Self-intersections of Immersed Manifolds

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Introduction. An immersion $f: M^k \rightarrow X^{k+r}$ of one C^∞ manifold into a second is called *n-normal* if for each n -tuple of distinct points $x_1, \dots, x_n$ of M with $f(x_1) = \dots = f(x_n)$ the images $\tilde{M}_{x_i}$ of the tangent spaces M_{x_i} under the differential of f have the minimum possible intersection in $X_y, y = f(x_i)$. Explicitly, it is required that $\dim \bigcap_{i=1}^n \tilde{M}_{x_i} = k - (n-1)r$. We will prove that any immersion of a closed manifold can be C_s approximated (any s) by an n -normal immersion.

If $f: M \rightarrow X$ is n -normal and $(f)^n: (M)^n \rightarrow (X)^n$ is the n -fold product map of f , then the restriction F of $(f)^n$ to the subspace of distinct n -tuples of the n -fold product space $(M)^n$ is t -regular in the sense of THOM [7] on the diagonal Δ of $(X)^n$. Then $F^{-1}(\Delta) = \Sigma_n$ is a manifold of dimension $k - (n-1)r$ which we call the n -self-intersection manifold of f . The reason for this terminology is as follows. Let $\pi_1: (M)^n \rightarrow M$ be the projection onto the first factor. Then π_1 restricted to Σ_n is an immersion $\pi_1: \Sigma_n \rightarrow M$ and $\pi_1(\Sigma_n)$ is the set of points of M which are mapped n (or more) to one by f . That is,

$$\pi_1(\Sigma_n) = \{x \in M; \exists x_2, \dots, x_n \in M, \text{ distinct, and } f(x) = f(x_i)\}.$$

If M is closed then Σ_n is closed; if M and X are orientable then so is Σ_n .

Assume now that M , and hence also Σ_n , is closed. Denote the image of the orientation of Σ_n under $\pi_{1*}: H_{k-(n-1)r}(\Sigma_n) \rightarrow H_{k-(n-1)r}(M)$ by Σ_n^* (use coefficients from Z_2 if Σ_n is non-orientable, otherwise use integer coefficients). We are able to compute this class as follows: Consider the composition

$$H_k(M) \xrightarrow{f^*} H_k(X) \xrightarrow{\lambda} H^r(X) \xrightarrow{f^*} H^r(M)$$

where λ is Poincaré duality (using cohomology with compact supports if X is not closed). Let M denote the orientation of M and $\tau = f^*\lambda f_*(M)$. If M is non-orientable then τ is defined with coefficients from Z_2 . Let $\bar{W}^r \in H^r(M)$ denote the r^{th} Stiefel-Whitney class of the normal bundle of M in X . It is an integral class unless this normal bundle is non-orientable or $r = 1$, in which cases it is defined with coefficients from Z_2 . We will prove

Theorem I: If $f: M^k \rightarrow X^{k+r}$ is an n -normal immersion of a closed, connected manifold M , then

$$(F_n) \quad \lambda \Sigma_n^* = \pm(\tau - \bar{W}^r)^{n-1}.$$

Here λ is Poincaré duality and the product on the right is the cup product. The formula is meant with integer coefficients unless $r = 1$ or M or X is non-orientable in which cases coefficients are taken mod 2. (The sign on the right will be $(-1)^{k(n-1)(k+r)}$ for our choice of orientation of Σ_n .)

WHITNEY in [10, p. 131] has given an early formulation of equation (F_n) in the orientable case when $n = 2$ and X is closed. In place of our τ he has the symbol $\{M', M\}$ which he defines geometrically. In place of Σ_2 , WHITNEY writes $D(M)$, "the distant intersection of M with itself". He speaks of $D(M)$ as a submanifold of M ; however, $D(M)$ may only be an immersed manifold in general.

In case X^{k+r} is Euclidean space E^{k+r} , it is an immediate consequence of its definition that $\tau = 0$, and we get, from Theorem I:

If $X^{k+r} = E^{k+r}$, then (F_n) becomes

$$(F'_n) \quad \lambda \Sigma_n^* = \pm(\bar{W}^r)^{n-1}.$$

In particular, the set $\pi_1(\Sigma_2)$ is the set of all points of M which are not mapped 1-1 by f , and we have

$$(F'_2) \quad \lambda \Sigma_2^* = \pm \bar{W}^r.$$

(For the case of (F'_2) where $r = k$, see also [4] and [11].)

Since $(\bar{W}^r)^2 = \bar{P}^{2r}$, r even, is the normal Pontrjagin class of the immersion (see [12]), we have

$$(F'_3) \quad \lambda \Sigma_3^* = \pm \bar{P}^{2r}, \quad r \text{ even.}$$

In the case $r = \frac{1}{2}k$, Σ_3 consists of isolated points, i.e. triple points, and $\bar{P}^{2k} \in H^k(M)$ is determined by duality theorems from the Pontrjagin classes of M [3]. Thus

Corollary A. *The number of triple points of an immersion $f: M^k \rightarrow E^{\frac{3}{2}k}$ of a closed manifold oriented with proper sign is independent of the immersion.*

(The sign of the triple points is determined as in 1.2. It is easy to obtain from this a geometric rule for attaching a $+$ or $-$ sign to an isolated triple point depending on how the pieces of M come together.)

This behavior of triple points contrasts with the number of double points of an immersion of M^k in E^{2k} , see [4] or [11].

The following example shows that the preceding theorems have some content. M. HIRSCH [2] has proved that the real projective 6-space P^6 can be immersed in E^7 . From the Whitney duality theorem and equation (F'_2) follows

Corollary B. *For every 7-normal immersion f of P^6 in E^7 , there exists $y \in E^7$ such that $f^{-1}(y)$ has at least 7 points.*

If $f: M^k \rightarrow X^{k+r}$ is 2-normal and at most 2 to 1 (e.g., if $r > \frac{1}{2}k$ and f is 3-normal), then Σ_2 is imbedded in M . The immersion f restricted to $\pi_1(\Sigma_2) = \Sigma_2 \subset M$ is a double covering. Let $\bar{\Sigma}_2$ be the image space $f(\Sigma_2)$. We will prove

Theorem II. *If $f: M^k \rightarrow X^{k+r}$ is 2-normal and at most 2 to 1, then Σ_2 is a non-orientable manifold when r is odd and an orientable manifold when r is even.*

In most cases it is very difficult to prove imbedding theorems from immersion theorems. However, from Theorem II it is very easy to prove the special theorem

Theorem III. *If k is even, then near any immersion of M^k in $E^{2k-1} \subset E^{2k}$, there is an imbedding of M in E^{2k} .*

WHITNEY has proved that every manifold M^k can be immersed in E^{2k-1} (for a modern proof see HIRSCH [2]).

