On Vitali's Theorem for Complex Spaces with Singularities

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The subject of complex spaces with singularities has been the scene of a great deal of recent activity in the theory of functions of several complex variables; and several function-theoretic results have been extended from complex manifolds to these more general spaces. The maximum modulus and Vitali convergence theorems have been treated in [3], for example; but there use was made of a rather explicit local representation of the analytic spaces as analytic subvarieties. The present note intends to furnish a simpler proof of these extensions, which will be valid in considerably more general circumstances, and which will be self-contained in so far as properties of spaces with singularities are concerned. It will be seen in this approach that these theorems follow from the corresponding theorems for complex manifolds as essentially topological rather than analytical consequences. As an application, it will be demonstrated that the full group of analytic automorphisms of a compact complex space with singularities is a Lie group.

The first section is devoted to the proof of the maximum modulus and convergence theorems for a large class of branched coverings of complex manifolds; the approach taken here involves a rather more extensive use of the order of branching than customary and a definition of holomorphic functions which is particularly suited to the theorems to be proved. In the second section this definition of holomorphic functions is re-examined on a restricted class of branched coverings, for which the definition can be put in a perhaps more natural form. The relationship between this restricted class of branched coverings and the branched coverings introduced by BEHNKE & STEIN [1] is discussed in the third section. The last section is devoted to the demonstration of the local compactness of the group of analytic automorphisms of a compact analytic space, from which the theorem that that group is a Lie group follows as an immediate consequence of known results.

1. Let V be a separable, connected, Hausdorff space; π be a continuous open mapping of V onto a bounded open subset D of the space of n complex variables;

and o be a mapping of V into the set of strictly positive integers. It will be assumed that the mapping π is *o-disciplined*, meaning that each point $p_0 \in V$ has arbitrarily small open neighborhoods $U(p_0)$ such that, for every $p \in U(p_0)$,

$$(1) \quad \sum_{\{q \in U(p_0) \mid \pi(q) = p\}} o(q) = o(p_0).$$

Such a collection of spaces and maps will be called a *local semi-analytic space*.

An open neighborhood $U(p_0)$ of p_0 with the property (1) will be called a *uniform neighborhood* of p_0 ; its projection $\pi(U(p_0))$ is an open neighborhood of $\pi(p_0)$ in D . For each point $z \in \pi(U(p_0))$ let $\{p_i(z)\}$ be an indexed set whose elements are the points of $\pi^{-1}(z) \cap U(p_0)$, but where a point p with $o(p) = s$ is listed altogether s times; thus the set $\{p_i(z)\}$ contains precisely $o(p_0)$ elements. If $f(p)$ is any complex-valued function on the set $U(p_0)$ and $\phi(x_1, \dots, x_r)$ is any symmetric function of $r = o(p_0)$ complex variables, then $\phi_r(z) = \phi(f(p_1(z)), \dots, f(p_r(z)))$ is a well-defined complex-valued function on $\pi(U(p_0))$. For example, the power-sums $\sigma^k(x_1, \dots, x_r) = \sum_{i=1}^r x_i^k$ lead to the functions $\sigma_r^k(z) = \sum_{i=1}^r [f(p_i(z))]^k$. This notation will be used without further reference throughout the present section.

Lemma 1. *A function $f(p)$ on V is continuous if and only if for each uniform neighborhood $U(p_0)$ of any point $p_0 \in V$ and for each symmetric polynomial $\phi(x_1, \dots, x_r)$, where $r = o(p_0)$, the function $\phi_r(z)$ is continuous on $\pi(U(p_0))$.*

