Generalized Heat Transfer between Solids and Gases under Nonlinear Boundary Conditions

AVNER FRIEDMAN*

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Introduction. Let B be a solid body with a given temperature $\psi(x)$ surrounded by a gas with temperature 1. Then heat will transfer through the surface $\dot{B}$ of B obeying the Newton Law of Cooling

$$\frac{\partial u}{\partial \nu} = g(u)$$

where u is the temperature of the body, $\partial u / \partial \nu$ is the thermal gradient at the surface evaluated from the interior in the direction of the outward normal and $g(u) = K(1 - u)$. For moderate temperatures K is approximately constant; however, for very high or low temperatures K will change much and nonlinearly with the temperature u (for more details and references see [7]). We thus get the following boundary value problem:

$$\begin{aligned} \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} &= 0 \quad \text{for } x \in B, \quad t > 0, \\ (0) \quad u(x, 0) &= \psi(x) \quad \text{for } x \in \bar{B}, \\ \frac{\partial u}{\partial \nu} &= g(u) \quad \text{for } x \in \dot{B}, \quad t > 0, \end{aligned}$$

where $g(u)$ is nonlinear in u .

In what follows we shall not make use of the special form of g but only make use of the following physically evident facts: $g(u)$ is continuous and decreasing in u and $g(1) = 0$. Furthermore, instead of considering the system (0) we shall

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consider the system

$$\begin{aligned} (0') \quad & Lu = f(x, t) \quad \text{for } x \in B, \quad t > 0, \\ & u(x, 0) = \psi(x) \quad \text{for } x \in \bar{B}, \\ & \frac{\partial u}{\partial T} = g(x, t, u) \quad \text{for } x \in \dot{B}, \quad t > 0, \end{aligned}$$

where L is a second order linear parabolic operator with coefficients depending on x and t (see (1.1)) and $\partial u / \partial T$ is defined in (1.4).

The system (0) for the special case $\psi(x) \equiv 0$, $n = 1$ and B a semi-infinite interval was considered by MANN & WOLF [7] using special methods. They proved uniqueness, existence and some other properties of the solution. Their results were slightly improved by ROBERTS & MANN [15]. Very recently PADMAVALLY [9] extended the results of [7], [15] to the case where the temperature of the gas is a function $\varphi(t)$ of the time.

Our methods are completely different. They involve comparison methods with use of the maximum principle [8] and generalized simple layers which have recently been studied by POGORZELSKI [12]. We proceed to describe briefly the main results of this paper.

In §1 we prove uniqueness (Theorem 1) and boundedness (Theorem 2) of the solutions of (0') in case $f \equiv 0$; the bound depends only on l.u.b. $|\psi(x)|$.

In §2 we prove the existence of a solution of the system (0') in case $f \equiv 0$. This theorem, as well as its proof, overlaps with recent results of POGORZELSKI. More details are given in the remark which follows Theorem 3.

In §3 we prove (Theorem 4) that every solution of the system (0') with $f \equiv 0$ tends to the temperature 1 of the gas as $t \rightarrow \infty$. If the coefficients of L are independent of t and if $\psi(x) \equiv \text{const.}$, the solutions are monotone functions of t (Theorem 5). In proving Theorem 4 we make use of the methods of §1 and of the existence theorem of §2.

In §4 we generalize the results of §§1, 2 and 3 to the following cases: (a) The temperature of the gas is a function $\varphi(t)$ of t . (b) The temperature of the gas varies at each point x of the surface of the body and at each time t in a certain way which depends only on the original temperature of the gas, and on the temperature of the body at the point x and at the same time t . (c) Heat transfer takes place only on a part of the surface of the body, the rest of the surface is kept under a given temperature.

In §5 we generalize most of our previous results (in particular, Theorems 1–4) to the case $f \not\equiv 0$, that is, to the general non-homogeneous systems (0').

The results of §§1, 2 and 3 for the special system (0) have appeared in a University of California Technical Report [5].

1. Uniqueness and Boundedness. Let B be a bounded domain in the n -dimensional Euclidean space of variables $x = (x_1, \dots, x_n)$ and denote by $\dot{B}$ the boundary of B . Let D_τ ($D_\infty = D$) be the topological product of B with

$0 < t < \tau$ and S_τ ($S_\infty = S$) the topological product of $\dot{B}$ with $0 < t \leq \tau$. $\bar{D}_\tau$ and $\bar{S}_\tau$ will denote the closures of D_τ and S_τ respectively.

We shall assume throughout this paper that $\dot{B}$ is of class $C^{1+\delta}$ for some $0 < \delta < 1$. This implies that $\dot{B}$ is of class C^1 and that the angle between tangent hyperplanes to $\dot{B}$ at any two points P and Q is bounded by $\text{const. } |P - Q|^\delta$. Thus the Liapounoff conditions [12] are satisfied and the theorems of POGORZELSKI concerning simple layers are valid. Some of our results (for instance, Theorems 1, 2) remain true if $\dot{B}$ is only assumed to be C^1 . On the other hand, there appear a few remarks in this paper where we make use of [3; Theorem 2]. We then have to assume, in addition to the above, that to every point x of $\dot{B}$ there corresponds a sphere V such that $V \subset B$ and $x \in \bar{V}$.

We shall consider the differential equation

$$(1.1) \quad Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = 0 \quad \text{in } D \quad \text{or } D_\tau$$

under the following conditions:

(A) L is parabolic in $\bar{D}$, that is, for every point (x, t) of $\bar{D}$ and for every real vector $\xi = (\xi_1, \dots, \xi_n) \neq 0$

$$\sum a_{ij}(x, t)\xi_i\xi_j > 0.$$

(B) The coefficients a_{ij} , b_i , c of L are continuous in $\bar{D}$ and $c(x, t) \leq 0$.

Together with equation (1.1) we shall consider the initial condition

$$(1.2) \quad u(x, 0) = \psi(x) \quad \text{on } \bar{B}$$

and the nonlinear boundary condition

$$(1.3) \quad \left. \frac{\partial u}{\partial T} \right|_{(x^0, t^0)} = g(x^0, t^0, u(x^0, t^0)) \quad \text{on } S \quad \text{or } S_\tau$$

where

$$(1.4) \quad \left. \frac{\partial u}{\partial T} \right|_{(x^0, t^0)} = \lim_{x \rightarrow x^0} \sum a_{ij}(x, t^0) \cos(\nu, x_j) \frac{\partial u(x, t^0)}{\partial x_i}$$

and (ν, x_j) is the angle which the outward normal to $\dot{B}$ at x^0 forms with the x_j -axis. We shall call $\partial u / \partial T$ the transversal derivative of u . Using (A) we easily find that $\partial / \partial T$ is a gradient directed outward to $\dot{B}$.

The function $g(x, t, u)$ in (1.3) will be assumed to satisfy some of the following conditions:

(G₁) $g(x, t, u)$ is continuous for $(x, t) \in \bar{S}$, $-\infty < u < \infty$.

(G₂) For each fixed (x, t) , g is a (strictly) decreasing function of u .

(G₃) $g(x, t, 1) \leq 0, g(x, t, 0) > 0$.

We shall need the following lemma.

Lemma 1. *Let L satisfy (A), (B) and let w be continuous in $\bar{D}_\tau$ and satisfy $Lw \leq 0$ in D_τ , $w \geq 0$ on B . If w assumes a negative minimum in $\bar{D}_\tau$ then it assumes it only at points P on S_τ and at such points $\partial w / \partial T \leq 0$.*

Proof. Let P be a point where w takes its negative minimum. If P is not on S_τ , then it lies in D_τ and by the strong maximum principle [8] it follows that $w(Q) = w(P) < 0$ at all points Q which can be connected to P by a continuous curve whose t -coordinate increases monotonically. In particular, $w(Q) < 0$ on B which is a contradiction. We have thus proved that P must lie on S_τ and that $w(Q) > w(P)$ at all points Q in D_τ . Noting that transversal directions are outwardly directed we conclude that $\partial w / \partial T \leq 0$ at P .

