_{S(t)} u_i^2 dS = 0.$$

To prove this we return to (4.4), assume $k > 0$, and set 2α equal to k . We now rewrite the integral over G_{12} as an iterated integral as before and the integral over S_1^2 as an iterated integral taken first around the cone on the boundary $l(t)$ of $S(t)$ and then along the cone; with this done, (4.4) has the form

$$(4.8) \quad t_2^{2k} \int_{S(t_2)} (u_i^2 + \sum u_i^2 - cu^2) dS - t_1^{2k} \int_{S(t_1)} (u_i^2 + \sum u_i^2 - cu^2) dS \\ + \int_{t_1}^{t_2} t^{2k} \int_{l(t)} \left(\frac{\sum (u_i \eta_i - u_i \eta_i)^2}{\eta_i^2} - cu^2 \right) dl dt \\ - 2k \int_{t_1}^{t_2} t^{2k-1} \int_{S(t)} (\sum u_i^2 - cu^2) dS dt = 0.$$

Again we let t_1 approach zero and obtain in the limit

$$\begin{aligned}
 (4.9) \quad & t_2^{2k} \int_{S(t_2)} (u_i^2 + \sum u_i^2 - cu^2) dS \\
 & + \int_0^{t_2} t^{2k} \int_{l(t)} \left(\frac{\sum (u_i \eta_i - u_i \eta_t)^2}{\eta_t^2} - cu^2 \right) dl dt \\
 & - 2k \int_0^{t_2} t^{2k-1} \int_{S(t)} (\sum u_i^2 - cu^2) dS dt = 0.
 \end{aligned}$$

By neglecting the first term of the second integral and applying the mean value theorem for integrals to the remainder of the second term and to the third term separately, we obtain the inequality

$$\begin{aligned}
 (4.10) \quad & t_2^{2k} \int_{S(t_2)} (u_i^2 + \sum u_i^2 - cu^2) dS \\
 & \leq \frac{t_2^{2k-1}}{2k} \int_{l(\bar{t})} cu^2 dl + t_2^{2k} \int_{S(\bar{t})} (\sum u_i^2 - cu^2) dS
 \end{aligned}$$

where $\bar{t}$, $\bar{l}$ are values between 0 and t_2 . On dividing by t_2^{2k} we have

$$\begin{aligned}
 (4.11) \quad & \int_{S(t_2)} u_i^2 dS \leq \frac{t_2}{2k} \int_{l(\bar{t})} cu^2 dl \\
 & - \int_{S(\bar{t})} (\sum u_i^2 - cu^2) dS - \int_{S(t_2)} (\sum u_i^2 - cu^2) dS.
 \end{aligned}$$

Each term on the right has a well defined limit as t_2 approaches zero; the first vanishes due to the factor t_2 and the remaining two cancel since $0 \leq \bar{t} \leq t_2$. Thus for $k > 0$ we have

$$(4.12) \quad \lim_{t \rightarrow 0^+} \int_{S(t)} u_i^2 dS = 0,$$

which was to be shown. For $k = 0$ (4.12) follows trivially from the hypotheses of the theorem.

Let us note that if we assume $k > 0$, the hypotheses of Theorem 2 are satisfied by any solution which satisfies those of Theorem 1.

With the aid of Theorem 2 we can now prove a uniqueness theorem.

Theorem 3. *If u^k satisfies the hypotheses of Theorem 2 and, in addition, $u^k(x, 0) = 0$, and if $\lim_{t \rightarrow 0^+} u_i^k = 0$, then u^k is identically zero.*

To prove this theorem we need to show that u^k is identically zero in a neighborhood of the hyperplane $t = 0$. Once this is done, the uniqueness for larger t follows from the uniqueness of the regular Cauchy problem; see, for example, [9].

Here we start from the identity (4.3). Integration of this identity over G_{12} and application of the theorem of Gauss yield the equation

$$(4.13) \quad \int_{S(t_2)} (u_i^2 + \sum u_i^2) dS - \int_{S(t_1)} (u_i^2 + \sum u_i^2) dS \\ + \int_{S_1} \frac{\sum (u_i \eta_i - u_i \eta_i)^2}{\eta_i} dS + \int_{t_1}^{t_2} \int_{S(t)} \left(\frac{2k}{t} u_i^2 - 2cuu_i \right) dS dt = 0.$$

