On the Order of Zeros of One-signed Solutions of Elliptic Equations*

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Communicated by J. KELLER

Let

$$(1) \quad L = \sum_{k=0}^m \sum_{i_1 + \dots + i_n = k} a_{i_1 \dots i_n}(x) \frac{\partial^k}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}$$

be an elliptic operator of (even) order m whose coefficients are defined in a neighborhood of the origin in euclidean n -space, $x = (x_1, \dots, x_n)$.

FRIEDMAN [5] proved that a non-negative function which satisfies $Lu \leq 0$ cannot have a zero of infinite order unless it is identically zero. This result had previously been proved for arbitrary solutions of various second order elliptic equations in [1], [3], [4], [6], [8], and [9].

FRIEDMAN also raised the question: Does there exist a constant k , depending only on m and n , such that zero is the only one-signed solution of $Lu = 0$ which has a zero of order greater than or equal to k ?

In this paper we show that if we modify FRIEDMAN's conjecture so as to allow k to depend on the coefficients of the m^{th} derivatives of L , then the answer is yes. We also construct an example which shows that this is a necessary modification.

Theorem. *Suppose that L is uniformly elliptic and that the coefficients of the i^{th} derivatives in L are of class C^i . There exists a constant, $k = k(L) \geq m$, such that if u is a one-signed solution of $Lu = 0$ whose j^{th} derivatives have zeros at $x = 0$ of order $k - j$, $j = 0, 1, \dots, m$, then $u \equiv 0$. Moreover, k depends only on the modulus of ellipticity and on the bounds of the coefficients of the m^{th} derivatives of L .*

For example, if $L = \Delta^p$, the polyharmonic operator, then $k = 2p$ (or equivalently $2p - 2 + \epsilon$ for any $2 > \epsilon > 0$). For by the mean value theorem for polyharmonic functions, see NICOLESCO [10], the mean of a solution u of $\Delta^p u = 0$

* This paper was written while the author was a Temporary Member of the Institute of Mathematical Sciences, New York University, during the academic year 1957-58. This Temporary Membership Plan is supported by the National Science Foundation.

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over the sphere of radius R centered at the origin is given by

$$(2) \quad \frac{1}{\omega_n R^{n-1}} \int_{r=R} u \, dS = \sum_{j=0}^{p-1} c_j(n, p) R^{2k} \Delta^k u(0).$$

If u has a zero of order $2p$ at the origin, the right hand side of (2) vanishes. Hence u cannot retain its sign in a neighborhood of the origin unless it vanishes identically.

The following corollary is a restatement of the maximum principle which is valid for elliptic equations of arbitrary order.

Corollary. *If $a_{00\dots 0} \equiv 0$, then a non-constant solution of $Lu = 0$ cannot have an interior maximum or minimum of order k .*

Proof of theorem. It follows from a lemma of BERS [2] that (under weaker assumptions than ours) there exists a solution of $Lu = 0$ of the form

$$u = r^{m-2} + O(r^m).$$

This shows that $k \geq m$.

Since the coefficients of the i^{th} derivatives in L are of class C^i , the formal adjoint L^* is defined. If u is a solution of $Lu = 0$, it follows from Green's identity that for any $v \in C^m$

$$(3) \quad \int_{r \leq R} u L^* v \, dV = \int_{r=R} M[u, v] \, dS.$$

We recall that L^* is an elliptic operator having the same m^{th} order terms as L and that $M[u, v]$ is a bilinear combination of the first $m - 1$ derivatives of u and v . We observe that if the j^{th} derivatives of u have zeros at $x = 0$ of order $k - j$, then (3) is valid for $v = r^{-p}$ for any $p \leq k + n - 1 - m$. Thus

$$(4) \quad \int_{r \leq R} u L^*(r^{-p}) \, dV = \int_{r=R} M[u, r^{-p}] \, dS.$$