Corollary. *If w is continuous in $\bar{D}_\tau$ and if it satisfies $Lw \leq 0$ in D_τ , $w \geq 0$ on B and $\partial w / \partial T > 0$ on S_τ , then $w \geq 0$ in $\bar{D}_\tau$.*

Theorem 1. *Let L satisfy (A), (B) and let $g(x, t, u)$ satisfy (G₂). For every τ there exists at most one solution $u(x, t)$ in D_τ of the system (1.1), (1.2), (1.3) which is continuous in $\bar{D}_\tau$.*

Proof. Suppose u and v are two solutions and write $w = u - v$. Then $Lw = 0$ and

$$(1.5) \quad \frac{\partial w}{\partial T} = g(x, t, u) - g(x, t, v).$$

If $w \not\equiv 0$ then w must assume either positive or negative values in D_τ . It will be enough to consider the case that w assumes negative values. In that case w assumes a negative minimum in $\bar{D}_\tau$, say at a point P . By Lemma 1 P belongs to S_τ and $\partial w / \partial T \leq 0$ at P . On the other hand, $u < v$ at P and using (1.5) and the property (G₂) of g we conclude that $\partial w / \partial T > 0$ at P , which is a contradiction.

Theorem 2. *Let L satisfy (A), (B) and let $g(x, t, u)$ satisfy (G₂), (G₃). If u is a solution in D_τ of the system (1.1), (1.2), (1.3) and if it is continuous in $\bar{D}_\tau$, then*

$$(1.6) \quad |u(x, t)| \leq \max(1, \max_{x \in B} |\psi(x)|);$$

more precisely, (a) for $t > 0$, $u > \min(0, \text{g.l.b. } \psi(x))$; (b) if $\text{l.u.b. } \psi(x) \leq 1$ then $u \leq 1$; (c) if $\text{l.u.b. } \psi(x) > 1$ then for $t > 0$, $u < \text{l.u.b. } \psi(x)$. The above statements remain true if we replace both of the inequality signs $>$, $<$ by $\geq$, $\leq$.

Proof. If (a) is not true then u must take in D_τ nonpositive values which are $\leq \text{g.l.b. } \psi(x)$. Thus u must assume its nonpositive minimum in D_τ and by the strong maximum principle that minimum is assumed at a point P on S_τ . We clearly have $\partial u / \partial T \leq 0$ at P . Hence, by (1.3),

$$g(P, u(P)) = \frac{\partial u}{\partial T} \leq 0.$$

On the other hand, using (G_2) , (G_3) and the fact that $u(P) \leq 0$ we get $g(P, u(P)) \geq g(P, 0) > 0$ which is a contradiction.

To prove (b) assume on the contrary that u takes values > 1 . Then by the strong maximum principle u assumes its maximum in $\bar{D}_\tau$ at a point P on S_τ and $u(Q) < u(P)$ for all Q in D_τ .

We then have $\partial u / \partial T \geq 0$ at P . Hence, by (1.3),

$$g(P, u(P)) = \frac{\partial u}{\partial T} \geq 0.$$

On the other hand, using (G_2) , (G_3) and the fact that $u(P) > 1$ we get

$$g(P, u(P)) < g(P, 1) \leq 0$$

which is a contradiction.

The proof of (c) is similar to the proof of (b) and will be omitted. We also omit the proof of the last assertion of the theorem.

Remark 1. If $c(x, t) \equiv 0$ in (1.1) and if $g(x, t, u) > 0$ for each $u < 1$ then we can prove a statement more precise than (a), namely, (a') if g.l.b. $\psi(x) < 1$ (≥ 1) then, for $t > 0$, $u > \text{g.l.b. } \psi(x)$ (≥ 1).

Remark 2. If $\psi(x) \equiv 0$ then Theorem 2 says that $0 < u(x, t) \leq 1$ for $t > 0$. If, however, $\tau = \infty$ in Theorem 2 then on using [3; Theorem 2] we can prove that $0 < u(x, t) < 1$ for $t > 0$.

By the method used to prove Theorem 2 we shall now prove the following lemma to be used later.

Lemma 2. Let L satisfy (A), (B) and let g and g' satisfy (G_2) . Let u be a solution in D_τ of the system (1.1), (1.2), (1.3), let u' be a solution in D_τ of the system (1.1), (1.2), (1.3) with g and ψ replaced by g' and ψ' and assume that

$$(1.7) \quad g'(x, t, v) \geq g(x, t, v), \quad \psi'(x) \geq \psi(x).$$

If u and u' are continuous in $\bar{D}_\tau$, then $u' \geq u$ in D_τ .

Proof. We have to show that the function $w = u - u'$ is nonpositive in D_τ . In the contrary case it follows on using Lemma 1 that w assumes a positive maximum in $\bar{D}_\tau$ at a point P on S_τ and $\partial w / \partial T \geq 0$ at P . Noting, however, that $u(P) > u'(P)$ and using (1.7) we get

$$\frac{\partial w}{\partial T} = g(P, u(P)) - g'(P, u'(P)) < g(P, u'(P)) - g'(P, u'(P)) \leq 0$$

which is a contradiction.

2. Existence Theorem. 2.1. In this section we replace (B) by the stronger assumption:

(B') The coefficients a_{ij} , b_i and c of L are Hölder continuous in compact subsets of $\bar{D}$ and $c(x, t) \leq 0$.

We shall need the following assumption on $\psi(x)$:

(Ψ) $\psi(x)$ is defined and continuous on $\bar{B}$ and has a compact support in B , that is, it vanishes near the boundary $\bar{B}$.

Theorem 3. *Let L satisfy the assumptions (A), (B'), let $g(x, t, u)$ satisfy the assumptions (G_1) , (G_2) , (G_3) and let $\psi(x)$ satisfy the assumption (Ψ). Then there exists a solution in D of the system (1.1), (1.2), (1.3) which is continuous in $\bar{D}$.*

Remark. In [5] we gave a proof of this theorem in the special case of the heat equation. In that proof we made extensive use of generalized simple layers and our arguments can easily be generalized to equations (1.1) with sufficiently regular coefficients (so that the fundamental solutions constructed by DRESSEL [1], [2] exist). Since then we found that fundamental solutions and simple layers have been thoroughly studied by POGORZELSKI [11], [12] under the very weak assumptions (A), (B') on the coefficients of L . He has also proved existence of solutions for some nonlinear boundary value problems [10], [13], [14]. In particular, he solved in D_τ the system (1.1), (1.2), (1.3) with ψ any continuous function (and with $\bar{B}$ replaced by B in (1.2)) provided $g(x, t, u)$ ($-\infty < u < \infty$, $0 \leq t \leq \tau$, $x \in \bar{B}$) is continuous and bounded. His solutions need not be continuous or even bounded near $\bar{B}$ and thus uniqueness is not guaranteed. Without uniqueness we cannot continue solutions defined in each D_τ into a solution in D . Below we give a simplification of his method using our results of §1.

Proof. We first prove existence of a solution in each D_σ . Let $\Gamma(x, t; \xi, \tau)$ be a fundamental solution in D_σ of equation (1.1) as constructed in [11], and define

$$(2.1) \quad \varphi(x, t) = \int_B \rho(\xi, 0) \Gamma(x, t; \xi, 0) \psi(\xi) d\xi$$

where

$$(2.2) \quad \rho(\xi, \tau) = (\det a^{ij}(\xi, \tau))^{\frac{1}{2}} / 2^n \pi^{\frac{1}{2}n}$$

and (a^{ij}) is the matrix inverse to (a_{ij}) . As was proved in [12], $\varphi(x, t)$ is a solution of (1.1) in D_σ , continuous in $\bar{D}_\sigma$, and $\varphi(x, 0) = \psi(x)$. We also observe that

$$(2.3) \quad \frac{\partial \varphi}{\partial T} \text{ is continuous on } \bar{S}_\sigma$$

(here we make use of the assumption (Ψ)).