Let us note that since η_i is positive and k is non-negative, the omission of the third integral and the first part of the fourth gives the inequality

$$(4.14) \quad \int_{S(t_2)} (u_i^2 + \sum u_i^2) dS \\ \leq \int_{t_1}^{t_2} \int_{S(t)} 2cuu_i dS dt + \int_{S(t_1)} (u_i^2 + \sum u_i^2) dS.$$

Theorem 2 and the hypothesis that u_i converges to zero as t_1 approaches zero imply that the second integral on the right converges to zero as t_1 approaches zero. That the first integral on the right also converges as t_1 approaches zero follows from an application of Gauss' theorem, which shows that

$$\int_{t_1}^{t_2} \int_{S(t)} 2cuu_i dS dt = \int_{S(t_2)} cu^2 dS - \int_{S(t_1)} cu^2 dS + \int_{S_1} cu^2 \eta_i dS,$$

and the observation that the right side of this equation converges. Thus, taking the limit, we have the inequality

$$(4.15) \quad \int_{S(t_2)} (u_i^2 + \sum u_i^2) dS \leq \int_0^{t_2} \int_{S(t)} 2cuu_i dS dt.$$

We now determine an upper bound on the right hand side of this inequality in terms of the left hand side by an extension of a technique of FRIEDRICHS & LEWY [9].

The integrand on the right is bounded above by $M(u^2 + u_i^2)$ where M is a bound on $|c|$ in $S(0)$. We now need a bound for $\int_0^{t_2} \int_{S(t)} u^2 dS dt$ which can be obtained as follows: For each x we may write

$$u(t) - u(\epsilon) = \int_{\epsilon}^t u_i dt \quad (t \geq \epsilon > 0);$$

then, on applying the inequality of Schwarz, we obtain

$$[u(t) - u(\epsilon)]^2 \leq (t - \epsilon) \int_{\epsilon}^t u_i^2 dt.$$

Integration of the last expression over $S(t)$ yields

$$\int_{S(t)} [u(t) - u(\epsilon)]^2 dS \leq (t - \epsilon) \int_{S(t)} \int_{\epsilon}^t u_i^2 dt dS.$$

By enlarging the region of integration of the right hand integral we obtain the

inequality

$$\int_{S(t)} [u(t) - u(\epsilon)]^2 dS \leq (t - \epsilon) \int_{\epsilon}^t \int_{S(\tau)} u_i^2 dS d\tau.$$

Now letting ϵ approach zero we have, by Theorem 2 and the hypothesis that $u(x, 0)$ vanishes, that

$$(4.16) \quad \int_{S(t)} u^2 dS \leq t \int_0^t \int_{S(\tau)} u_i^2 dS d\tau.$$

Integrating the expressions in (4.16) from zero to t_2 leads to

$$(4.17) \quad \int_0^{t_2} \int_{S(t)} u^2 dS dt \leq \frac{1}{2} t_2^2 \int_0^{t_2} \int_{S(t)} u_i^2 dS dt.$$

Thus for the right hand side of (4.15) we now have the estimate that

$$(4.18) \quad \int_0^{t_2} \int_{S(t)} 2cuu_i dS dt \leq M(1 + \frac{1}{2}t_2^2) \int_0^{t_2} \int_{S(t)} u_i^2 dS dt.$$

Addition of the positive term $\sum u_i^2$ to the integrand on the right in (4.18) only improves the inequality and leads to that given by

$$(4.19) \quad \int_{S(t_2)} (u_i^2 + \sum u_i^2) dS dt \leq M(1 + \frac{1}{2}t_2^2) \int_0^{t_2} \int_{S(t)} (u_i^2 + \sum u_i^2) dS dt.$$

Let t^* be a fixed positive value of t_2 , then for any t , $0 \leq t \leq t^*$, we can conclude from (4.19) that

$$(4.20) \quad \int_{S(t)} (u_i^2 + \sum u_i^2) dS d\tau \leq M^* \int_0^t \int_{S(\tau)} (u_i^2 + \sum u_i^2) dS d\tau$$

where $M^* = M(1 + \frac{1}{2}t^{*2})$. By integrating both sides of (4.20) from zero to t , $0 \leq t \leq t^*$, we arrive at an inequality given by

$$(4.21) \quad \int_0^t \int_{S(\tau)} (u_i^2 + \sum u_i^2) dS d\tau \leq M^* t \int_0^t \int_{S(\tau)} (u_i^2 + \sum u_i^2) dS d\tau.$$

But (4.21) cannot hold for arbitrarily small t unless both u_i and $\sum u_i^2$ vanish identically inside the cone in a neighborhood of $t = 0$. Thus, inside the cone with vertex at x_0, t_0 and sufficiently close to $t = 0$, u is constant and equal to its initial value zero. This proves Theorem 3.