Proof. Suppose, firstly, that $f(p)$ is continuous; select a point $z' \in \pi(U(p_0))$ and let $x'_i = f(p_i(z'))$. To any $\epsilon > 0$ there corresponds a $\delta > 0$ such that $|\phi(x_1, \dots, x_r) - \phi(x'_1, \dots, x'_r)| < \epsilon$ whenever $|x_i - x'_i| < \delta$. If $\{q_i\}$ are the distinct points of the set $\{p_i(z')\}$, there are disjoint uniform open neighborhoods $W_i = W(q_i)$ about them such that $|f(p) - f(q_i)| < \delta$ whenever $p \in W_i$. Then $A = \bigcap \pi(W_i)$ is an open neighborhood of z' ; and since the neighborhoods W_i are uniform each W_i contains $o(q_i)$ elements of the set $\{p_i(z)\}$ for any $z \in A$, as well as $o(q_i)$ elements of the set $\{p_i(z')\}$. Thus, after suitable renumbering, $|f(p_i(z)) - f(p_i(z'))| < \delta$ whenever $z \in A$, so that $|\phi_r(z) - \phi_r(z')| < \epsilon$ also whenever $z \in A$.

Secondly, to prove the continuity of $f(p)$, select a point $p_0 \in V$ and a uniform open neighborhood $U(p_0)$ of p_0 . By assumption the functions $\tau^k(z) = \tau_r^k(z)$ are continuous in $\pi(U(p_0))$, where $\tau^k(x_1, \dots, x_r)$ are the elementary symmetric polynomials in $r = o(p_0)$ variables. The values $f(p_i(z))$ are precisely the r roots of the polynomial equation $P(w; z) = w^r - \tau^1(z)w^{r-1} + \dots + (-1)^r \tau^r(z) = 0$; but it follows immediately from Rouché's theorem in one complex variable that the roots of such an equation vary continuously with the coefficients of the polynomial. Since $P(w; z) = 0$ has a multiple root $w_0 = f(p_0)$ of order r at $z = \pi(p_0)$, there consequently corresponds to every $\epsilon > 0$ an open neighborhood A of $\pi(p_0)$ such that $|w_i(z) - w_0| < \epsilon$ whenever $z \in A$ and $P(w_i(z); z) = 0$; thus $|f(p_i(z)) - f(p_0)| < \epsilon$ whenever $z \in A$.

A complex-valued function $f(p)$ on the set V is called *holomorphic* if for each

uniform neighborhood $U(p_0)$ of any point $p_0 \in V$ and for each symmetric polynomial $\phi(x_1, \dots, x_r)$ of $r = o(p_0)$ variables, the function $\phi_r(z)$ is holomorphic in $\pi(U(p_0))$ in the customary sense. It follows from Lemma 1 that every such function is continuous on V ; two further properties of such functions are contained in the following theorems.

Theorem 1 [Maximum Modulus Theorem]. *If $f(p)$ is holomorphic on a local semi-analytic space and if $|f(p)| \leq |f(p_0)|$ for all points p in a neighborhood of p_0 , then $f(p) = f(p_0)$ for all points p in a neighborhood of p_0 .*

Proof. Let $U(p_0)$ be a uniform neighborhood of p_0 sufficiently small so that $|f(p)| \leq |f(p_0)|$ for all $p \in U(p_0)$, and set $r = o(p_0)$. The power-sums $\sigma_r^k(z) = \sum_{i=1}^r [f(p_i(z))]^k$ are holomorphic for $z \in \pi(U(p_0))$, while, furthermore, $\sigma_r^k(z_0) = rf(p_0)^k$, so that $|\sigma_r^k(z)| \leq |\sigma_r^k(z_0)|$ for all $z \in \pi(U(p_0))$; but the ordinary maximum modulus theorem [2] implies that $\sigma_r^k(z) = \sigma_r^k(z_0)$ for all z in a neighborhood of z_0 . Since the power sums generate all symmetric polynomials, it therefore follows that $f(p_i(z)) = f(p_0)$ for all z in a neighborhood of z_0 .