Denote by Z the Banach space of all functions $u(x, t)$ continuous in $\bar{D}_\sigma$ with the norm

$$||u|| = \text{l.u.b.}_{D_\sigma} |u(x, t)|$$

and denote by Z_R the set $\{u; u \in Z, ||u|| < R\}$. We define a transformation $v = Tu$ for $u \in Z_R$ in the following manner: v is a solution of (1.1) in D_σ , con-

tinuous in $\bar{D}_\sigma$, and it satisfies $v = \psi$ on $\bar{B}$ and $\partial v / \partial T = g(x, t, u)$ on S_σ . By Theorem 1, v is uniquely determined. We try to find it in the form

$$(2.4) \quad v(x, t) = \int_0^t \int_{\bar{B}} 2\rho(\xi, \tau) \Gamma(x, t; \xi, \tau) h(\xi, \tau) dW_\xi d\tau + \varphi(x, t)$$

where dW_ξ is the element of surface on $\bar{B}$ at the point ξ , $\rho(\xi, \tau)$ is defined by (2.2) and h is still to be determined. The integral in (2.4) is a generalized simple layer with density h . Using [12] we conclude: v is a solution of the equation (1.1) in D_σ , it is continuous in $\bar{D}_\sigma$, it satisfies $v(x, 0) = \psi(x)$ on $\bar{B}$ and

$$(2.5) \quad \left. \frac{\partial v}{\partial T} \right|_{(x, t)} = h(x, t) + \int_0^t \int_{\bar{B}} 2\rho(\xi, \tau) \frac{\partial}{\partial T} \Gamma(x, t; \xi, \tau) h(\xi, \tau) dW_\xi d\tau + \left. \frac{\partial \varphi}{\partial T} \right|_{(x, t)} \quad ((x, t) \in S_\sigma).$$

Putting $\partial v / \partial T = g(x, t, u)$ we obtain for h the integral equation

$$(2.6) \quad h(x, t) + \int_0^t \int_{\bar{B}} 2\rho(\xi, \tau) \frac{\partial}{\partial T} \Gamma(x, t; \xi, \tau) h(\xi, \tau) dW_\xi d\tau = g(x, t, u(x, t)) - \left. \frac{\partial \varphi}{\partial T} \right|_{(x, t)}.$$

Since the kernel is integrable [12], this singular Volterra type integral equation can be solved (and uniquely) by successive iterations (compare [11; pp. 41–44]). We then get that $h(x, t)$ is continuous on $\bar{S}_\sigma$ and

$$(2.7) \quad \text{l.u.b.}_{S_\sigma} |h(x, t)| \leq A + \text{l.u.b.}_{S_\sigma} |g(x, t, u(x, t))|$$

where A is a constant independent of u .

By Theorem 2 any possible solution is bounded by $\lambda \equiv \max(1, \text{l.u.b. } |\psi|)$. It follows that if we define

$$G(x, t, u) = \begin{cases} g(x, t, u) & \text{if } |u| \leq \lambda \\ g(x, t, \pm\lambda) & \text{if } \pm u > \lambda \end{cases}$$

and prove the existence of a solution u of (1.1), (1.2), (1.3) with g replaced by G then u will also be a solution of the original problem. Since G is bounded for $(x, t) \in D_\sigma$ and $-\infty < u < \infty$, we conclude that without loss of generality we may assume from the beginning that

$$(2.8) \quad |g(x, t, u)| \leq A_1 \quad ((x, t) \in D, -\infty < u < \infty)$$

where A_1 is a constant.

Substituting (2.8) into (2.7) we conclude that for every $u \in Z_R$

$$(2.9) \quad \text{l.u.b.}_S |h(x, t)| \leq A + A_1.$$

Substituting (2.9) into (2.4) we get

$$\text{l.u.b.}_{D_\sigma} |v(x, t)| \leq K$$

where K is a constant independent of R and of the particular u in Z_R . Taking $R \geq K$ it follows that $v = Tu$ maps Z_R into itself.

To prove that $v = Tu$ is a continuous transformation let $u_m \in Z_R$, $\|u_m - u\| \rightarrow 0$ as $m \rightarrow \infty$ and denote by h_m the corresponding solutions of (2.6). It then follows that

$$\text{l.u.b.} |h_m - h| \leq \text{const. l.u.b.} |g(x, t, u_m) - g(x, t, u)|$$

and by (G₁) the right side tends to zero as $m \rightarrow \infty$. Denoting $v_m = Tu_m$ and using (2.4) with h, v replaced by h_m, v_m we conclude that $\|v_m - v\| \rightarrow 0$ as $m \rightarrow \infty$ and the continuity of T is thereby proved.

We shall now prove that T maps Z_R into a compact set in Z . The functions

$$(2.10) \quad \int_0^t \int_{\bar{B}} 2\rho(\xi, \tau) \Gamma(x, t; \xi, \tau) h(\xi, \tau) dW_\xi d\tau, \quad u \in Z_R,$$

satisfy a uniform Hölder condition with any exponent $\vartheta < 1$ in x and $\frac{1}{2}\vartheta$ in t . This follows from [12] and (2.9). Hence the set of functions (2.10) is a compact set in Z . If we add to each function in (2.10) the function $\varphi(x, t)$ (which is continuous in $\bar{D}_\sigma$) then clearly the resultant set of functions is also a compact subset of Z . Thus the set $\{v; v = Tu, u \in Z_R\}$ is compact.

Having proved that T is a continuous transformation which maps the closed bounded convex set Z_R of the Banach space Z into a compact subset of itself, we can apply the Schauder fixed point theorem [16] and thus conclude that there exists $u \in Z_R$ for which $u = Tu$. This proves the existence of a solution in D_σ .

We now note that if u_σ and u_τ are solutions of (1.1), (1.2), (1.3) in D_σ and D_τ respectively and if $\sigma \geq \tau$ then $u_\sigma \equiv u_\tau$ in D_τ . Indeed, this follows from Theorem 1. Using this remark we define a unique solution in D in the following way: In each D_σ , $u = u_\sigma$ where u_σ is the solution of (1.1), (1.2), (1.3) in D_σ .

2.2. The assumption (Ψ) was needed in order to conclude (2.3). However, if the coefficients of L satisfy additional regularity conditions then we can weaken the assumption (Ψ) . We shall make the following assumption:

(B'') The coefficients of L are defined and Hölder continuous in compact subsets of the closure of a cylinder D' with base B' which contains $\bar{B}$, and they have first continuous x -derivatives in some D' -neighborhood of $\bar{B}$. Furthermore, the functions $\partial a_{ij}/\partial x_\lambda$ are Hölder continuous (in (x, t)) in that D' -neighborhood. Finally, $c(x, t) \leq 0$.

By a D' -neighborhood of $\bar{B}$ we mean a set of points (x, t) which satisfy $(x, t) \in \bar{D}'$, distance from x to $\bar{B}$ is $\leq \epsilon$ and $t \leq \epsilon$ for some $\epsilon > 0$.

We shall need the following assumption (weaker than (Ψ)):

(Ψ') $\psi(x)$ is continuous in $\bar{B}'$ and it is continuously differentiable in some neighborhood of $\bar{B}$.

We can now prove

Theorem 3*. *Theorem 3 remains true if we replace (B') and (Ψ) by (B'') and (Ψ') .*

Proof. We denote by $\Gamma(x, t; \xi, \tau)$ the fundamental solution of POGORZELSKI constructed in D'_σ instead of D_σ . Then the proof of Theorem 3* proceeds as that of Theorem 3. If we prove (2.3) for the function

$$\varphi(x, t) = \int_{B'} \rho(\xi, 0) \Gamma(x, t; \xi, 0) \psi(\xi) d\xi$$

then the proof of Theorem 3* is completed. For simplicity we shall first give the proof only for the case of the heat equation; the proof of the general case follows along the same lines if we make use of [11; pp. 41–44]. Since $\Gamma(x, t; \xi, \tau)$ is differentiable in x whenever $x \neq \xi$, it is enough to prove that the function

$$(2.11) \quad \vartheta(x, t) = \int_{B''} \frac{\psi(\xi)}{t^{\frac{n}{2}}} \exp \left\{ -\frac{|x - \xi|^2}{4t} \right\} d\xi$$

is continuously x -differentiable in a small D' -neighborhood of $\dot{B}$. Here B'' is any neighborhood of $\dot{B}$ where $\psi(x)$ is differentiable. We now differentiate $\vartheta(x, t)$ with respect to x and integrate the resultant integral by parts. We obtain

$$\frac{\partial \vartheta}{\partial x} = \int_{B''} \frac{\partial \psi(\xi)}{\partial \xi} \frac{1}{t^{\frac{n}{2}}} \exp \left\{ -\frac{|x - \xi|^2}{4t} \right\} d\xi + \text{boundary integrals},$$

from which our assertion readily follows.