In the regions $|x_i| > t \geq 0$ in which our solution $u_\lambda^k(x, t, f)$ is valid it is clear that when f satisfies the condition (3.14) and is always at least once continuously differentiable, $u_\lambda^k(x, t, f)$ does in fact satisfy the differentiability hypotheses of Theorem 1 and thus also those of Theorems 2 and 3. Consequently for k real and non-negative our solution is unique.

For negative k , to any solution of our Cauchy may be added a problem function of the form $t^{1-k}g(x)$, where $g(x)$ is any solution of the equation

$$\sum_{i=1}^m \left(\frac{\partial^2 g}{\partial x_i^2} + \frac{\lambda_i}{x_i^2} g \right) = 0,$$

so that for negative k the solution $u_\lambda^k(x, t, f)$ is not unique.

Thus the function $u_\lambda^k(x, t, f)$ which we have defined for any k ($\neq -1, -3, -5, \dots$), provided f is sufficiently differentiable, is the (unique) analytic continuation with respect to k of the unique solution of the singular Cauchy problem for non-negative real k .

5. Related differential equations. In this section we examine some forms which equation (1.1) takes under transformation of the function u and the relationships which these equations and their solutions have with our Cauchy problem.

To begin with, we observe that the expression

$$(5.1) \quad \frac{\partial^2 v}{\partial y^2} + \frac{2\alpha}{y} \frac{\partial v}{\partial y} \quad (\alpha = \text{constant})$$

under the transformation $v = wy^{-\alpha}$ yields

$$(5.2) \quad y^{-\alpha} \left\{ \frac{\partial^2 w}{\partial y^2} + \frac{\alpha(1-\alpha)}{y^2} w \right\}.$$

By setting $\bar{v} = wy^{\alpha-1} = vy^{2\alpha-1}$ we obtain another expression given by

$$(5.3) \quad y^{1-2\alpha} \left\{ \frac{\partial^2 \bar{v}}{\partial y^2} + \frac{2(1-\alpha)}{y} \frac{\partial \bar{v}}{\partial y} \right\},$$

in which the differential operator has the same form as (5.1). The transformation of (5.1) into (5.3) was noticed by WEINSTEIN and, in fact, is the basis of the correspondence principle that was used in Section 2 to obtain v_λ^k .

Note that if we set $\lambda = \alpha(1-\alpha)$ and require that α be real, then λ can take no value larger than $\frac{1}{4}$. However, if we consider λ as given, we may write $\alpha = \frac{1}{2} \pm \frac{1}{2} \sqrt{1-4\lambda}$; then for real λ greater than $\frac{1}{4}$ the values of α are complex.

There are several implications for equation (1.1) in these observations.

If we set $\lambda_0 = \frac{1}{4}k(1-k)$, $k = 2\alpha_0$, then for $t \neq 0$ (1.1) is transformed into the equation

$$(5.4) \quad \frac{\partial^2 w}{\partial t^2} + \frac{\lambda_0}{t^2} w = \sum_{i=1}^m \left(\frac{\partial^2 w}{\partial x_i^2} + \frac{\lambda_i}{x_i^2} w \right)$$

for which the Huygens' principle for the regular Cauchy problem was obtained by STELLMACHER [10]. On the other hand, each of the terms on the right hand side of (1.1) may be transformed from the λ -form to an α -form. The most interesting form obtained this way, also considered by STELLMACHER in connection with

(5.4), is given (in our notation) by

$$(5.5) \quad \frac{\partial^2 v}{\partial t^2} + \frac{2\alpha_0}{t} \frac{\partial v}{\partial t} = \sum_{i=1}^m \left(\frac{\partial^2 v}{\partial x_i^2} + \frac{2\alpha_i}{x_i} \frac{\partial v}{\partial x_i} \right),$$

in which the quantities α_i are just those already introduced in equation (2.10). Here we could as well have used the quantities β_i of (2.10) or any selection of α 's and β 's. While equation (5.4) is not itself suitable for a singular Cauchy problem, it will be convenient to use the parameter λ_0 later when we compare the criterion for the Huygens' principle for the singular Cauchy problem for equation (1.1) with that for the regular Cauchy problem for equation (5.4).