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1. Normal immersions. If $M \rightarrow M^k$ is a manifold of dimension k , M_x will denote the tangent space of M at x . All manifolds considered will be C^∞ with a countable base, but not necessarily closed or connected. The following definition of t -regular has been used by THOM [7]. If $N = N^q$ is a submanifold of $X = X^n$ and $f: M \rightarrow X$ is a C^∞ map, f is said to be t -regular on N if the composition

$$M_x \xrightarrow{f^*} X_x \rightarrow X_x/N_x$$

is onto for all $y \in N$ and $x \in f^{-1}(y)$. If f is t -regular on N , then $Q = f^{-1}(N)$ is a submanifold of M of the same codimension ($n - q$) as N . Moreover, f induces a map of a tubular neighborhood of Q onto a tubular neighborhood of N which can be considered as a bundle map of normal vectors over Q into the space of normal vectors over N . (For this purpose one uses a Riemannian metric and constructs the tubular neighborhood from geodesics orthogonal to the submanifold and hence corresponding to normal vectors.)

A C^∞ map $f': M \rightarrow X$ is an (r, ϵ) -approximation of another C^∞ map $f: M \rightarrow X$ if, roughly speaking, f' and the first r derivatives of f' are within ϵ of f and the first r derivatives of f respectively. See WHITNEY [9] for a precise definition. WHITNEY calls such a map an (f, η, M, r) -approximation.) An n -tuple, of C^∞ maps, $f'_i: M \rightarrow X, i = 1, 2, \dots, n$, is an (r, ϵ) -approximation of a second n -tuple $f_i: M \rightarrow X$, if, for each i , f'_i is an (r, ϵ) -approximation of f_i .

The following theorem can be found in THOM [7]. See also WHITNEY [9] for the case of an immersion.

Theorem 1.1. *Let $f: M \rightarrow X$ be a C^∞ map, N a submanifold of X and $\epsilon > 0$. Then there exists an (r, ϵ) -approximation (any r) of f , say $f': M \rightarrow X$, such that f' is t -regular on N and f' agrees with f except possibly on $f^{-1}(V_\epsilon(N))$, where $V_\epsilon(N)$ is an ϵ neighborhood of N .*

The basic class of a closed connected orientable manifold $M = M^k$ is a choice of generator of $H_k(M)$. We usually denote this class by M itself. The basic class of a closed connected non-orientable manifold M is the non-zero element of $H_k(M, \mathbb{Z}_2)$. The basic class of a non-connected closed manifold M^k is the direct sum of the basic classes of the components, if each component is an orientable manifold; otherwise it is the direct sum of the generators of $H_k(M, \mathbb{Z}_2)$.

Lemma 1.2. *Let $f: M \rightarrow X$ be C^∞ and t -regular on a manifold $N \subset X$, where M , X and N are oriented. Then the orientations of M , X and N determine an orientation of $Q = f^{-1}(N)$.*

Proof. The map $f: M \rightarrow X$ induces $f_*: M_q \rightarrow X_p$, $q \in Q$, $p = f(q) \in N$. That f is t -regular means the quotient map $M_q/Q_q \rightarrow X_p/N_p$ is an isomorphism onto. From the orientations of X_p and N_p we get an orientation of X_p/N_p and hence M_q/Q_q . Then, since M_q is oriented, an orientation of Q_q is determined. This in turn gives an orientation of Q . (To be precise, take an ordered basis for N_p , the order determined by the orientation of N , and extend it to an ordered basis for X compatible with the orientation of X . The image of this basis in X_p/N_p is an ordered basis which determines the orientation of the factor space. An ordered basis for Q is then chosen so as to give the same result for M_q/Q_q .)

An orientation of a manifold determines a basic class in a natural fashion. Under the conditions of 1.2 we will always suppose Q is oriented as in this proof, and therefore the basic class of Q is determined.

The following theorem is known.

Theorem 1.3: *Let $f: M^n \rightarrow X^p$ be a proper C^∞ map, t -regular on a closed submanifold $N^{p-q} \subset X$, with M and X connected. Let $Q = f^{-1}(N)$ and write N^* and Q^* for the images of the basic classes of N and Q in X and M respectively. Then*

$$\lambda Q^* = f^* \lambda N^*.$$

Here integer coefficients are meant in the case that M , X , N and hence Q (by 1.2) are oriented. Otherwise coefficients are to be in $\mathbb{Z}_2$. The symbol λ always stands for Poincaré duality between singular homology and Čech cohomology with compact supports, in both M and X .

Proof. Let V_0 and V'_0 be tubular neighborhoods of Q and N respectively, and, using a Riemannian metric on M and X , we consider $f: V_0 \rightarrow V'_0$ to be a bundle map. Consider the following diagram (cohomology with compact supports and singular homology):

$$\begin{array}{ccccccc}
 H_{n-q}(Q) & \xleftrightarrow{\lambda_Q} & H^0(Q) & \xleftarrow{f^*} & H^0(N) & \xleftrightarrow{\lambda_N} & H_{p-q}(N) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_{n-q}(V_0) & \xleftrightarrow{\lambda_{V_0}} & H^q(V_0) & \xleftarrow{f^*} & H^q(V'_0) & \xleftrightarrow{\lambda_{V'_0}} & H_{p-q}(V'_0) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_{n-q}(M) & \xleftrightarrow{\lambda_M} & H^q(M) & \xleftarrow{f^*} & H^q(X) & \xleftrightarrow{\lambda_X} & H_{p-q}(X)
 \end{array}$$

Here ϕ and ϕ' are isomorphisms as in THOM [6], and the other vertical maps are all inclusions. The subscripts on λ indicate duality in the different spaces. Various naturality theorems show that each smaller square commutes (e.g. see [1] and [6]). The theorem is proved by commutativity of the large rectangle: Starting from the upper right hand corner, the basic class of N is mapped across the top row onto the basic class of M by 1.2 and the fact that f^* takes the sum of the generators of $H^0(N)$ onto the sum of the generators $H^0(Q)$. These basic classes then map down into $N^* \in H_{p-q}(X)$ and $Q^* \in H_{m-q}(M)$ respectively. Commutativity then implies that $f^*\lambda_X N^* = \lambda_M Q^*$.