In the general case the function $\vartheta(x, t)$ in (2.11) has to be replaced by (see [11])

$$\begin{aligned} & \int_{B''} \psi(\xi) w^{\xi, 0}(x, t; \xi, 0) d\xi \\ & + \int_{B''} \psi(\xi) \left[\int_0^t \int_{B'} w^{\nu, \vartheta}(x, t; y, \vartheta) \Phi(y, \vartheta; \xi, 0) dy d\vartheta \right] d\xi. \end{aligned}$$

The first integral can be treated as in the previous case of the heat equation. In the second integral we may without loss of generality replace B' by B'' . However, one cannot apply directly the previous method to the second integral with B' replaced by B'' since $\partial \Phi / \partial y$ and $\partial \Phi / \partial \xi$ have nonintegrable singularities at $\vartheta = 0$. We therefore consider first the integral.

$$(2.12) \quad I(\epsilon) = \int_{B''} \psi(\xi) \left[\int_\epsilon^t \int_{B''} w^{\nu, \vartheta}(x, t; y, \vartheta) \Phi(y, \vartheta; \xi, 0) dy d\vartheta \right] d\xi,$$

where $\epsilon > 0$. We have

$$(2.13) \quad \frac{\partial I(\epsilon)}{\partial \epsilon} = \int_{B''} \psi(\xi) \left[\int_\epsilon^t \int_{B''} \frac{\partial}{\partial x} w^{\nu, \vartheta}(x, t; y, \vartheta) \Phi(y, \vartheta; \xi, 0) dy d\vartheta \right] d\xi.$$

We now integrate by parts noting that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)w^{y,\vartheta}(x, t; y, \vartheta), \quad \left(\frac{\partial}{\partial y} + \frac{\partial}{\partial \xi}\right)\Phi(y, \vartheta; \xi, 0)$$

contribute terms of smaller order of magnitude. Here we make use of the assumption (B'') and of some formulas of [11; pp. 41-44]. We obtain

$$(2.14) \quad \begin{aligned} \frac{\partial I(\epsilon)}{\partial x} &= \int_{B''} \frac{\partial \psi(\xi)}{\partial \xi} \left[\int_{\epsilon}^t \int_{B''} w^{y,\vartheta}(x, t; y, \vartheta) \Phi(y, \vartheta; \xi, 0) dy d\vartheta \right] d\xi \\ &+ \int_{B''} \psi(\xi) \left[\int_{\epsilon}^t \int_{B''} W dy d\vartheta \right] d\xi + \int_{B''} \psi(\xi) \left[\int_{\epsilon}^t \int_{\dot{B}''} R dS_y d\vartheta \right] d\xi \end{aligned}$$

where $\dot{B}''$ is the boundary of B'' , dS_y is the surface element of B'' at y and R , W are appropriate expressions (when $\epsilon = 0$ the singularity of W is absolutely integrable).

Comparing (2.13) with (2.14) and taking $\epsilon \rightarrow 0$, and then noting that the two last integrals on the right side of (2.14) are continuous functions in a D' -neighborhood of $\bar{B}$ when $\epsilon = 0$, the proof is completed.

2.3. The assumptions on ψ can also be relaxed if we make more assumptions on g . We shall need the following assumption on g :

(G₄) There exists a positive constant A such that $g(x, t, u)$ satisfies the Lipschitz condition (in u)

$$|g(x, t, u') - g(x, t, u'')| \leq A |u' - u''|$$

for all $(x, t) \in \bar{D}$ and $|u'|, |u''| \leq \max(1, \text{l.u.b. } |\psi|)$.

We can now prove the following theorem:

Theorem 3. *Let L satisfy the assumptions (A), (B'), let $g(x, t, u)$ satisfy the assumptions (G₁), (G₂), (G₃), (G₄) and let $\psi(x)$ be a continuous function on $\bar{B}$. Then there exists a solution in D of the system (1.1), (1.2), (1.3) which is continuous in $\bar{D}$.*

Proof. We first construct a solution in D_σ , for σ sufficiently small. The method we use is that of successive approximation. We define $v_1(x, t) = \varphi(x)$ and then define $v_{m+1}(x, t)$ in terms of $v_m(x, t)$ as follows: Let $h = h_{m+1}$ be a solution of (2.6) with $u(x, t)$ replaced by $v_m(x, t)$. Then substitute $h = h_{m+1}$ in (2.4) and let $v = v_{m+1}$.

If we use the inequalities

$$\begin{aligned} \left| \frac{\partial \varphi(x, t)}{\partial T} \right| &\leq \frac{\text{const.}}{t^\gamma}, \\ |\Gamma(x, t; \xi, \tau)| &\leq \frac{\text{const.}}{(t - \tau)^\mu} \frac{1}{|x - \xi|^{n-2\mu}} \end{aligned}$$

for $\frac{1}{2} < \gamma < \mu < 1$, then we can show by induction that

$$(2.15) \quad \text{l.u.b.}_{x \in B} |v_{m+1}(x, t) - v_m(x, t)| \leq K^m t^{(1-\mu)m} \quad (m = 1, 2, \dots)$$

for all $0 < t \leq \sigma$, where the constant K is an increasing function of σ . Here we have made use of (G_4) . $h_{m+2} - h_{m+1}$ is also bounded by $\text{const. } t^{(1-\mu)m}$. (Actually we first modify the definition of g for $|u| > \max(1, \text{l.u.b. } |\psi|)$ so as to get a new function which satisfies the Lipschitz condition for all u .) From (2.15) it follows that for σ sufficiently small the sequence $\{v_m\}$ converges to a solution $u(x, t)$.

We can now repeat the above argument and construct step by step a solution u in the whole cylinder D . Here we make use of Theorem 1.

3. Further Properties of Solutions. In this section we shall prove some properties of the solution of the system (1.1), (1.2), (1.3). The first theorem is concerned with the existence of $\lim_{t \rightarrow \infty} u(x, t)$. We shall need the following assumptions:

$$(G'_3) \quad g(x, t, 1) \equiv 0.$$

$$(G'_4) \quad \text{There exist positive constants } \eta, A \text{ such that for all } (x, t) \in \bar{D} \text{ and for all } u' > u'' \text{ bounded by } \max(1, \text{l.u.b. } |\psi|)$$

$$-A(u' - u'') \leq g(x, t, u') - g(x, t, u'') \leq -\eta(u' - u'').$$

$$(C) \quad \text{There exist positive constants } K_1, K_2 \text{ such that}$$

$$a_{11}(x, t) \geq K_1 > 0, \quad -b_1(x, t) \leq K_2 \quad \text{for all } (x, t) \in D.$$

We shall prove the following theorem:

Theorem 4. *Let L satisfy (A), (B'), (C), let $c(x, t) \equiv 0$ and let g satisfy (G_1) , (G_2) , (G'_3) , (G'_4) . If $u(x, t)$ is a solution of (1.1), (1.2), (1.3) in D and is continuous in $\bar{D}$, then $\lim_{t \rightarrow \infty} u(x, t) = 1$ uniformly in $x \in \bar{B}$. More precisely, there exists a positive constant μ such that*

$$(3.1) \quad |1 - u(x, t)| = O(\exp \{-\mu t\})$$

uniformly in $x \in \bar{B}$, as $t \rightarrow \infty$.

Proof. We first consider the special case that $\text{l.u.b. } \psi \leq 1$. In that case we know, by Theorem 2, that $u(x, t) \leq 1$. The function $v = 1 - u$ is nonnegative in D and satisfies $Lv = 0$ in D and

$$(3.2) \quad \begin{aligned} \frac{\partial v}{\partial t} + \eta v &= -g(x, t, u) + \eta v \\ &= g(x, t, 1) - g(x, t, u) + \eta v \leq -\eta(1 - u) + \eta v = 0. \end{aligned}$$

Here we made use of (G'_4) .