For equation (5.5) we consider the singular Cauchy problem with data

$$(5.7) \quad v(x, 0) = h(x), \quad v_i(x, 0) = 0.$$

The solution, which we shall denote by $v_{2\alpha}^{2\alpha_0}(x, t, h)$, can be written down immediately as

$$(5.8) \quad v_{2\alpha}^{2\alpha_0}(x, t, h) = \prod_{i=1}^m x_i^{-\alpha_i} u_{\lambda(\alpha)}^{2\alpha_0}(x, t, f(h)).$$

The procedure here is to transform the data (5.7), by multiplying by $\prod_{i=1}^m x_i^{\alpha_i}$, to data (1.2) for equation (1.1), to solve this singular Cauchy problem, and then to transform the solution back by dividing it by the same product. This is indicated by writing $f(h)$ and $\lambda(\alpha)$.

For $\text{Re}(2\alpha_0) > m - 1$, $v_{2\alpha}^{2\alpha_0}(x, t, h)$ can be written by using the formula (3.1) in the form

$$(5.9) \quad v_{2\alpha}^{2\alpha_0}(x, t, h) = \frac{\Gamma(\alpha_0 + \frac{1}{2})}{\pi^{\frac{1}{2}m} \Gamma(\alpha_0 - \frac{1}{2}m + \frac{1}{2})} |t|^{1-2\alpha_0} \cdot \int_{S(x, t)} f(\xi) \prod_{i=1}^m \left(\frac{\xi_i}{x_i} \right)^{\alpha_i} \Gamma^{\alpha_0 - \frac{1}{2}m - \frac{1}{2}} F_B(\alpha, 1 - \alpha, \alpha_0 - \frac{1}{2}m + \frac{1}{2}, z) d\xi.$$

The specialization of formula (5.9) to the case in which $m = 1$ coincides with a recent result obtained by E. T. COPSON [15, equation (4.4)].

For equation (5.5) each term on the right hand side of the form

$$\frac{\partial^2 u}{\partial x_i^2} + \frac{2\alpha_i}{x_i} \frac{\partial u}{\partial x_i}$$

presents the possibility of applying each of the two correspondence principles of WEINSTEIN. To make the notation simpler we will for the moment suppress all indications of the parameters of the equation (5.5) which are not of immediate interest. With this convention we can write the correspondence principles as follows:

$$(5.10) \quad x_i^{1-2\alpha_i} v_{2-2\alpha_i} = v_{2\alpha_i},$$

$$(5.11) \quad \frac{1}{x_i} \frac{\partial}{\partial x_i} v_{2\alpha_i} = v_{2\alpha_i+2}.$$

If the quantities $v_{2-2\alpha_i}$ and $v_{2\alpha_i}$ which appear on the left in (5.10) and (5.11) are themselves solutions of singular Cauchy problems, then the quantities on the right are also solutions of singular Cauchy problems with related data. The particular way in which these solutions are related is summarized by the equations

$$(5.12) \quad x_i^{1-2\alpha_i} v_{2-2\alpha_i}^{2\alpha_0}(x, t, h) = v_{2\alpha_i}^{2\alpha_0}(x, t, h x_i^{1-2\alpha_i})$$

and

$$(5.13) \quad \frac{1}{x_i} \frac{\partial v_{2\alpha_i}^{2\alpha_0}}{\partial x_i}(x, t, h) = v_{2\alpha_i+2}^{2\alpha_0}\left(x, t, \frac{1}{x_i} \frac{\partial}{\partial x_i} h\right).$$

The first is obtained by writing out the expression for the left hand side by using (5.8) and by observing that it exactly equals the right hand side. The second is obtained at first for α_0 real and satisfying $2\alpha_0 > m - 1$ by the use of (5.8) and (3.4) to express the left hand side. After the indicated differentiation is performed, the left hand side is given by