Theorem 2 [Convergence Theorem]. *Every uniformly bounded sequence of functions holomorphic on a local semi-analytic space contains a subsequence converging uniformly on compact subsets to a limit function which is also holomorphic.*

Proof. For each $z \in D$ the set $\pi^{-1}(z)$ in V has no limit points and must therefore be countable since V is separable; thus if $\{z_i\}$ is a countable dense subset of D , then $\pi^{-1}\{z_i\}$ is a countable dense subset of V . Any uniformly bounded sequence of functions contains a subsequence $\{f_k(z)\}$ which converges at all points of the set $\pi^{-1}\{z_i\}$. Select an arbitrary point $p_0 \in V$ and a uniform neighborhood $U(p_0)$ of p_0 , and set $r = o(p_0)$. For any symmetric polynomial $\phi(x_1, \dots, x_r)$ the functions $\phi_{f_k}(z)$ are holomorphic and uniformly bounded in $\pi(U(p_0))$, and the sequence of these functions converges on a dense subset of $\pi(U(p_0))$; therefore by the ordinary convergence theorem [2] the sequence of these functions converges everywhere on $\pi(U(p_0))$, indeed uniformly on compact subsets, to a holomorphic function $\phi(z)$. For the particular polynomials $\sigma^1(x_1, \dots, x_r) = x_1 + \dots + x_r$ of course $\sigma^1_{f_k}(z_0) = rf_k(p_0)$, thus showing that the values $f_k(p_0)$ converge to a limit $f(p_0)$; that is to say, the subsequence selected is pointwise convergent. Returning to a general symmetric polynomial, then it is obvious that $\phi(z) = \phi_r(z)$; therefore the limit function is holomorphic on V . Finally, to verify the uniformity of the convergence, it suffices to consider a uniform compact subset K of one uniform neighborhood $U(p_0)$. Suppose given $\epsilon > 0$, and let $P(w; z) = \prod_{i=1}^r [w - f(p_i(z))]$ and $P_k(w; z) = \prod_{i=1}^r [w - f_k(p_i(z))]$. The coefficients of $P_k(w; z)$ converge uniformly on $\pi(K)$ to the coefficients of $P(w; z)$; therefore, again using the fact that the roots of a polynomial vary continuously with the coefficients, there exist numbers $\delta > 0$ and k_0 such that each root of the equation $P_k(w; z_1) = 0$ lies within an $\epsilon/4r$ -neighborhood of some root of $P(w; z_2) = 0$ whenever $k > k_0$ and $|z_1 - z_2| < \delta$, $z_1, z_2 \in K$. Select finitely many points $p_i \in K$ and open neighborhoods W_i of the p_i such that each $\pi(W_i)$ is

contained in a δ -neighborhood of $\pi(p_i)$ and the W_i cover K ; and choose k_0 sufficiently large that in addition $|f_k(p_i) - f(p_i)| < \epsilon/4r$ whenever $k > k_0$ and for all i . For each i take $\epsilon/4r$ -neighborhoods around the roots of the equation $P_k(w; \pi(p_i)) = 0$, and let X_i be that connected component of the union of these neighborhoods which contains $f(p_i)$; clearly $|x_1 - x_2| < \frac{1}{2}\epsilon$ whenever $x_1, x_2 \in X_i$. Note further that $|f_k(p_i) - f(p_i)| < \epsilon/4r$, so $f_k(p_i) \in X_i$ whenever $k > k_0$; but since $f_k(W_i)$ is connected, then $f_k(W_i) \subset X_i$. Now, for any point $p \in K$, there will be some i for which $p \in W_i$; thus $f_k(p) \in X_i$ for all $k > k_0$, so that $f(p) \in \bar{X}_i$ and $|f(p) - f_k(p)| < \epsilon$ for all $k > k_0$.