Consider now the function

$$(3.3) \quad w = e^{\alpha R} - e^{\alpha x_1}.$$

If R is sufficiently large, then $w > 0$ in $\bar{B}$. Using (C) it follows that we can fix α (independently of R) such that

$$(3.4) \quad Lw = (-\alpha^2 a_{11} - \alpha b_1)e^{\alpha x_1} + cw \leq -e^{\alpha x_1}.$$

We now choose R sufficiently large such that also

$$(3.5) \quad \frac{\partial w}{\partial T} + \eta w = -\alpha e^{\alpha x_1} \frac{\partial x_1}{\partial T} + \eta(e^{\alpha R} - e^{\alpha x_1}) > 0 \quad \text{on } \dot{B}$$

and put

$$(3.6) \quad \lambda = \min_{\bar{B}} \frac{e^{\alpha x_1}}{e^{\alpha R} - e^{\alpha x_1}} > 0.$$

From (3.4), the definition (3.3) and (3.6) it follows that the function

$$(3.7) \quad z(x, t) = e^{-\lambda t} w(x)$$

satisfies $Lz \leq 0$. From (3.5) it follows that

$$(3.8) \quad \frac{\partial z}{\partial T} + \eta z > 0 \quad \text{on } S.$$

Taking K to be such that $Kz(x, 0) > u(x, 0)$, we shall prove that

$$(3.9) \quad v(x, t) \leq Kz(x, t) = Ke^{-\lambda t}(e^{\alpha R} - e^{\alpha x_1})$$

from which (3.1) would follow.

To prove (3.9) define $y = v - Kz$. Using (3.2), (3.8) it follows that

$$(3.10) \quad \frac{\partial y}{\partial T} + \eta y < 0 \quad \text{on } S.$$

Furthermore, $Ly \geq 0$ in D and $y \leq 0$ on $\bar{B}$. We have to prove that $y \leq 0$ in D . In the contrary case, y assumes positive values in D and therefore, for some $\sigma > 0$, y has a positive maximum in $\bar{D}$ which, by the maximum principle, is assumed at a point P on S_σ . At that point $\partial y / \partial T \geq 0$ and $\eta y > 0$. Thus $\partial y / \partial T + \eta y > 0$ at P which contradicts (3.10).

Having proved Theorem 4 in case l.u.b. $\psi \leq 1$, we remark that a similar proof can be constructed in the case where g.l.b. $\psi > 1$. We have only to define $v = u - 1$ and then proceed as before.

To prove Theorem 4 for general functions $\psi(x)$ we introduce two constant functions

$$\psi_1(x) \equiv \min(0, \text{g.l.b. } \psi), \quad \psi_2(x) \equiv \max(1, \text{l.u.b. } \psi).$$

Then

$$(3.11) \quad \psi_1(x) \leq \psi(x) \leq \psi_2(x) \quad \text{for } x \in \bar{B}$$

and

$$(3.12) \quad \psi_1(x) \leq 1 \leq \psi_2(x) \quad \text{for } x \in \bar{B}.$$

Since ψ_1 and ψ_2 are continuous functions, there exist by Theorem $\bar{3}$ solutions u_1 and u_2 of the system (1.1), (1.2), (1.3) with ψ replaced by ψ_1 and ψ_2 respectively. From Lemma 2 it follows that

$$(3.13) \quad u_1(x, t) \leq u(x, t) \leq u_2(x, t).$$

By what we have already proved

$$(3.14) \quad 1 - u_i(x, t) = O(\exp \{-\lambda_i t\}) \quad \text{as } t \rightarrow \infty \quad (\lambda_i > 0).$$

Using (3.14) in (3.13), the proof of (3.1) is completed.

Remark. In §5 we shall see that Theorem 4 remains true if the assumption $c(x, t) \equiv 0$ is replaced by the weaker assumption $\lim c(x, t) = 0$ ($t \rightarrow \infty$).

The next theorem considers the question of monotonicity of u in t . We shall need the following assumption on g :

(G₅) $g(x, t, u)$ is a non-decreasing function of t .

Theorem 5. Let the coefficients of L satisfy (A), (B) and be independent of t . Let g satisfy (G₂), (G₃), (G₅). If $u(x, t)$ is continuous in $\bar{D}$ and satisfies the system (1.1), (1.2), (1.3) and if $\psi(x) \equiv \text{const.} < 1$ and ≥ 0 then $u(x, t)$ is nondecreasing in t .

Proof. Let $v(x, t) = u(x, t + h) - u(x, t)$ where h is an arbitrary positive constant. Then $Lv = 0$ in D and, by Theorem 2, v is positive on $t = 0$. If v is not zero or positive at each point of D , then, for some $\sigma > 0$, v assumes a negative minimum in $\bar{D}_\sigma$. From Lemma 1 we conclude that the minimum is assumed at a point P on S_σ and $\partial v / \partial T \leq 0$ at P . On the other hand, using (G₂), (G₅) we have

$$\frac{\partial v}{\partial T} = g(x, t + h, u(x, t + h)) - g(x, t, u(x, t)) > 0$$

at P , since $u(x, t + h) < u(x, t)$ at P . We have thus arrived at a contradiction which proves the theorem.

Remark 1. Theorem 5 remains true for solutions $u(x, t)$ with $\psi(x) \equiv \text{const.} < 0$ provided $c(x, t) \equiv 0$.

Remark 2. A similar theorem can be proved if $\psi(x) \equiv \text{const.} > 1$. If $\psi(x) \equiv \text{const.}$ Theorem 5 need not be true. Take, for instance, the heat equation and assume that $\psi(x^0) < 1$ ($x^0 \in B$) and that $\psi(x)$ decreases very fast as $|x - x^0|$ increases. Since u has the physical meaning of the heat distribution (see Introduction) it is clear (from physical experience—and it can also be checked mathematically) that $u(x^0, t)$ does not increase monotonically in t .

Remark 3. Using [3; Theorem 2] we can prove, under the assumptions of Theorem 5, that $u(x, t)$ is strictly increasing in t .

4. Varying Temperature of the Gas. In the previous sections we considered generalized heat transfer problems under the condition that the temperature of

the gas is 1, that is, $g(x, t, 1) \equiv 0$ (sometimes the weaker condition $g(x, t, 1) \leq 0$). In this chapter we shall mainly be concerned with generalized heat transfer problems in the case that the gas temperature varies with time. We shall consider two different situations:

- (a) The temperature of the gas is a function $\varphi(t)$ of the time t ;
- (b) The temperature of the gas depends on t and on the temperature u of the body.

Although the situation (a) might very well occur in nature, the situation (b) is much more common. The precise *a priori* dependence of the gas temperature (near the body) on u should be very complicated and we shall consider here an extremely simple model which perhaps represents a typical situation.

4.1. Problem (a). To discuss the problem (a), we assume that a function $g(x, t, u, v)$ ($(x, t) \in \bar{S}$, $-\infty < u, v < \infty$) is given, and we shall make use of the following assumptions:

- $(\bar{G}_1)$ $g(x, t, u, v)$ is continuous for $(x, t) \in \bar{S}$, $-\infty < u, v < \infty$.
- $(\bar{G}_2)$ $g(x, t, u, v)$ is (strictly) decreasing in u and (strictly) increasing in v .
- $(\bar{G}_3)$ $g(x, t, u, u) \leq 0$ and $g(x, t, u - \alpha, u) > 0$ for some $\alpha > 0$ and for all $(x, t) \in \bar{S}$, $-\infty < u < \infty$.
- $(\bar{G}_4)$ There exists a positive constant A such that

$$|g(x, t, u', v) - g(x, t, u'', v)| \leq A |u' - u''|$$

for all $(x, t) \in \bar{D}$, $|v| \leq \text{l.u.b. } |\varphi(t)|$ and $|u'|, |u''| \leq \max(1, \text{l.u.b. } |\psi|)$.

In what follows we shall assume that $\varphi(t)$, the gas temperature, is a continuous function of t . We now replace (1.3) by

$$(4.1) \quad \frac{\partial u}{\partial T} = g(x, t, u, \varphi(t))$$

and proceed to discuss the extensions of Theorems 1–5.

Theorem 1'. *Theorem 1 remains true for solutions of the system (1.1), (1.2), (4.1) (g is assumed only to satisfy $(\bar{G}_2)$).*

The proof follows word by word the proof of Theorem 1.

Theorem 2'. *Assume that g satisfies $(\bar{G}_2)$, $(\bar{G}_3)$ and that L satisfies (A), (B). If u is a solution of (1.1), (1.2), (4.1) in D_τ and is continuous in $\bar{D}_\tau$, then*

$$(4.2) \quad |u(x, t)| \leq \max \left(\text{l.u.b.}_{0 < t < \tau} \varphi(t), \text{g.l.b.}_{0 < t < \tau} (\varphi(t) - \alpha), \text{l.u.b.}_{x \in B} |\psi(x)| \right).$$

One can easily formulate statements analogous to (a), (b), (c). We mention the following example: If $c(x, t) \equiv 0$ and $\psi(x) \leq \varphi(t)$ for all $x \in \bar{B}$ and $t \geq 0$ then

$$\min_{0 < t < \tau} \{ \text{g.l.b.}_{x \in B} \varphi(t) - \alpha, \text{g.l.b.}_{x \in B} \psi(x) \} \leq u(x, t) \leq \text{l.u.b.}_{0 < t < \tau} \varphi(t).$$

In particular, if $\psi(x) \equiv A \leq \varphi(0)$ and if $\varphi(t)$ increases monotonically with t , then $\min(A, \varphi(0) - \alpha) \leq u(x, t) \leq \varphi(t)$.