$$(5.14) \quad \begin{aligned} & -\frac{\alpha_i}{x_i^{\frac{1}{2}}} v_{2\alpha_i}^{2\alpha_0}(h) + \frac{1}{x_i} v_{2\alpha_i}^{2\alpha_0}\left(\frac{\partial h}{\partial x_i}\right) + \frac{\alpha_i}{x_i} v_{2\alpha_i}^{2\alpha_0}\left(\frac{h}{x_i}\right) \\ & + \frac{1}{x_i} \prod_{i=1}^m x_i^{-\alpha_i} \frac{\Gamma(\alpha_0 + \frac{1}{2})}{\Gamma(\alpha_0 - \frac{1}{2}m + \frac{1}{2})} \\ & \cdot \int_{\sum \eta_i^2 \leq 1} h \prod_{i=1}^m (x_i + \eta_i |t|)^{\alpha_i} (1 - \sum \eta_i^2)^{\alpha_0 - \frac{1}{2}m - \frac{1}{2}} \\ & \cdot \frac{\partial F_B}{\partial z_i} \frac{\partial}{\partial x_i} \left\{ \frac{t^2(1 - \sum \eta_i^2)}{4x_i(x_i + \eta_i |t|)} \right\} d\eta. \end{aligned}$$

For $t = 0$ the first and third terms cancel, the fourth vanishes since the x_i -derivative is $O(t^2)$, and the second yields $x_i^{-1} \partial h / \partial x_i$. The uniqueness of the solution of the singular Cauchy problem requires that the expression (5.14) equal that given by (5.8) with initial data $x_i^{-1} \partial h / \partial x_i$. However, (5.13) can be regarded as an identity in α_0 , all other quantities being regarded as fixed. Since it holds whenever α_0 is real and $2\alpha_0 > m - 1$, it holds for all α_0 for which both sides are given meaning by analytic continuation in α_0 . Here, of course, we require that h have one more derivative than is usual.

6. Huygens' principle. This section is devoted to the determination of the necessary and sufficient conditions that our solution $u_\lambda^k(x, t, f)$ has Huygens' property in the sense of "Huygens' minor premise," as given by HADAMARD [11, pp. 53-55]. It will become clear that it is *essential* to distinguish between the regular Cauchy problem and the singular Cauchy problem for equation (1.1) when the Huygens' property is considered.

From the development of the solution $u_\lambda^k(x, t, f)$ in earlier sections, it is apparent that in general this solution depends on the values of f throughout the region

of integration $S(x, t)$. If, however, it should happen that for some values of the parameters k, λ and for arbitrary f , sufficiently differentiable, the solution depends only on the values of f and its derivatives up to some order on the boundary of S , we will say that the solution has Huygens' property. Here we will usually speak of Huygens' property for the solution $u_\lambda^k(x, t, f)$ rather than of Huygens' principle for the singular Cauchy problem (1.1), (1.2). This is convenient since, as has already been pointed out, the solution of the problem may be non-unique, as for instance for negative k . When the solution is unique, however, Huygens' property for the solution $u_\lambda^k(x, t, f)$ does imply the Huygens' principle for the singular Cauchy problem.

We use the criterion of WEINSTEIN for the Huygens' property for the solution of the singular Cauchy problem for the EPD equation [2].* This states that for $k \neq -1, -3, -5, \dots$ the solution $u_0^k(x, t, f)$ has Huygens' property if and only if, for some $n = 0, 1, 2, \dots$, k satisfies

$$(6.1) \quad k = m - 2n - 1.$$

From this criterion it is clear that the solution $u_\lambda^k(x, t, f)$ given by (3.12) will also have the Huygens' property if (6.1) is satisfied and, in addition, each of the factors $C(\lambda, \rho, x)$ multiplying $u_0^{k+2\nu}(x, t, g_\rho)$ vanishes whenever $k + 2\nu > m - 1$. That is, the non-Huygens terms disappear and the series (3.12) terminates. However, $C(\lambda, \rho, x)$ vanishes if and only if the product of the form $\prod_{i=1}^m [\lambda_i, \rho_i]$ vanishes. Thus it is required that each of these products vanish whenever $\sum_{i=1}^m \rho_i \geq n + 1$. The necessary and sufficient condition for this is that for each i $\lambda_i = \alpha_i(1 - \alpha_i)$, where each α_i is a positive integer not greater than $n + 1$, and together the α_i satisfy the inequality $\sum_{i=1}^m \alpha_i \leq n + m$.