An immersion $f: M^k \rightarrow X^{k+r}$ is a C^∞ map with Jacobian of rank k at every point. A property P of n -tuples of immersions $f_i: M_i \rightarrow X$, $i = 1, \dots, n$, with M closed, will be called *generic* (after THOM [8]) if both the following conditions hold:

- (1) Given an n -tuple $f_i: M_i \rightarrow X$, there exists an $\epsilon > 0$ such that every $(2, \epsilon)$ -approximation of $\{f_i\}$ has property P .
- (2) Given any n -tuple $f_i: M_i \rightarrow X$ and any $\epsilon > 0$, there exists an (r, ϵ) -approximation of $\{f_i\}$ by an n -tuple with property P . (That the M_i be closed is not necessary, it only simplifies matters.)

In the case $n = 1$, generic describes properties of immersions, and this case will be our main concern. We proceed to describe one such property; the proof that it is generic will be given later (1.5).

If $f: M \rightarrow X$ is an immersion of one manifold in a second, let $\bar{M}_x$ denote the image of M_x , $x \in M$, under the differential f_* of f . Then an immersion $f: M^k \rightarrow X^{k+r}$ is called *n-normal* if, for any n distinct points $x_1, \dots, x_n \in M$ with $f(x_1) = \dots = f(x_n)$,

$$\dim \bigcap_{i=1}^n \bar{M}_{x_i} = k - (n - 1)r.$$

Instead of 2-normal we may say simply *normal*. An immersion is *completely normal* if it is n -normal for every n .

Let $(M)^n$ denote the n -fold Cartesian product of M with itself, Δ the diagonal of $(X)^n$ and $R = \{(x_1, \dots, x_n) \in (M)^n \mid x_i = x_j \text{ for some distinct } i \text{ and } j\}$.

Lemma 1.4. *The n -fold product map of an immersion $f: M \rightarrow X$, restricted to $(M)^n - R$, is t -regular on Δ if and only if f is n -normal.*

Proof. Let $F: (M)^n - R \rightarrow (X)^n$ be the restriction of $(f)^n$. The map f is n -normal if and only if $\dim \bigcap \bar{M}_{x_i} = k - (n - 1)r$ for all n -tuples $x_1, \dots, x_n$ of distinct points of M with $f(x_1) = \dots = f(x_n)$. Let $x_1, \dots, x_n$ be such an n -tuple with $f(x_i) = y$. Let V be the set of vectors Y of $(M)_{(x)}$, $(x) = (x_1, \dots, x_n)$, with $F_* Y \in \Delta_{(y)}$, $(y) = (y, y, \dots, y)$, and let $\bar{V} = F_*(V)$. Then $\dim \bigcap \bar{M}_{x_i} = k - (n - 1)r$ if and only if $\dim \bar{V} = k - (n - 1)r$. Since F_* is an immersion, $\dim V = \dim \bar{V}$. Now F is t -regular on Δ at $(x_1, \dots, x_n)$ means that $nk - \dim V = n(k + r) - \dim \Delta = (n - 1)(k + r)$ or $\dim V = k - (n - 1)r$. Putting these statements together we obtain 1.4.

If f is n -normal (using 1.4) we define $\Sigma_n = F^{-1}(\Delta)$, where $F: (M)^n \rightarrow R \rightarrow (X)^n$ is the restriction of the n -fold product map, to be the n -self-intersection manifold of f . It has the following properties:

- (1) Σ_n is a manifold of dimension $k - (n - 1)r$.
- (2) If $\pi: (M)^n \rightarrow M$ is projection onto the first factor then π restricted to Σ_n is an immersion.

Proof. We will show that the differential π_* is 1-1. Let $\pi: (M)^n \rightarrow R \rightarrow M$ and $\pi': (X)^n \rightarrow X$ be projections onto the first factor. Then $\pi'F = f\pi$. Now $F(\Sigma_n) \subset \Delta$ and π'_* restricted to Δ is 1-1. Since F_* is 1-1 this implies π_* is 1-1.

(3) $\pi(\Sigma_n) \subset M$ is the set of points of M which are mapped n or more to 1 by f . In particular, $\pi(\Sigma_2)$ is the set of points at which f is not 1-1.

(4) If $r > k/n$ then $\dim \Sigma_{n+1} < 0$, and hence f is at most n to 1.

(5) Let $p_n: \Sigma_n \rightarrow \Sigma_{n-1}$, $p_n(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. Then p_n is 2-normal, if f is $(n + 1)$ -normal. Denote by K_n the self-intersection manifold of p_n ($K_n \subset \Sigma_n \times \Sigma_n$). Then there exists a diffeomorphism $\alpha_n: K_n \rightarrow \Sigma_{n+1}$ such that $\pi\alpha_n = \pi_1$, where π_1 is projection onto the first coordinate of $\Sigma_n \times \Sigma_n$ followed by $\pi: \Sigma_n \rightarrow M$. In particular, if Σ_3 is empty then Σ_2 is imbedded by π in M .

Proof. The fact that p_n is 2-normal is easily checked. To define α_n , let $[(x_1, \dots, x_n), (y_1, \dots, y_n)] \in K_n$. Then

$$\alpha_n[(x_1, \dots, x_n), (y_1, \dots, y_n)] = (x_1, \dots, x_n, y_n).$$

Note that since $p_n(x_1, \dots, x_n) = p_n(y_1, \dots, y_n)$, $x_i = y_i$, $i = 1, \dots, n - 1$. The details are left to the reader.

- (6) n -normal is generic for closed M . (See 1.5 for proof.)
- (7) Σ_n is a closed subset of $(M)^n$, and hence Σ_n is closed if M is closed.
- (8) Σ_n is orientable if M and X are orientable. (This follows from 1.2.)

An n -tuple of immersions $f_i: M_i^{k_i} \rightarrow X^s$, $i = 1, \dots, n$, is called *mutually normal* if for each n -tuple of points $x_1, \dots, x_n$, such that $x_i \in M_i$ and $f_1(x_1) = \dots = f_n(x_n)$, $\dim \bigcap_{i=1}^n f_{i*}(M_{x_i}) = \sum_i k_i - (n - 1)s$.

Let $f: M \rightarrow X$ be an immersion and $i: N \rightarrow X$ an imbedding of a submanifold in X . Then the following are equivalent:

- (1) f is t -regular on N .
- (2) f and i are mutually normal.

The proof is very easy and left to the reader.

Theorem 1.5. *The property n -normal is generic for immersions of closed manifolds.*

Proof. We first show that n -normal satisfies property 1 of generic. If f' is close to a given n -normal immersion $f: M \rightarrow X$ then the n -fold product map of f' will be close to the n -fold product of f , and hence F' will be close to F (the

restrictions of the n -fold product maps to $(M)^n - R$. By 1.4, F is t -regular on Δ , and it follows (see THOM [7]) that F' will be also, since any map sufficiently close to a t -regular map is again t -regular. Application of 1.4 again yields that f' is n -normal if it is sufficiently close to f .