The proof of Theorem 2' can be carried out along the lines of proof of Theorem 2 and will be omitted.

Lemma 2'. Let L satisfy (A), (B) and let g satisfy $(\bar{G}_2)$. Let u be a function in D_τ satisfying (1.2), (4.1) and let u' be a function in D_τ satisfying the same system but with φ' and ψ' replacing φ and ψ . If $\varphi' \geq \varphi$, $\psi' \geq \psi$ and if u and u' are continuous in $\bar{D}_\tau$ and $Lu' \leq Lu$ in D_τ , then $u' \geq u$ in D_τ .

The proof of the lemma will be omitted.

With the aid of Theorems 1', 2' we can now proceed to prove the analogues of Theorems 3, 3* and $\bar{3}$ for the system (1.1), (1.2), (4.1), noting (for instance, in proving the analogue of Theorem 3) that as in the proof of Theorem 3 (see (2.8)) we may assume without loss of generality that in each D_σ

$$(4.3) \quad |g(x, t, u, \varphi(t))| \leq A_1$$

where A_1 is independent of R and of the specific u in Z_R . We thus conclude

Theorem 3'. Theorems 3, 3* (and $\bar{3}$) remain true for this system (1.1), (1.2), (4.1) with $g(x, t, u, v)$ satisfying $(\bar{G}_1)$, $(\bar{G}_2)$, $(\bar{G}_3)$ (and $(\bar{G}_4)$) and $\varphi(t)$ continuous.

To generalize Theorem 4 we shall assume that

$$(4.4) \quad \varphi(\infty) = \lim_{t \rightarrow \infty} \varphi(t) \text{ exists.}$$

We shall also need the following assumptions:

$(\bar{G}_3')$ $g(x, t, u, u) = 0$ for all u and all $(x, t) \in \bar{D}$.

$(\bar{G}_4')$ There exist positive constants η, A such that for all $(x, t) \in \bar{D}$, for all $u' > u''$ bounded by $\max(1, \text{l.u.b. } |\psi(x)|)$ and for all v bounded by $\text{l.u.b. } |\varphi(t)|$,

$$-A(u' - u'') \leq g(x, t, u', v) - g(x, t, u'', v) \leq -\eta(u' - u'').$$

Theorem 4'. Let L satisfy (A), (B'), (C) and let g satisfy $(\bar{G}_1)$, $(\bar{G}_2)$, $(\bar{G}_3')$, $(\bar{G}_4')$. If $\varphi(t)$ satisfies (4.4) and if $u(x, t)$ is a solution of (1.1), (1.2), (4.1) in D which is continuous in $\bar{D}$, then

$$\lim_{t \rightarrow \infty} u(x, t) = \varphi(\infty)$$

uniformly with respect to x in $\bar{B}$.

Proof. Let ϵ be an arbitrary positive number. We have to prove that

$$(4.5) \quad |u(x, t) - \varphi(\infty)| < \epsilon \quad \text{if } t \geq t_1, x \in \bar{B};$$

this will complete the proof of the theorem. Because of (4.4) we can find σ such that

$$(4.6) \quad \varphi(\infty) - \epsilon_- \leq \varphi(t) \leq \varphi(\infty) + \epsilon_+ \quad \text{if } t \geq \sigma$$

where $\epsilon_+ = \min(\epsilon, \text{l.u.b. } \varphi(t) - \varphi(\infty))$, $\epsilon_- = \min(\epsilon, \varphi(\infty) - \text{g.l.b. } \varphi(t))$.

Denote by u_+ the solution of the system

$$\begin{aligned} Lu_+ &= 0 \quad \text{in } D - D_\sigma, \\ u_+(x, \sigma) &= \text{l.u.b.}_{\xi \in B} u(\xi, \sigma) \quad \text{for } x \in \bar{B}, \end{aligned}$$

$$\frac{\partial u_+}{\partial T} = g(x, t, u_+, \varphi(\infty) + \epsilon_+) \quad \text{on } S - S_\sigma.$$

The existence of u_+ follows from Theorem $\bar{3}$.

Using Lemma 2' we get

$$(4.7) \quad u(x, t) \leq u_+(x, t) \quad \text{in } D - D_\sigma.$$

We can apply Theorem 4 (with 1 replaced by $\varphi(\infty) + \epsilon_+$) to the function u_+ and thus get

$$(4.8) \quad u_+(x, t) - \varphi(\infty) - \epsilon_+ = O(\exp \{-\lambda_1 t\}) \quad (\lambda_1 > 0).$$

Combining (4.7) with (4.8) we get

$$(4.9) \quad u(x, t) < \varphi(\infty) + \epsilon_+ + O(\exp \{-\lambda_1 t\}).$$

In a similar manner we can prove that

$$(4.10) \quad u(x, t) > \varphi(\infty) - \epsilon_- + O(\exp \{-\lambda_2 t\}) \quad (\lambda_2 > 0).$$

Combining (4.9) with (4.10) and taking t_1 to be sufficiently large, the proof of (4.5) is completed.

Remark. The above proof can easily be modified to show that if $A_1 \leq \varphi(t) < A_2$, then for any $\epsilon > 0$ there exists t_1 such that $A_1 - \epsilon \leq u(x, t) \leq A_2 + \epsilon$ for $x \in \bar{B}$, $t > t_1$.

In extending Theorem 5 we shall clearly assume that $\varphi(t)$ is monotone in t , otherwise (as physical experience shows—it can also be checked mathematically) $u(x, t)$ need not be monotone in t . We shall need the following assumption:

$(\bar{G}_5)$ $g(x, t, u, v)$ is non-decreasing in t .

Theorem 5'. Let the coefficients of L be independent of t and satisfy (A), (B). Let g satisfy $(\bar{G}_1)$, $(\bar{G}_2)$, $(\bar{G}_3')$, $(\bar{G}_5)$. Finally, let $\varphi(t)$ be monotone increasing in t and let $\psi(x) = \text{const.} \leq \varphi(0)$. If $u(x, t)$ is a solution of (1.1), (1.2), (4.1) and is continuous in $\bar{D}$, then it is non-decreasing in t .

Remark. Theorem 5' remains true if $(\bar{G}_3')$ is replaced by the weaker assumption $(\bar{G}_3)$, provided $\alpha \leq \varphi(0) - \psi$ (ψ being a constant function).

The proof of Theorem 5' is similar to the proof of Theorem 5 via Theorem 2' and will be omitted.

Remark. Theorems 1'–5' can easily be extended to the case that $\varphi = \varphi(x, t)$ is a continuous function in (x, t) on $\bar{S}$.

4.2. **Problem (b).** We shall consider the case where $\psi(x) \equiv 0$ and the following simple model of (b):

(Φ) The temperature of the gas at each point x at the surface $\bar{B}$ of the body and at each time t is given by

$$(4.11) \quad \varphi = \varphi(t, u) = 1 - \rho(t)u$$

where u is the temperature of the body at (x, t) and $\rho(t)$ is a continuous function of t which satisfies:

$$(4.12) \quad \rho(0) = 0, \quad \rho(t) \geq 0 \quad \text{for all } t \geq 0, \quad \lim_{t \rightarrow \infty} \rho(t) = 0.$$

The assumption (Φ) corresponds to the case that the original temperature of the gas (before the body was inserted) was 1. The cooling of the gas by the body at each point (x, t) is $-\rho(t)u(x, t)$. When $t = 0$ there is not yet any cooling ($\rho(0) = 0$) while as $t \rightarrow \infty$ we would expect the temperature u to tend to that of the gas if the volume of the gas is large, so the cooling effect will reduce to zero ($\lim_{t \rightarrow \infty} \rho(t) = 0$ as $t \rightarrow \infty$).