Thus for $k \neq -1, -3, -5, \dots$ a sufficient condition for $u_\lambda^k(x, t, f)$ to have Huygens' property is that for some non-negative integer n the parameters k and λ satisfy the conditions

$$(6.2) \quad \left\{ \begin{array}{ll} k = m - 2n - 1, & n \in \{0, 1, 2, \dots\} \\ \lambda_i = \alpha_i(1 - \alpha_i), & \alpha_i \in \{1, 2, \dots, n + 1\}, \\ \sum_{i=1}^m \alpha_i \leq m + n. \end{array} \right.$$

That these conditions are also necessary can be proved as follows: Let $\bar{k}, \bar{\lambda}$ be fixed values of k and λ respectively and let p be the smallest integer, positive, negative, or zero, such that $m - 2p - 1 \leq \operatorname{Re}(\bar{k})$. Recall that, for f in the class $C^{(p+2)}$, $u_{\bar{\lambda}}^{\bar{k}}(x, t, f)$ is always given by (3.12). Now let x, t be fixed. If $u_{\bar{\lambda}}^{\bar{k}}(x, t, f)$ has Huygens' property it must hold that $u_{\bar{\lambda}}^{\bar{k}}(x, t, \bar{f})$ has the same value for any other function $\bar{f}$ which is in the class $C^{(p+2)}$ and agrees with f and its derivatives up to the order p in the normal direction on the boundary of $S(x, t)$. Conse-

* Independently DIAZ & WEINBERGER [5] stated this criterion, but only for non-negative values of k .

quently, it is enough to examine the implications of the statement

$$(6.3) \quad u_{\lambda}^{\bar{k}}(x, t, g) = 0$$

when $g(x)$ is in the class $C^{(p+2)}$ and satisfies the requirement that g and its derivatives up to the order p vanish on the boundary of $S(x, t)$. Under these restrictions on g we see that $u_{\lambda}^{\bar{k}}(x, t, g)$ is always given by the integral (3.4) with f replaced by g . For $\bar{k} > m - 1$ (i.e. p negative) this is immediate, while for $\bar{k} \leq m - 1$ it is a consequence of the fact that (3.12) is the analytic continuation in k of (3.4) and that the integral of (3.4) is convergent. To verify the last statement it is only necessary to recognize that $|g(x)|$ can be bounded by an expression of the form $N|\Gamma|^{p+1}$, N a positive constant, in a neighborhood of the boundary of $S(x, t)$, from which it follows that the integrand of (3.4) is of the order of $(1 - \sum \eta_i^2)^{\delta}$ with $\delta \geq 0$ in a neighborhood of the sphere $\sum \eta_i^2 = 1$. The integral in (3.4) is thus uniformly convergent for $k \geq \bar{k}$, is an analytic function of k , and thus coincides with (3.12). With this understood, the necessary condition for Huygens' property is that the equation

$$(6.4) \quad \frac{\Gamma[\frac{1}{2}(\bar{k} + 1)]}{\pi^{\frac{1}{2}m} \Gamma[\frac{1}{2}(\bar{k} - m + 1)]} \cdot \int_{\sum \eta_i^2 \leq 1} g(x + \eta | t) (1 - \sum \eta_i^2)^{\frac{1}{2}(\bar{k} - m - 1)} F_B(\bar{\alpha}, 1 - \bar{\alpha}, \frac{1}{2}(\bar{k} - m + 1), z) d\eta = 0$$

hold for every function g which satisfies the conditions imposed above. By the fundamental lemma of the calculus of variations, equation (6.4) implies that

$$(6.5) \quad \frac{F_B(\bar{\alpha}, 1 - \bar{\alpha}, \frac{1}{2}(\bar{k} - m + 1), z)}{\Gamma[\frac{1}{2}(\bar{k} - m + 1)]} \equiv 0$$

in the interior of $S(x, t)$. That is, the equation

$$(6.6) \quad \sum_{\nu=0}^{\infty} \left(\frac{\Gamma}{4}\right)^{\nu} \frac{1}{\Gamma(\frac{1}{2}(\bar{k} - m + 1) + \nu)} \sum_{\substack{\rho_1, \rho_2, \dots, \rho_m=0 \\ \sum \rho_i = \nu}}^{\nu} \left(\prod_{i=1}^m \frac{[\bar{\lambda}_i, \rho_i]}{(x_i \xi_i)^{\rho_i}} \right) \equiv 0$$

is satisfied interior to $S(x, t)$. Since the first term in the ν -summation is completely independent of $\bar{\lambda}$, we must have for (6.6) to hold that for some non-negative integer q the index $\bar{k}$ satisfies

$$(6.7) \quad \bar{k} = m - 2q - 1$$

and that for those terms in which $\nu > q$ each product $\prod_{i=1}^m [\bar{\lambda}_i, \rho_i]$ vanishes. But, except for notation, these are exactly the conditions (6.2) already found to be sufficient for Huygens' property.