We next prove that 2-normal satisfies property 2 of generic. The proof will then be finished by induction. Let $f: M \rightarrow X$ be a given immersion of a closed manifold M . For $i = 1, \dots, m$ let $W_i, \bar{W}_i$ be open sets of M such that f is 1-1 on $W_i, \bar{W}_i \subset W'_i$, and $\cup W_i = M$. Given $\epsilon > 0$, choose $\epsilon_i, i = 1, \dots, m$, such that $\sum_{i=1}^m \epsilon_i < \epsilon$ and $\epsilon_i < \frac{1}{3} d(X - f(W'_i), f(\bar{W}_i))$, d being the distance function in X . Let $f'_1: M - \bar{W}_1 \rightarrow X$ be given by Theorem 1.1 where we take ϵ_1 for ϵ , $f(W_1)$ for N , and $f|_{M-\bar{W}_1}$ for the given map. Then $f'_1 = f$ except on $f^{-1}(V_{\epsilon_1}(f(W_1)))$; in particular, $f'_1 = f$ on $\{x \in W'_1 \mid d(f(x), f(\bar{W}_1)) \geq \epsilon_1 \text{ and } d(f(x), X - f(W'_1)) \geq \epsilon_1\}$. Hence if we define $f_1 = f$ on $\{x \in W'_1 \mid d(f(x), X - f(W'_1)) \geq \epsilon_1\}$ and $f_1 = f'_1$ on the complement of this set, f_1 will be a C^∞ map of M into X . The immersion f_1 will have the property that $f_{1|W_1}$ and $f_{1|M-\bar{W}_1}$ are mutually normal. For some $\epsilon'_2, (r, \epsilon'_2)$ -approximations of f_1 will also have this property and will still be 1-1 on $\bar{W}_1$. Define $f_2: M \rightarrow X$ in a fashion similar to f_1 using $\min(\epsilon_2, \epsilon'_2)$ instead of ϵ_1 and W_2 in place of W_1 . Iteration finally yields an immersion $f_m: M \rightarrow X$ such that $f_{m|W_i}$ and $f_{m|M-\bar{W}_i}$ are mutually normal and f_m is 1-1 on $\bar{W}_i, i = 1, \dots, m$. It follows that f_m is an (r, ϵ) -approximation of f which is 2-normal.

Proceeding by induction, let $f: M \rightarrow X$ be an $(n - 1)$ -normal immersion. It is sufficient to find an (r, ϵ) -approximation of f which is n -normal. For the proof we need the following lemma.

Lemma 1.6. *Let $g: V \rightarrow X$ be an immersion and $g': N \rightarrow X$ an imbedding. Then there exists an (r, ϵ) -approximation $g'_1: N \rightarrow X$ of g' such that g and g'_1 are mutually normal and g'_1 is given by hg' where $h: X \rightarrow X$ is a diffeomorphism of X equal to the identity outside any prescribed tubular neighborhood of $g'(N)$.*

Proof. This is essentially in THOM [7]. (Compare the above lemma with 1.1.) In fact, THOM's result gives a diffeomorphism $h_1: X \rightarrow X$ which is the identity outside a prescribed tubular neighborhood of $g'(N)$ and such that $h_1g: V \rightarrow X$ is t -regular on $g'(N)$. As noted above, this means that h_1g and g' are mutually normal. But since $h = h_1^{-1}$ is a diffeomorphism, $g: V \rightarrow X$ and $hg': N \rightarrow X$ are again mutually normal, and again h is the identity outside the prescribed tubular neighborhood of $g'(N)$.

Returning to the proof of the theorem, let $W_i, \bar{W}_i$ be as before and let Σ_{n-1} be the $(n - 1)$ -self-intersection manifold of f . By the above lemma we can find an (r, ϵ_1) -approximation f_1 to $f|_{W_1}$ by means of a diffeomorphism $h: X \rightarrow X$ which is the identity outside of $V_{\epsilon_1}(f(W_1))$ such that $f_1 = hf|_{W_1}$ and $f\pi: \Sigma_{n-1} \rightarrow X$ are mutually normal. Now $hf = f$ on $\{x \in W'_1 \mid d(f(x), f(\bar{W}_1)) \geq \epsilon_1 \text{ and } d(f(x), X - f(W'_1)) > \epsilon_1\}$. Hence we can extend f_1 to M by letting $f_1 = f$ on the complement of $\{x \in W'_1 \mid d(f(x), f(\bar{W}_1)) > \epsilon_1\}$. Then f_1 has the following property. Suppose $x_1, \dots, x_n$ are distinct points of M such that $f_1(x_1) = \dots =$

$f_1(x_n)$, and that, for some i , $x_i \in W_1$, at most one $x_i \in W'_1$ since f_1 is 1-1 on W'_1 . Hence $(x_1, \dots, \hat{x}_i, \dots, x_n) \in \Sigma_{n-1}$, since $f_1(x_i) = f(x_i)$, $j \neq i$. But the mutual normality of $f_{1|W_1}$ and $f\pi: \Sigma_{n-1} \rightarrow X$ implies that

$$\begin{aligned} \dim \bigcap_{i=1}^n f_1(M_{x_i}) &= \dim \bigcap_{i \neq 1} f(M_{x_i}) \cap f_1(M_{x_1}) \\ &= k - (n-2)r + k - (k+r) = k - (n-1)r. \end{aligned}$$

This property is equivalent to saying that the map F_1 , the cut down of the n -fold product map of f_1 to $W_1 \times (M - \bar{W}_1)^{n-1} - R$, is t -regular on the diagonal Δ of $(X)^n$. But for sufficiently small ϵ'_2 , (r, ϵ'_2) -approximations of f_1 will again have this property and will still be 1-1 on $\bar{W}_1$. We now proceed by induction on i , obtaining a map f_m which will have the desired property for any n distinct points since $\cup W_i = M$ and f_m will be 1-1 on each $\bar{W}_i$. This completes the proof of the theorem.

Theorem 1.7. *Mutually normal is generic for n -tuples of immersions of a closed manifold.*

Proof. Property 2 of generic is satisfied as follows. Let $f_i: M_i \rightarrow X$ be an n -tuple of immersions where each $M_i = M$, $i = 1, \dots, n$. Let M^* be the disjoint union of the M_i and $F: M^* \rightarrow X$ be defined by $F(x) = f_i(x)$ where $x \in M_i$. By Theorem 1.5, there exists an (r, ϵ) -approximation $G: M^* \rightarrow X$ of F which is n -normal. Let $f'_i: M_i \rightarrow X$ be the restriction of G to M_i . It is immediate that $\{f'_i\}$ is an (r, ϵ) -approximation of $\{f_i\}$ and is mutually normal. Property 1 of generic is proved in a similar fashion.