We shall consider the following system:

$$(4.13) \quad \begin{aligned} Lu &= 0 \quad \text{in } D, \\ u(x, 0) &= 0 \quad \text{on } \bar{B}, \\ \frac{\partial u}{\partial T} &= g(x, t, u, 1 - \rho(t)u) \end{aligned}$$

where $g(x, t, u, v)$ is the function which appears in (4.1).

Using the fact that $\rho(t) \geq 0$ we can easily extend Theorem 1' to the present case.

To extend Theorem 2' we first note that $u(x, t) > 0$ if $t > 0$. Indeed, in the contrary case we have a point P on S_τ where u takes its nonpositive minimum and where $\partial u / \partial T \leq 0$. On the other hand, using (4.13), the properties $(\bar{G}_2)$, $(\bar{G}_3)$ with $\alpha = 1$ and (4.12) we get

$$\frac{\partial u}{\partial T} = g(x, t, u, 1 - \rho(t)u) > 0$$

and we arrive at a contradiction.

In a similar way we can prove that $u(x, t) \leq 1$ if $t > 0$. Thus if L satisfies (A), (B) and g satisfies $(\bar{G}_2)$, $(\bar{G}_3)$ with $\alpha = 1$ and if (Φ) is satisfied, then any solution $u(x, t)$ of (4.13) in D_τ which is continuous in $\bar{D}_\tau$ must satisfy

$$0 < u(x, t) \leq 1 \quad \text{for } 0 < t \leq \tau, \quad x \in \bar{B}.$$

If $\tau = \infty$ then we have the stronger inequality $u(x, t) < 1$ in $\bar{D}$.

The proof of the last statement follows on using [3; Theorem 2].

Using the extensions of Theorems 1', 2' which we have just proved we can

proceed to extend Theorem 3' to the system (4.13). The proof follows almost word by word the proof of Theorem 3'.

Theorem 4' can also be very easily extended to the system (4.13). Indeed, since $0 < u < 1$,

$$(4.14) \quad \varphi = \varphi(x, t) = 1 - \rho(t)u(x, t) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly in $x \in B$ and hence the proof of Theorem 4' can be used to conclude that for every $\epsilon > 0$

$$(4.15) \quad 0 < 1 - u(x, t) \leq \epsilon + O(\exp \{-\lambda t\}) \quad \text{if } t \geq t_1, \quad x \in \bar{B}$$

where t_1 depends on ϵ and $\lambda > 0$ is independent of ϵ .

Theorem 5' can also be extended to the present situation provided we assume that

$$\text{if } u < w \text{ then } g(x, t, u, 1 - \rho(t)u) > g(x, t, w, 1 - \rho(t)w)$$

and that $g(x, t, u, v)$ is non-decreasing in t .

4.3. Partial Transfer of Heat. We shall briefly mention the case where heat transfer takes place only on a part $\dot{B}_1$ of $\dot{B}$ while the temperature on the complementary part $\dot{B}_2$ is given *a priori*; denote it by $k(x, t)$. We shall also make the assumption that $\dot{B}_2$ is a portion of an hyperplane, that k vanishes near the boundary of $\dot{B}_2$, and that the coefficients of L are defined and sufficiently smooth in a cylinder with the base E to be defined below.

Theorems 1, 2 can easily be extended. To extend Theorem 3 we denote by E an n -dimensional rectangle which contains B and one of whose faces contains $\dot{B}_2$. Denote by $G(x, t; \xi, \tau)$ the Green's function (constructed in [4]) corresponding to L in the cylinder with base E . We then replace the right side of (2.4) in the proof of Theorem 3 by

$$\begin{aligned} & \int_0^t \int_{\dot{B}_1} G(x, t; \xi, \tau) h(\xi, \tau) dW_\xi d\tau \\ & + \int_0^t \int_{\dot{B}_2} \frac{\partial}{\partial T} G(x, t; \xi, \tau) k(\xi, \tau) dW_\xi d\tau + \varphi(x, t) \end{aligned}$$

where φ is defined by (2.1) with Γ replaced by G and $\partial/\partial T$ is the transversal derivative with respect to (ξ, τ) .

We also modify the definition of the Banach space Z . Z is the set of all functions u which are continuous in $\bar{D}_\sigma$ and which satisfy $u(x, t) = k(x, t)$ for $x \in \dot{B}_2$, $0 \leq t \leq \sigma$.

With the above modifications it is now easy to extend the proof of Theorem 3 to the present situation.

We can similarly extend Theorems 1', 2', 3', 3*, $\bar{3}$ to the present case.

We can also extend Theorem 4'. We have the following result: If

$$\lim_{t \rightarrow \infty} \varphi(t) = A, \quad \lim_{t \rightarrow \infty} k(x, t) = A \quad \text{uniformly in } x \in \bar{B},$$

then the solution $u(x, t)$ (which satisfies $\partial u / \partial T = g$ for $x \in \dot{B}_1$, $t > 0$ and $u = k$ for $x \in \dot{B}_2$, $t > 0$) satisfies

$$\lim_{t \rightarrow \infty} u(x, t) = A \quad \text{uniformly in } x \in \bar{B}.$$

5. Non-homogeneous Equations. In this chapter we generalize results obtained in previous chapters to the case of the non-homogeneous equation

$$(5.1) \quad Lu = f(x, t).$$

The uniqueness results (Theorems 1, 1') clearly hold for solutions of (5.1).

We proceed to generalize Theorem 2, assuming that L satisfies (A), (B) and that $g(x, t, u)$ satisfies (G_2) , (G_3) . We first introduce a function $\zeta(x)$ which satisfies (for some $\alpha > 0$)

$$(5.2) \quad \begin{aligned} \zeta(x) &\geq \alpha > 0 \quad \text{on } B, \\ L\zeta(x) &\leq -1 \quad \text{in } D_\tau \end{aligned}$$

and let

$$(5.3) \quad \left| \frac{\partial \zeta(x)}{\partial T} \right| \leq \beta < \infty \quad \text{on } S_\tau.$$

We may, for instance, take $\zeta(x) = \gamma(e^{\delta R} - e^{\delta x_1})$ where γ and δ are appropriate constants and R is such that $x_1 < R$ for all $x = (x_1, \dots, x_n)$ in $\bar{B}$.

Define

$$(5.4) \quad K = \max \left(\text{l.u.b.}_D |f(x, t)|, \frac{1}{\alpha} \text{l.u.b.}_B |\psi(x)| \right).$$

Then the function $w(x, t) = K\zeta(x) - u(x, t)$ satisfies

$$(5.5) \quad Lw(x, t) \leq 0 \quad \text{in } D_\tau,$$

$$(5.6) \quad w(x, 0) \geq 0 \quad \text{on } \bar{B},$$

$$(5.7) \quad \frac{\partial w}{\partial T} = K \frac{\partial \zeta}{\partial T} - g(x, t, K\zeta - w) \equiv \tilde{g}(x, t, w) \quad \text{on } S_\tau.$$

We shall assume that the following (physically obvious) condition is satisfied:

$$(5.8) \quad \lim_{u \rightarrow \pm \infty} g(x, t, u) = \mp \infty \quad \text{uniformly in } (x, t) \in \bar{D}_\tau.$$

It then follows from (5.7) that

$$(5.9) \quad \tilde{g}(x, t, w) \geq 0 \quad \text{if } w < -H$$

where H is sufficiently large (depending on g , K and β).

The method used to prove Theorem 2 can be applied to the present situation, on making use of (5.5), (5.6), (5.7) and (5.9). We then conclude that

$$w(x, t) > -H \quad \text{in } D_\tau.$$

Using the definition of w we get

$$(5.10) \quad u(x, t) \leq H + K\zeta(x) \leq H' \quad \text{in } D_\tau.$$

In a similar manner we obtain $u(x, t) \geq -H''$ in D_τ . Defining $H_0 = \max(H', H'')$, we have thus proved

Theorem 2''. *Let L satisfy (A), (B) and let $g(x, t, u)$ satisfy (G_2) , (G_3) and (5.8). If $u(x, t)$ is a solution in D_τ of (5.1), (1.2), (1.3) and if it is continuous in $\bar{D}_\tau$, then*

$$(5.11) \quad |u(x, t)| \leq H_0 \quad \text{in } D_\tau$$

where H_0 depends only on g , the diameter of B , l.u.b. $|f(x, t)|$ in D_τ , l.u.b. $|\psi(x)|$ in B , upper bounds on the coefficients $a_{11}(x, t)$, $-b_1(x, t)$ of L in D_τ and g.l.b. $a_{11}(x, t)$ in D_τ .