The conditions (6.2) can be written in a somewhat simpler form by setting $\nu_i = \alpha_i - 1$. With this (6.2) becomes

$$(6.8) \quad \left\{ \begin{array}{l} k = m - 2n - 1, \quad n \in \{0, 1, 2, \dots\}, \\ \lambda_i = -\nu_i(\nu_i + 1), \quad \nu_i \in \{0, 1, 2, \dots, n\}, \\ \sum_{i=1}^m \nu_i \leq n. \end{array} \right\}$$

Several specializations of equation (1.1) are of interest. For $\lambda_i = 0$, *i.e.* $\alpha_i = 0$ for each i , equation (1.1) becomes simply the EPD equation and the conditions (6.2) give back the Huygens' conditions (6.1) for the singular Cauchy problem for that equation. For $k = 0$ the Cauchy problem (1.1), (1.2) is no longer singular. In this case the equations (6.8) can be written

$$(6.9) \quad \left\{ \begin{array}{l} m = 2n + 1, \quad n \in \{0, 1, 2, \dots\}, \\ \lambda_i = -\nu_i(\nu_i + 1), \quad \nu_i \in \{0, 1, 2, \dots, n\}, \\ \prod_{i=1}^m \nu_i \leq n = \frac{1}{2}(m - 1). \end{array} \right\}$$

In this case it is important to remember that the initial conditions (1.2) are not general Cauchy data since $u_i(x, 0) = 0$.

The criterion (6.2) can be written in still another form for comparison with the criterion of STELLMACHER for equation (5.4) in the regular case. As in (6.8) we set $\nu_i = \alpha_i - 1$ ($i = 1, 2, \dots, m$). Then define $\alpha_0 = \frac{1}{2}k$, $\nu_0 = \alpha_0 - 1$ and $\lambda_0 = -\nu_0(\nu_0 + 1)$. With these conventions we can rewrite (6.2) in the form

$$(6.10) \quad \left\{ \begin{array}{l} \lambda_i = -\nu_i(\nu_i + 1) \quad (i = 0, 1, 2, \dots, m), \\ \nu_0 = \frac{1}{2}(m - 2n - 3), \quad \nu_i \in \{0, 1, 2, \dots, n\} \\ \quad (i = 1, 2, \dots, m), \quad n \in \{0, 1, 2, \dots\}, \\ \sum_{i=0}^m \nu_i \leq \frac{1}{2}(m - 3). \end{array} \right\}$$

The criterion for equation (5.4) is

$$(6.11) \quad \left\{ \begin{array}{l} m = 2q + 1, \quad q \in \{0, 1, 2, \dots\}, \\ \lambda_i = -\nu_i(\nu_i + 1), \quad \nu_i \in \{0, 1, 2, \dots, q - 1\} \quad (i = 0, 1, 2, \dots, m), \\ \sum_{i=0}^m \nu_i \leq \frac{1}{2}(m - 3) = q - 1. \end{array} \right\}$$

The most striking difference between (6.10) and (6.11) is that (6.11) requires that m be odd while (6.10) is indifferent to the parity of m . Thus for even m a large class of equations (1.1) is of Huygens' type for the singular problem while there are *none* for the regular problem. (We note that since the odd negative values of k have been excluded, the Huygens cases for even m occur only when k is positive, and thus under our assumptions on f the solution is unique, con-

sequently the equation is said to be of Huygens' type for the singular problem.)

Even if m is odd (6.10) contains more cases than (6.11). This is due to the freedom allowed by the number n in (6.10) and may be observed as follows: In (6.11) we choose first $\nu_0 = q - 1 - l$, for some $l \in \{0, 1, 2, \dots, q - 1\}$. Then we observe that with this choice of l the restriction $\sum_{i=0}^m \nu_i \leq \frac{1}{2}(m - 3) = q - 1$ requires that $\sum_{i=1}^m \nu_i \leq l$ so that $\nu_i \in \{0, 1, 2, \dots, l\}$ ($i = 1, 2, \dots, m$). These, however, are exactly the same conditions that are specified in (6.10) for n equal to l . Thus the cases allowed under (6.11) are just those of (6.10) when n takes the values $1, 2, \dots, q - 1$. However, if in (6.10) n is chosen larger than $q - 1$, there is no corresponding case in (6.11). It is interesting to observe that if n is greater than $q - 1$, then $k (= m - 2n - 1)$ is either zero or a negative even integer. Thus the "additional" cases beyond those for the regular Cauchy problem are those in which either the equation (1.1) is non-singular ($k = 0$) and thus the data (1.2) are not completely general or the solution is non-unique. No such "weakness" is present when m is even.