2. The main theorem. The goal of this section is to prove Theorem I of the introduction. For simplicity, we will assume M and X are connected in the rest of this paper, also that M is closed. If $f: M^k \rightarrow X^{k+r}$ is any continuous map of the manifold M into the manifold X , we denote the image of the basic class M under the composition

$$H_k(M) \xrightarrow{f^*} H_k(X) \xrightarrow{\lambda} H^r(X) \xrightarrow{f^*} H^r(M) \xrightarrow{\lambda} H_{k-r}(M)$$

by $\gamma(f)$; i.e. $\gamma(f) \in H_{k-r}(M)$. If M or X is non-orientable it is defined with coefficients $\mathbb{Z}_2$ and otherwise with integer coefficients. We denote the dual of the cup product in the manifold M by $\circ$; i.e. if $x, y \in H_*(M)$, $x \circ y = \lambda(\lambda x \cup \lambda y)$. Theorem I may then be stated as follows:

Theorem 2.1. *If $f: M^k \rightarrow X^{k+r}$ is an n -normal immersion of a closed connected manifold M in a connected manifold X , then the n -self intersection class*

$$\Sigma_n^* = \pm(\gamma(f) - \lambda \bar{W}^r)^{n-1},$$

where $\bar{W}^r$ is the normal Stiefel-Whitney class, λ is duality, $\gamma(f)$ is the class defined above, and the product is $\circ$ product.

We start the proof of 2.1 by choosing immersions $f^i: M \rightarrow X, i = 1, \dots, n-1$, (r, ϵ) -approximations of f , by 1.5 and 1.7 (we will choose the ϵ later), such that $f, f^1, \dots, f^{n-1}$ are mutually normal and each is n -normal. Write $f^0 = f$.

Let V_0 be the normal bundle of vectors of magnitude $< \delta$. Then V_0 can be immersed as an (immersed) tubular neighborhood of M in X . Let $f_0: M \rightarrow V_0$ be the zero-cross-section. Further, if $j_0: V_0 \rightarrow X$ is the above immersion, let $f_0^i: M \rightarrow V_0, i = 1, \dots, n-1$ be the cross-section uniquely defined by the condition

$$j_0 f_0^i = f^i$$

(this is well defined if f^i is sufficiently close to f). In order to use a unified notation, we set $f_1^i = f, i = 0, \dots, n-1$, and $V_1 = X$. Then if we let $\alpha = (\epsilon_1, \dots, \epsilon_{n-1})$ where $\epsilon_i = 0$ or 1 we define $F_\alpha: (M)^n \rightarrow M \times V_\alpha$, where $V_\alpha = V_{\epsilon_1} \times \dots \times V_{\epsilon_{n-1}}$ by $F_\alpha = i \times f_{\epsilon_1}^1 \times \dots \times f_{\epsilon_{n-1}}^{n-1}$ ($i: M \rightarrow M$ the identity). Further, we define $G_\alpha: (M)^n \rightarrow M \times V_\alpha$ by $G_\alpha = i \times f_{\epsilon_1}^0 \times \dots \times f_{\epsilon_{n-1}}^0$.

Now let $\Gamma_\alpha \subset M \times V_\alpha$ be the image of M under the composition

$$M \xrightarrow{d} (M)^n \xrightarrow{G_\alpha} M \times V_\alpha,$$

where d is the diagonal map.

Lemma 2.2. F_α is t -regular on Γ_α .

Proof. Let $i \times j_\alpha: M \times V_\alpha \rightarrow M \times (X)^{n-1}, j_\alpha = j_{\epsilon_1} \times \dots \times j_{\epsilon_{n-1}}, j_0: V_0 \rightarrow X, j_1: X \rightarrow X$ the identity. Then if $(1) = (1, \dots, 1), \Gamma_{(1)} = (i \times j_\alpha) \Gamma_\alpha$ and $F_{(1)} = (i \times j_\alpha) F_\alpha$. Since $i \times j_\alpha$ is an immersion, it follows that F_α is t -regular on Γ_α , for all α , if and only if $F_{(1)}$ is t -regular on $\Gamma_{(1)}$. Now if $(x) = (x_0, \dots, x_{n-1}) \in (M)^n$ and $F_{(1)}(x) \in \Gamma_{(1)}$ then $f^0(x_0) = f^1(x_1) = \dots = f^{n-1}(x_{n-1})$. Let $V_{(x)}$ be the subspace of $(M)_{(x)}^n$ such that $V_{(x)} = F_{(1)}^{-1}(\Gamma_{(1)})_{(y)}, (y) = (x_0, f^1(x_1), \dots, f^{n-1}(x_{n-1}))$. We see immediately that $\dim F_{(1)} V_{(x)} = \dim \bigcap_{i=0}^{n-1} f^i(M_{x_i})$, and hence

$$\dim V_{(x)} = \dim \bigcap f^i(M_{x_i}) = k - (n-1)r.$$

But then

$$\dim (M)_{(x)}^n - \dim V_{(x)} = kn - (k - (n-1)r) = (n-1)(k+r),$$

and

$$\dim (M \times (X)^{n-1})_{(y)} - \dim \Gamma_{(1)} = k + (n-1)(k+r) - k = (n-1)(k+r).$$

Thus since $F_{(1)}$ is an immersion, it is t -regular on $\Gamma_{(1)}$.

At this point we stop to review some facts concerning the "slant operation" (see [5]). If X and Y are two topological spaces, then, using singular chains and cochains, let $z \in C^p(X \times Y), y \in C_r(Y)$; then we define a cochain $z/y \in C^{p-r}(X)$ by the property

$$z/y(x) = z(x \otimes y), \quad x \in C_{p-r}(X).$$

Then

$$\delta(z/y) = \delta z/y + (-1)^{p-r-1} z/\partial y;$$

and hence we get a pairing $H^p(X \times Y) \otimes H_r(Y) \rightarrow H^{p-r}(X)$, $w \otimes v \rightarrow w/v$, $w \in H^p(X \times Y)$, $v \in H_r(Y)$, $w/v \in H^{p-r}(X)$, which again has the property

$$(A) \quad w/v(u) = w(u \otimes v), \quad u \in H_{p-r}(X).$$

Here we use singular homology and cohomology with coefficients in any groups G, G_1, G_2 for $X \times Y, X, Y$ respectively; and then w/v will have coefficients in $G \otimes G_2$ and the Kronecker index in (A) will have values in $G \otimes G_1 \otimes G_2$.