Theorem 2'' can easily be generalized to the case where the temperature of the gas $\varphi(t)$ varies in time.

With the aid of the uniqueness theorem and Theorem 2'' we can now proceed to prove the existence of a solution in D of the system (5.1), (1.2), (1.3) provided that, in addition to the assumptions made in the previous existence theorems, we assume also that g satisfies (5.8) and that $f(x, t)$ is Hölder continuous in compact subsets of $\bar{D}$. The proof of the extension of Theorem 3 is similar to the proof of Theorem 3 with one modification: In defining the transformation $v = Tu$ we add to the right side of (2.4) the generalized potential of volume distribution

$$-\int_0^t \int_B 2\rho(\xi, \tau) \Gamma(x, t; \xi, \tau) f(\xi, \tau) d\xi d\tau$$

whose properties have been studied in [12]. Theorems 3*, $\bar{3}$, 3' can also be generalized to the non-homogeneous equation (5.1). In what follows we shall refer to these extensions of Theorems 3, 3*, $\bar{3}$, 3' as Theorem 3''.

To prove an extension of Theorem 4' we shall need the following assumptions:

$(\bar{G}_4'')$ For any $K > 0$ there exist positive constants η, A such that for all (x, t) in $\bar{D}$, $|v| \leq \text{l.u.b. } |\varphi(t)|$ and $u' > u'', |u'| \leq K, |u''| \leq K$

$$-A(u' - u'') \leq g(x, t, u', v) - g(x, t, u'', v) \leq -\eta(u' - u'').$$

$(\bar{G}_6)$ For every fixed v

$$\lim_{u \rightarrow \pm\infty} g(x, t, u, v) = \mp\infty$$

uniformly with respect to (x, t) in compact subsets of $\bar{D}$.

Theorem 4''. *Let L satisfy (A), (B'), (C). Let $g(x, t, u, v)$ satisfy $(\bar{G}_1)$, $(\bar{G}_2)$, $(\bar{G}_3)$, $(\bar{G}_4'')$, $(\bar{G}_6)$. Let $f(x, t)$ be a bounded function in $\bar{D}$ which satisfies*

$$(5.11) \quad \lim_{t \rightarrow \infty} f(x, t) = 0 \quad \text{uniformly in } x \in \bar{B}$$

and assume that

$$\varphi(\infty) = \lim_{t \rightarrow \infty} \varphi(t) \text{ exists.}$$

If $u(x, t)$ is a solution in D of the system (5.1), (1.2), (4.1) and if it is continuous in $\bar{D}$, then

$$(5.12) \quad \lim_{t \rightarrow \infty} u(x, t) = \varphi(\infty) \text{ uniformly in } x \in \bar{B}.$$

Proof. Let ϵ be an arbitrary positive number and let σ be such that

$$(5.13) \quad |f(x, t)| < \epsilon \text{ if } t \geq \sigma, \quad x \in \bar{B},$$

$$(5.14) \quad |\varphi(t) - \varphi(\infty)| < \epsilon \text{ if } t \geq \sigma.$$

Let $|u(x; \sigma)| \leq C_1$ for $x \in \bar{B}$ and denote by v the solution of the system

$$(5.15) \quad \begin{aligned} Lv &= -\epsilon \text{ in } D - \bar{D}_\sigma, \\ v(x, 0) &= C_1 \text{ for } x \in \bar{B}, \\ \frac{\partial v}{\partial T} &= g(x, t, v, \varphi(\infty) + \epsilon_+) \text{ on } S - \bar{S}_\sigma, \end{aligned}$$

where $\epsilon_+ = \min(\epsilon, \text{l.u.b. } \varphi(t) - \varphi(\infty))$. The existence of v follows from Theorem 3''. By a simple comparison argument we derive the inequality

$$(5.16) \quad u(x, t) \leq v(x, t) \text{ in } D - D_\sigma,$$

and we have to estimate v . To do that define

$$(5.17) \quad w = v + \epsilon \vartheta(x)$$

where $\vartheta(x)$ is any function which satisfies $L\vartheta \geq 1$.

Because of (5.15) w satisfies the system

$$(5.18) \quad \begin{aligned} Lw &\geq 0 \text{ in } D - \bar{D}_\sigma, \\ w(x, \sigma) &\leq C_2 \text{ for } x \in \bar{B} \quad (C_2 = C_1 + \epsilon \text{ l.u.b. } |\vartheta(x)|), \\ \frac{\partial w}{\partial T} &= g(x, t, w - \epsilon \vartheta, \varphi(\infty) + \epsilon_+) + \epsilon \frac{\partial \vartheta}{\partial T} \equiv g_0(x, t, w) \text{ on } S - \bar{S}_\sigma. \end{aligned}$$

Using $(\bar{G}_4')$ it follows that there exists a sufficiently large constant N such that

$$(5.19) \quad g_0(x, t, w) \leq g(x, t, w - N\epsilon, \varphi(\infty) + \epsilon_+).$$

We now denote by $s(x, t)$ the solution of the system

$$(5.20) \quad \begin{aligned} Ls &= 0 \text{ in } D - \bar{D}_\sigma, \\ s(x, \sigma) &= C_2 \text{ for } x \in \bar{B}, \\ \frac{\partial s}{\partial T} &= g(x, t, s - N\epsilon, \varphi(\infty) + \epsilon_+). \end{aligned}$$

The existence of s follows from Theorem $\bar{3}$.

Using (5.19) and applying Lemma 2' we get

$$(5.21) \quad w(x, t) \leq s(x, t) \quad \text{in } D - D_\sigma.$$

Applying Theorem 4 to the solution $s(x, t)$ of the system (5.20) we get

$$(5.22) \quad s(x, t) - (\varphi(\infty) + \epsilon_+ + N\epsilon) = O(\exp \{-\mu t\}) \quad (\mu > 0)$$

provided $t \geq t'$ and t' is sufficiently large.

Combining (5.22) with (5.21), (5.17) and (5.16) we get

$$(5.23) \quad u(x, t) \leq \varphi(\infty) + C_3\epsilon \quad (C_3 > 0)$$

provided $t \geq t_1$ and t_1 is sufficiently large (C_3 is independent of ϵ and t_1). In a similar way we get $u(x, t) \geq \varphi(\infty) - C_4\epsilon$, and the proof of Theorem 4'' is thereby completed.

We remark that the proof of Theorem 4'' remains valid under weaker assumptions than $(\bar{G}_4')$ and $(\bar{G}_6)$. Thus it is enough to assume that the inequalities in $(\bar{G}_4')$ hold only for an appropriate large K and, instead of assuming $(\bar{G}_6)$, to assume only that g becomes sufficiently large (small) for some $u > 0$ ($u < 0$).

Theorem 4'' was proved under the assumption $c(x, t) \equiv 0$. We can now generalize Theorem 4'' by replacing this assumption by the weaker assumption

$$(5.24) \quad \lim_{t \rightarrow \infty} c(x, t) = 0 \quad \text{uniformly in } x \in \bar{B}.$$

Indeed, we have just to transfer $c(x, t)u(x, t)$ from the left side of (5.1) to the right side, note (on using Theorem 2'') that $\lim_{t \rightarrow \infty} c(x, t)u(x, t) = 0$ and apply Theorem 4''.

We can similarly extend Theorems 4, 4' to the case that $c(x, t)$ satisfies (5.24). We remark that if $\lim_{t \rightarrow \infty} c(x, t) \neq 0$ and if $\varphi(\infty) \neq 0$ then $v(x) \equiv \varphi(\infty)$ is not a solution of the system $(\lim_{t \rightarrow \infty} L)v = 0$ in B , $\partial v / \partial T = 0$ on $\bar{B}$ and therefore we should not expect that $\lim_{t \rightarrow \infty} u(x, t) = \varphi(\infty)$.

Theorem 5' can also be extended to the case of the non-homogeneous equation (5.1), provided f satisfies

$$f(x, t) \leq 0, \quad f(x, t + h) \leq f(x, t)$$

for all $(x, t) \in \bar{D}$ and $h > 0$.

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University of California
Berkeley, California
and
Indiana University
Bloomington, Indiana