The convergence theorem admits a simple extension, which will be used subsequently. Let $\pi_1: V_1 \rightarrow D_1$, $o_1: V_1 \rightarrow (\text{integers})$ and $\pi_2: V_2 \rightarrow D_2$, $o_2: V_2 \rightarrow (\text{integers})$ be two local semi-analytic spaces; and let $\phi: V_1 \rightarrow V_2$ be a continuous mapping from V_1 into V_2 . The coordinates in the domain D_2 can be considered as complex-valued functions on D_2 , and these induce complex-valued functions on the space V_1 by means of the mapping $\pi_2\phi: V_1 \rightarrow D_2$; the mapping ϕ will be called *analytic* if these functions are holomorphic functions on the space V_1 . In particular, an *analytic homeomorphism* $\phi: V_1 \rightarrow V_2$ will be a topological homeomorphism between the spaces V_1 and V_2 such that the mappings ϕ and ϕ^{-1} are both analytic. As a convenient abbreviation, a local semi-analytic space will be called *finite* if there is a constant $r_0 < \infty$ such that, for each $z \in D$, $\sum_{\{p \in V \mid \pi(p) = z\}} o(p) = r_0$. Both spaces V_1 and V_2 are of course metric spaces, with metrics ρ_1 and ρ_2 say.

Theorem 3. *Every sequence of analytic mappings between two finite local semi-analytic spaces contains a subsequence converging uniformly on compact subsets to a limit mapping which is also analytic.*

Proof. Since D_2 is a bounded domain it follows from Theorem 2 that, for some subsequence ϕ_k of mappings, the sequence $\pi_2\phi_k$ converges uniformly on compact subsets to an analytic mapping $\Psi: V_1 \rightarrow D_2$; but then for every point $p \in V_1$ the sequence $\phi_k(p)$ must have one of the finitely many points $\pi_2^{-1}\Psi(p)$ as a limit point, so that after suitable refining it is possible to assume that the sequence ϕ_k converges on a dense countable subset of V_1 . Now for an arbitrary point $p \in V_1$ select disjoint uniform open neighborhoods X_i around the points $p'_i \in \pi_2^{-1}\Psi(p)$; since these are uniform neighborhoods, it follows that there is an open neighborhood A of $\Psi(p)$ such that $\pi_2^{-1}(A) \subset \bigcup_i X_i$. In addition, since the mappings $\pi_2\phi_k$ converge uniformly on compact subsets, there is an open neighborhood W of p such that $\pi_2\phi_k(W) \subset A$ whenever $k > k'_0$. If $q \in W$ is a point at which the sequence $\phi_k(q)$ converges, then $\phi_k(q) \in X_i$ for some fixed i whenever $k > k_0$; but then $\phi_k(W) \subset X_i$ for that same i whenever $k > k_0$, so that $\phi_k(p)$ converges to p'_i . Thus the mappings ϕ_k converge pointwise to some mapping ϕ for which $\pi_2\phi = \Psi$; it remains to verify the uniformity of this convergence, from which it follows that ϕ is continuous, hence an analytic mapping. The procedure is the same as that used in the proof of Theorem 2. Fix an $\epsilon > 0$ and a uniform compact subset $K \subset V_1$. There exists an $\eta > 0$ such that whenever

$z_1, z_2 \in \Psi(K)$ and $|z_1 - z_2| < \eta$, each point $p_i(z_1) \in \pi_2^{-1}(z_1)$ is contained in an $\epsilon/4r_0$ -neighborhood of some point $p_k(z_2) \in \pi_2^{-1}(z_2)$; to see this, simply note that the function

$$\rho(z_1, z_2) = \min_{x \in S(r_0)} \max_{j=1, \dots, r_0} \rho_2[p_j(z_1), p_{\pi(j)}(z_2)],$$