We observe in the cases in which $u_\lambda^k(x, t, f)$ has Huygens' property that an associated equation of the form (5.5) is especially interesting. This is

$$(6.12) \quad \frac{\partial^2 v}{\partial t^2} + \frac{k}{t} \frac{\partial v}{\partial t} = \sum_{i=1}^m \left(\frac{\partial^2 v}{\partial x_i^2} + \frac{2\alpha_i}{x_i} \frac{\partial v}{\partial x_i} \right),$$

in which the constants α_i are those given by (6.2). Since each α_i is an integer the operator on the right is a multiply symmetric Laplacian of total dimension $m_1 = \sum_{i=1}^m (2\alpha_i + 1) \leq 2n + 3m$. Similarly, on the left the operator is for $k \geq 0$ also a simply symmetric Laplacian of dimension $m_2 = k + 1 = m - 2n$. Thus equation (6.12) is an axially symmetric operator of ultrahyperbolic type of total dimension $m_1 + m_2 \leq 4m$.

BIBLIOGRAPHY

1. A. WEINSTEIN, Sur le problème de Cauchy pour l'équation de Poisson et l'équation des ondes, *Comptes Rendus Acad. Sci. Paris*, **234** (1952), p. 2584.
2. A. WEINSTEIN, On the Cauchy problem for the Euler-Poisson-Darboux equation, *Bull. Amer. Math. Soc.*, **59** (1953), p. 454. On the wave equation of Euler-Poisson, *Proc. 5th Symposium in Appl. Math.*, 1952 (McGraw-Hill, 1954), p. 137.
3. P. APPELL & J. KAMPÉ DE FÉRIET, *Fonctions Hypergéométriques et Hypersphériques-Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.
4. F. BUREAU, Divergent integrals and partial differential equations, *Communications on Pure and Applied Mathematics*, **8** (1955), pp. 143-202.
5. J. DIAZ & H. WEINBERGER, A solution of the singular initial value problem for the Euler-Poisson-Darboux equation, *Proc. Amer. Math. Soc.*, **4** (1953), pp. 703-718.
6. W. WALTER, Über die Euler-Poisson-Darboux-Gleichung, *Math. Zeitschrift*, **67** (1957), pp. 361-376.
7. S. ZAREMBA, Sopra au teorema d'unicita relativo alla equazione delle onde sferiche, *Rendiconti Accad. Lincei, Ser. 5*, **24** (1915), p. 904.
8. L. ASGERSSON, Über eine Mittelwerteigenschaft von Lösungen homogener linearer partieller Differentialgleichungen, *Math. Annalen*, **113** (1937), pp. 321-346.

9. K. FRIEDRICHS & H. LEWY, Über die Eindeutigkeit und das Abhängigkeitsgebiet der Lösungen beim Anfangswertproblem linearer hyperbolischer Differentialgleichungen, *Math. Annalen*, **28** (1928), pp. 192–204.
10. K. STELLMACHER, Eine Klasse huyghenscher Differentialgleichungen und ihre Integration, *Math. Annalen*, **130** (1955), pp. 219–233.
11. J. HADAMARD, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Yale Univ. Press, 1923.
12. D. FOX, Sur le principe de Huygens pour un problème singulier de Cauchy, *Comptes Rendus Acad. Sci. Paris*, **246** (1958), p. 213.
13. J. DIAZ & G. LUDFORD, On the Euler-Poisson-Darboux equation, integral operators, and the method of descent. *Proceedings of the Conference on Differential Equations*, University of Maryland (1956), p. 73.
14. E. BLUM, The Euler-Poisson-Darboux equation in the exceptional cases. *Proc. Amer. Math. Soc.*, **5** (1954), p. 511.
15. E. T. COPSON, On a singular boundary value problem for an equation of hyperbolic type, *Archive for Rational Mechanics and Analysis*, **1** (1958), pp. 349–356.
16. D. FOX, Uniqueness theorems for singular Cauchy problems, *Notices, Amer. Math. Soc.*, **6** (1958), p. 341.

Institute for Fluid Dynamics and Applied Mathematics
University of Maryland, College Park, Maryland