Lemma 2.3. *Let $u \in H^p(X; Z)$ be an integral cohomology class of a space X . Then u is uniquely determined by its values (Kronecker index) on the homology classes $x \in H_p(X; G)$, where G runs through Z_n (integers mod n), for all n , as well as Z .*

This lemma is well known and we leave it to the reader to check.

Lemma 2.4. (Naturality Lemma.) *Let $f: X \rightarrow X', g: Y \rightarrow Y'$ be continuous maps. Then for $u \in H^p(X' \times Y')$, $y \in H_r(Y)$*

$$f^*(u/g_*y) = [(f \times g)^*(u)]/y.$$

This lemma follows immediately from the characterization (A) above and Lemma 2.3.

Lemma 2.5. *Let X^n and Y^m be closed connected oriented manifolds and λ be Poincaré duality in X, Y or $X \times Y$ as the context indicates. Let $\bar{Y}: H^p(X \times Y) \rightarrow H^{p-m}(X)$ and $\pi_{1*}: H_{n+m-p}(X \times Y) \rightarrow H_{n+m-p}(X)$, where π_{1*} is projection onto the first factor and $\bar{Y} \in H_m(Y)$ is the basic class. Then $\lambda \cdot (\bar{Y}) = \pi_{1*}\lambda$.*

Proof. Let $u \in H^p(X \times Y)$, $x \in H^{n+m-p}(X; G)$. Then

$$\begin{aligned} \lambda(u/\bar{Y})(x) &= u/\bar{Y}(\lambda x) = u(\lambda x \otimes \bar{Y}) = \lambda u(\lambda(\lambda x \otimes \bar{Y})) \\ &= \lambda u(x \otimes 1) = \lambda u(\pi_1^*(x)) = \pi_{1*}\lambda u(x). \end{aligned}$$

Note that $1 \in H^0(Y)$ and hence the sign here is $+1$. By 2.3 our result follows.

We now summarize some known facts (see CARTAN [1]) concerning duality in a paracompact connected manifold X . The coefficients are in an arbitrary abelian group if the manifold is oriented and integers mod 2 if it is non-oriented.

Let $H_v^\phi(X)$ be locally finite singular homology supports in ϕ , $H_\phi^p(X)$ cohomology with supports in ϕ .

- (a) If $\phi = C$ = all compact subsets of X , then
 $H_v^C(X)$ = ordinary singular homology.
- (b) If $\phi = F$ = all closed subsets of X , then
 $H_F^p(X)$ = ordinary singular cohomology.

(More generally, since X is H.L.C., $H_\phi^p(X)$ is singular cohomology with supports in ϕ .)

If X is n -dimensional, we have Poincaré duality:

$$(c) \quad \lambda_\phi : H_\phi^p(X) \cong H_{n-p}^\phi(X).$$

The *fundamental class* X of the manifold is the image by λ_F of a generator of $H_F^0(X)$, $X \in H_n^F(X)$ (integers or integers mod 2). From the *cup product*

$$H_\phi^p \otimes H_{\phi'}^q \rightarrow H_{\phi \cap \phi'}^{p+q},$$

we get the *cap product*

$$(d) \quad H_\phi^p(X) \otimes H_{\phi'}^q(X) \rightarrow H_{\phi \cap \phi'}^{p+q}(X).$$

Taking $\phi' = F$, $\phi \cap F = \phi$, and $X \in H_n^F(X)$ we get from (d)

$$H_\phi^p(X) \otimes X \rightarrow H_{n-p}^\phi(X).$$

This is precisely λ_ϕ [1, p. 88].

Further, the cup and cap products have the following properties:

(e) Let $u \in H_C^p(X)$, $v \in H_F^r(X)$, $x \in H_{p+r}^F(X)$; then

$$(v \cup u)(x) = u(v \cap x) \quad (\text{Kronecker index}),$$

$$v(u \cap x) = (u \cup v)(x).$$

(f) Then, for $u \in H_C^p(X)$, $v \in H_F^{n-p}(X)$,

$$v^{n-p}(\lambda_C u^p) = (-1)^{p(n-p)} u^p(\lambda_F v^{n-p}).$$

Proof.

$$\begin{aligned} v^{n-p}(\lambda_C u^p) &= v^{n-p}(u^p \cap X) = (u^p \cup v^{n-p})(X) = (-1)^{p(n-p)} (v^{n-p} \cup u^p)(X) \\ &= (-1)^{p(n-p)} u^p(v^{n-p} \cap X) = (-1)^{p(n-p)} u^p(\lambda_F v^{n-p}) \end{aligned}$$

Consequently, if $x \in H_{n-p}^C(X)$, $y \in H_p^F(X)$,

$$\lambda_F^{-1} y_p(x_{n-p}) = (-1)^{p(n-p)} \lambda_C^{-1} x_{n-p}(y_p).$$

(g) Let $i_* : H_C^p(X) \rightarrow H_p^F(X)$, $i^* : H_C^p(X) \rightarrow H_p^F(X)$ be the natural maps; then

$$i_* \lambda_C = \lambda_F i^*, \quad i^* \lambda_C^{-1} = \lambda_F^{-1} i_*.$$

Now let M be a closed connected manifold and $f: M \rightarrow X$ a continuous map. Then if $f_C^* : H_C^p(X) \rightarrow H^p(M)$ and $f_F^* : H_F^p(X) \rightarrow H^p(M)$ are the induced maps (with similar definitions for f_*^C and f_*^F),

$$(h) \quad f_F^* i^* = f_C^*, \quad i_* f_*^C = f_*^F.$$

Finally, from (f), (g), (h) we get the following lemma:

Lemma 2.6. Let M and X be connected manifolds, M compact. Let $f, g: M \rightarrow X$ be continuous maps. Let $z \in H_p(M)$, $w \in H_{n-p}(M)$, $n = \dim X$. Then (with notations as in (f), (g), (h) above)

$$f_*^C \lambda_C^{-1} g_*^C z(w) = (-1)^{p(n-p)} g_*^C \lambda_C^{-1} f_*^C w(z).$$

Proof.

$$\begin{aligned} f_*^C \lambda_C^{-1} g_*^C z(w) &= \lambda_C^{-1} g_*^C z(f_*^P w) = g_*^C z(\lambda_F^{-1} f_*^P w) (-1)^{p(n-p)} = g_*^C z(\lambda_F^{-1} i_* f_*^C w) (-1)^{p(n-p)} \\ &= g_*^C z(i_* \lambda_C^{-1} f_*^C w) (-1)^{p(n-p)} = z(g_F^* i_* \lambda_C^{-1} f_*^C w) (-1)^{p(n-p)} = z(g_F^* \lambda_C^{-1} f_*^C w) (-1)^{p(n-p)}. \end{aligned}$$

Lemma 2.7. $\pi_1 F_\alpha^{-1}(\Gamma_\alpha) = \gamma_\alpha = (-1)^{k(n-1)(k+r)} \gamma_{\epsilon_1} \circ \cdots \circ \gamma_{\epsilon_{n-1}}$ where $\gamma_{\epsilon_i} = \gamma(f_{\epsilon_i})$.