The right hand side of (4) may be rewritten

$$(5) \quad \int_{r=R} M[u, r^{-p}] \, dS = R^{-p} \sum_{i=0}^{m-1} p^i X_i$$

where each X_i is a sum of surface integrals which are independent of p . Since $\{p^i\}$ is a linearly independent set of polynomials in p , we may pick m values $p_0, p_1, \dots, p_{m-1}$, $p_i \leq n - 1$, such that the matrix $\{(p_i)^i\}$ is non-singular. Thus we obtain the system of equations

$$(6) \quad \sum_{i=0}^{m-1} p_i^j X_i = R^{p_i} \int_{r \leq R} u L^*(r^{-p_i}) \, dV, \quad j = 0, 1, \dots, m - 1.$$

Solving this system for the X_i 's and substituting into (5), (4) we obtain

$$(7) \quad \int_{r \leq R} u \left[L^*(r^{-p}) - \sum_{i=0}^{m-1} a_i p^i R^{-p+p_i} L^*(r^{-p_i}) \right] dV = 0$$

where the matrix $\{a_{ij}\}$ is the inverse of $\{(p_i)^i\}$.

From (7) we see that in order to prove our theorem it suffices to find a p and an R for which the quantity within the brackets is positive.

Since we have assumed that L is uniformly elliptic, there exist positive constants ϵ and M such that for every x and for every $\xi = (\xi_1, \dots, \xi_n)$

$$(8) \quad \sum_{i_1 + \dots + i_n = m} a_{i_1, \dots, i_n}(x) \xi_1^{i_1} \dots \xi_n^{i_n} \geq \epsilon |\xi|^m$$

and such that the coefficients of the m^{th} derivatives in L are uniformly bounded by M .

The terms in (7) which arise from the i^{th} derivatives of L^* are of order r^{-p-i} , since $R^{-p-p_i} \leq r^{-p+p_i}$ for $p_i \leq n-1$ and for $r \leq R$. One of the terms of order r^{-p-m} is

$$p^m r^{-p-2m} \sum_{i_1 + \dots + i_n = m} a_{i_1, \dots, i_n} x_1^{i_1} \dots x_n^{i_n} \geq \epsilon p^m r^{-p-m}$$

by (8). The rest of the terms of order r^{-p-m} have coefficients of order at most p^{m-1} . Hence there exists a p (depending only on ϵ and M) such that the terms of order r^{-p-m} are bounded from below by $\frac{1}{2}\epsilon p^m r^{-p-m}$. We now hold p fixed and note that the remaining terms are of order at worst r^{-p-m+1} . Hence for R sufficiently small

$$L^*(r^{-p}) - \sum_{i, i=0}^{m-1} a_i p^i R^{-p+p_i} L^*(r^{-p_i}) \geq \frac{1}{4}\epsilon p^m r^{-p-m}.$$

It therefore follows that a non-zero solution u of $Lu = 0$ which has a zero of order $k = p - n + 1 + m$ cannot retain its sign in a neighborhood of the origin.

Remark. The above proof offers a method for finding an upper bound to the best value of k for which the theorem is true. If L is a second order elliptic operator, we can show that $k = 2$ by choosing a coordinate system in which $L = \Delta$ (the Laplacian) at the origin and by choosing $p = n-1$, $p_0 = n-2$, and $p_1 = 0$. This agrees with FRIEDMAN's result. If L is of order greater than two, it is not immediately obvious which coordinate system and which values of $p_0, p_1, \dots, p_{m-1}$ yield the best estimate.

We now give an example which shows that k does indeed depend on the coefficients of the highest order derivatives of L .

An example. Consider the fourth order elliptic operator in two dimensions

$$L_A = \frac{\partial^4}{\partial x^4} + A \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}, \quad A > 0,$$

and the homogeneous polynomial of degree N in x^2 and y^2

$$p_N(x, y) = x^{2N} + \sum_{i=1}^{N-1} a_i x^{2i} y^{2(N-i)} + y^{2N}.$$

We will show that for any fixed $N \geq 2$ and for all sufficiently large A 's we can find a_i 's for which $p_N(x, y)$ is a solution of $L_A p_N = 0$. For A sufficiently large $p_N(x, y)$ will be positive for $(x, y) \neq 0$.