Proof. By 1.2 and 2.2 we have

$$(1) \quad F_\alpha^{-1}(\Gamma_\alpha) = \lambda F_\alpha^* \lambda(\Gamma_\alpha),$$

or

$$(1') \quad \pi_1 F_\alpha^{-1}(\Gamma_\alpha) = \pi_1 \lambda F_\alpha^* \lambda(\Gamma_\alpha).$$

Consider

$$\begin{array}{ccccc} H_K(M \times V_\alpha) & \xleftrightarrow{\lambda} & H^{(n-1)(k+r)}(M \times V_\alpha) & \xrightarrow{F_\alpha^*} & H^{(n-1)(k+r)}((M)^n) & \xrightarrow{\lambda} & H_{K-(n-1)r}((M)^n) \\ & & & \downarrow \lambda M & & & \downarrow \pi_i \\ & & H^{(n-1)r}(M) & \xrightarrow{\lambda} & H_{K-(n-1)r}(M) & & \end{array}$$

The above square commutes by Lemma 2.5. Hence

$$(2) \quad \pi_1 \lambda F_\alpha^* \lambda \Gamma_\alpha = \lambda(F_\alpha^* \lambda \Gamma_\alpha / \otimes^{n-1} M).$$

We will now show

$$(3) \quad \lambda(F_\alpha^* \lambda \Gamma_\alpha / \otimes^{n-1} M) = \gamma_{\epsilon_1} \circ \cdots \circ \gamma_{\epsilon_{n-1}}.$$

Combining (1'), (2), (3) we will have our lemma.

Proof of (3). Let $x \in H_{(n-1)r}(M)$. Then

$$\begin{aligned} (F_\alpha^* \lambda \Gamma_\alpha / \otimes^{n-1} M)(x) &= F_\alpha^* \lambda \Gamma_\alpha(x \otimes M \otimes M \otimes \cdots \otimes M) \\ &= F_\alpha^* \lambda G_{\alpha*} d_* M(x \otimes M \otimes M \otimes \cdots \otimes M). \end{aligned}$$

By Lemma 2.6 this gives

$$(-1)^{k(n-1)(k+r)} G_\alpha^* \lambda F_{\alpha*}(x \otimes M \otimes M \otimes \cdots \otimes M)(d_* M).$$

Let $\sigma = (-1)^{k(n-1)(k+r)}$. Then since $f_{\epsilon_i}^i$ is homotopic to $f_{\epsilon_i} = f_{\epsilon_i}^0$, and again since $\deg \lambda M = 0$, the above becomes

$$\begin{aligned} \sigma(\lambda x \otimes f_{\epsilon_1}^* \lambda f_{\epsilon_1*} M \otimes \cdots \otimes f_{\epsilon_{n-1}}^* \lambda f_{\epsilon_{n-1}*} M)(d_* M) \\ = \sigma(\lambda x \cup \lambda \gamma_{\epsilon_1} \cup \cdots \cup \lambda \gamma_{\epsilon_{n-1}})(M) = \sigma(\lambda \gamma_{\epsilon_1} \cup \cdots \cup \lambda \gamma_{\epsilon_{n-1}})(x) \\ = \sigma \lambda(\gamma_{\epsilon_1} \circ \cdots \circ \gamma_{\epsilon_{n-1}})(x). \end{aligned}$$

This completes the proof.

Let $R_\alpha = G_\alpha^{-1}(\Gamma_\alpha)$. Then

$$R_\alpha = \{(x_0, \dots, x_{n-1}) \in (M)^n \mid f(x_i) = f(x_0) \text{ all } i; \text{ and if } \epsilon_{i_\alpha} = 0 \text{ then } x_{i_\alpha} = x_0\}.$$

Now partially order the α 's by defining $\alpha < \alpha'$ if $\sum \epsilon_i < \sum \epsilon'_i$ and $\epsilon_i \leq \epsilon'_i$ for each i . Let $Q_{0\dots 0} = R_{0\dots 0}$ and $Q_\alpha = R_\alpha - \bigcup_{\alpha' < \alpha} Q_{\alpha'}$. Then (as is easily given by induction)

$$Q_\alpha = \{(x_0, \dots, x_{n-1}) \in (M)^n \mid f(x_i) = f(x_0) \text{ all } i, x_{i_\alpha} = x_0 \text{ if } \epsilon_{i_\alpha} = 0, \\ x_{i_p} \neq x_1 \text{ if } \epsilon_{i_p} = 1\}$$

and the Q_α are closed, disjoint and $\Sigma_n = Q_{1\dots 1}$.

Since the Q_α are closed we can take open neighborhoods T_α of the Q_α with $\bar{T}_\alpha \cap \bar{T}_{\alpha'} = \emptyset$ if $\alpha \neq \alpha'$. Let V be an open neighborhood of $\Gamma_{1\dots 1}$ such that $G_{1\dots 1}^{-1}(V) \subset \bigcup_\alpha T_\alpha$. This can be done since $G_{1\dots 1}((M)^n - \bigcup_\alpha T_\alpha)$ is compact and disjoint from $\Gamma_{1\dots 1}$. By our construction of $F_{1\dots 1}$ we can assume that $F_{1\dots 1}$ and $G_{1\dots 1}$ agree outside of $G_{1\dots 1}^{-1}(V)$. Let $Q'_\alpha = T_\alpha \cap F_{1\dots 1}^{-1}(\Gamma_{1\dots 1})$. Then Q'_α is a closed submanifold of $(M)^n$. Further, $Q'_{1\dots 1}$ is isotopic to $Q_{1\dots 1} = \Sigma_n$, since $F_{1\dots 1}$ and $G_{1\dots 1}$ are t -regular on $\Gamma_{1\dots 1}$, considered as maps of $T_{1\dots 1}$, at least if $F_{1\dots 1}$ is sufficiently close to $G_{1\dots 1}$.

Now let $R'_\alpha = F_\alpha^{-1}(\Gamma_\alpha)$. Then

$$(A) \quad Q'_\alpha = R'_\alpha - \bigcup_{\alpha' < \alpha} Q'_{\alpha'}.$$

Further, we have seen that $\pi_1 R'_\alpha$ represents the class γ_α . Let $\text{sgn}(\alpha) = +1$ or -1 according as α has an even or odd number of zeros.

Theorem 2.8. $\pi_1 Q'_\alpha$ represents $\text{sgn}(\alpha) \Sigma_{\alpha' \leq \alpha} \text{sgn}(\alpha') \gamma_{\alpha'}$ and, in particular, $\pi_1 \Sigma_n$ represents $\Sigma_\alpha \text{sgn}(\alpha) \gamma_\alpha$.

This result follows purely algebraically from relation (A) and the proof is left to the reader.

Theorem 2.9. $\Sigma_n^* = \sigma(\gamma_1 - \gamma_0)^{n-1}$.

This follows from 2.8 and a short algebraic argument which we again leave to the reader.

Then 2.1 follows immediately from 2.6 and the next lemma.

Lemma 2.7. $\gamma_0 = \lambda \bar{W}^r$.

Proof. Consider the diagram

$$\begin{array}{ccccc} H^r(V) & \xleftarrow{\quad} & H^r(V, \dot{V}) & \xleftarrow{\lambda} & H_k(V) \\ \uparrow & & \uparrow \phi & & \uparrow \\ H^r(M) & \xleftarrow{\cup \bar{W}^r} & H^0(M) & \xleftarrow{\lambda} & H_k(M) \end{array}$$

where V is the normal bundle of $f: M \rightarrow X$ of vectors with length $\leq \delta$, for some δ , and $\dot{V}$ its boundary. Each of these diagrams commutes according to THOM [6]. The proof of 2.7 is an immediate consequence of this picture.

3. An orientability theorem. We assume throughout this section that $f: M^k \rightarrow X^{k+r}$ is a 2-normal immersion without triple points and M is closed. If, for example, f is 3-normal and $r > \frac{1}{2}k$ then from Section 1 it follows that Σ_3 is empty and hence we have the above situation. We will identify $\pi_1(\Sigma_2)$ with Σ_2 since π_1 is an imbedding (Section 1). Then f restricted to Σ_2 , say $\tilde{f}$, is 2 to 1 and hence a double covering. Let $\tilde{\Sigma}_2 = \tilde{f}(\Sigma_2)$. Then if $\tilde{N}$ is a component of $\tilde{\Sigma}_2$, $\tilde{f}^{-1}(\tilde{N})$ can have either one or two components. If $\tilde{f}^{-1}(\tilde{N})$ has two components, then $\tilde{f}$ restricted to these components is a trivial double covering. This is a relatively simple case. We will now consider the second case.

Theorem 3.1. *Let $f: M^k \rightarrow X^{k+r}$ be a 2-normal immersion without triple points, M closed. Let $N = \tilde{f}^{-1}(\tilde{N})$ be connected where $\tilde{N}$ is a component of $\tilde{\Sigma}_2$. Then (1) if r is odd $\tilde{N}$ is non-orientable and (2) if r is even $\tilde{N}$ is orientable.*

Before we prove this we note a corollary.

Corollary 3.2. *If $k > 2$ is even and $f: M^k \rightarrow E^{2k-1}$ is an immersion, then M^k may be imbedded by a map h into E^{2k} arbitrarily close to $f: M^k \rightarrow E^{2k-1} \subset E^{2k}$.*

Proof of corollary. By Theorem 1.5 we may assume f is 2-normal and without triple points. Then $\tilde{\Sigma}_2$ is a disjoint union of circles. Since a circle is orientable and $r = k - 1$ is odd, by 3.1, each component of $\tilde{\Sigma}_2$ is trivially doubly covered by $\tilde{f}$. Let N_1, N_2 be the two circles of $\tilde{f}^{-1}(\tilde{N})$ where $\tilde{N}$ is a component of $\tilde{\Sigma}_2$. Let g be a C^∞ function which is not zero on N_1 and zero outside a small tubular neighborhood of N_1 . Let $h: M \rightarrow E^{k+r+1}$ be defined by having the same values as f on the first $k + r$ coordinates but with last coordinate g . Then h has no intersections near N_1 or N_2 , and will be close to f if g is chosen with small enough absolute value. Repeating this process gives the desired imbedding of the corollary.

We now prove 3.1. Let $t: M \times M \rightarrow M \times M$ be defined by $t(p, q) = (q, p)$. Then $t(\Sigma_2) = \Sigma_2$. We will write Σ for Σ_2 .

Lemma 3.3. *If $\Sigma_{(p,q)}^+$ is the positive orientation of the tangent space $\Sigma_{(p,q)}$, then if r is odd $t_*(\Sigma_{(p,q)}^+) = \Sigma_{(q,p)}^-$ and if r is even $t_*(\Sigma_{(p,q)}^+) = \Sigma_{(q,p)}^+$ where t_* is the differential of t .*

If 3.3 has been proved, then we show 3.1 as follows. Let $\phi: M \times M \rightarrow X'$ where $\phi = f\pi_1$; then $\phi t = \phi$ and ϕ restricted to N is our double covering. If r is odd then $\phi_* t_* N_{(p,q)}^+ = \phi_* N_{(q,p)}^+ = \phi_* N_{(p,q)}^+$ by the above diagram and 3.3. Hence $\tilde{N}$ is orientable. On the other hand, if r is even $\phi_* N_{(p,q)}^+ = \phi_* N_{(q,p)}^-$ and so $\tilde{N}$ is non-orientable.

We now prove 3.3. Let $t': X \times X \rightarrow X \times X$ be defined by $t'(x, y) = (y, x)$. Then $(f \times f) \circ t = t' \circ (f \times f)$.

Suppose r is odd.

Case I. k is even. Then t' reverses the orientation of $X \times X$ and t preserves the orientation of $M \times M$. Hence t reverses the orientation of Σ . (See Section 1 for the definition of the orientation of Σ .)

Case II. k is odd. t' preserves the orientation of $X \times X$ while t reverses the orientation of $M \times M$. Hence t reverses the orientation of Σ .

Suppose now that r is even.

Case I. k is even. Then t preserves the orientation of $M \times M$ and t' preserves the orientation of $X \times X$, so t preserves the orientation of Σ .

Case II. k is odd. Then t reverses the orientation of $M \times M$ and t' reverses the orientation of $X \times X$; so t preserves the orientation of Σ .

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