where $S(r_0)$ is the symmetric group on r_0 letters, is continuous on $\Psi(K) \times \Psi(K)$, as in Lemma 1, hence uniformly continuous since $\Psi(K) \times \Psi(K)$ is compact. Consequently, since the mappings $\pi_2 \phi_k$ converge to Ψ uniformly on K , there exists a $\delta > 0$ and a k_0 such that whenever $p_1, p_2 \in K$, $\rho_1(p_1, p_2) < \delta$, and $k > k_0$, then $\phi_k(p_1)$ lies in an $\epsilon/4r_0$ -neighborhood of some point of $\pi_2^{-1}\Psi(p_2)$. Select finitely many points $p_i \in K$ such that open δ -neighborhoods W_i around the p_i cover K ; and choose k_0 sufficiently large so that, in addition, $\rho_2(\phi_k(p_i), \phi(p_i)) < \epsilon/4r_0$ whenever $k > k_0$, for all i . For each i take $\epsilon/4r_0$ -neighborhoods around the points of $\pi_2^{-1}\Psi(p_i)$, and let X_i be that connected component of the union of these neighborhoods which contains $\phi(p_i)$; clearly $\rho_2(q_1, q_2) < \frac{1}{2}\epsilon$ whenever $q_1, q_2 \in X_i$. Note further that $\rho_2(\phi_k(p_i), \phi(p_i)) < \epsilon/4r_0$, so that $\phi_k(p_i) \in X_i$ whenever $k > k_0$; but since $\phi_k(W_i)$ is connected and is contained within the $\epsilon/4r_0$ -neighborhoods of the points of $\pi_2^{-1}\Psi(p_i)$, then $\phi_k(W_i) \subset X_i$. Now for any point $p \in K$ there will be some i for which $p \in W_i$; thus $\phi_k(p) \in X_i$ for all $k > k_0$, so that $\phi(p) \in \bar{X}_i$ and $\rho_2(\phi(p), \phi_k(p)) < \epsilon$ for all $k > k_0$.

2. A point p of a local semi-analytic space V is called a *branch point* if $o(p) > 1$; the set of all non-branch points forms an open subset $V' \subset V$, which is a complex manifold with a natural complex structure induced by the projection $\pi: V' \rightarrow D$, since π is a local homeomorphism on V' . A local semi-analytic space V will be called a *local analytic space* if $\pi(V - V')$ is contained in a complex analytic subvariety of the domain D , and the mapping π is, in addition, proper (meaning that $\pi^{-1}(K)$ is compact whenever K is compact).

Theorem 4. *A complex-valued function on a local analytic space V is holomorphic if and only if it is continuous on V and is holomorphic on the complex manifold V' in the usual sense.*

Proof. Suppose that $f(p)$ is continuous on V and holomorphic on V' ; select an arbitrary point $p_0 \in V$ and a uniform neighborhood $U(p_0)$ of p_0 , and set $r = o(p_0)$. For any symmetric polynomial $\phi(x_1, \dots, x_r)$ the function $\phi_f(z)$ is continuous in $\pi(U(p_0))$ and holomorphic in $\pi(U(p_0)) \cap \pi(V - V')$; therefore by the ordinary removable singularities theorem [2] $\phi_f(z)$ is holomorphic on $\pi(U(p_0))$, and hence $f(p)$ is holomorphic on V . The converse is obvious.

The above result identifies the previously introduced notion of a holomorphic function on a local analytic space with the more usual approach to such functions. The manifold V' appearing in the discussion is not intrinsically defined however, since it is not necessarily preserved under analytic homeomorphisms between local analytic spaces; it is desirable to remedy this defect.

Lemma 2. *If $\phi: V_1 \rightarrow V_2$ is an analytic homeomorphism between two local analytic spaces and f is a holomorphic function on V_2 , then $f\phi$ is a holomorphic function on V_1 .*

Proof. Take any open neighborhood $U \subset V_1'$ and let $U' = \phi^{-1}(\phi(U) \cap V_2')$; then the restriction $\phi: U' \rightarrow \phi(U')$ is an analytic homeomorphism between two complex manifolds, so that $f\phi$ is holomorphic on U' . If U is sufficiently small there exists a function g holomorphic on $\phi(U)$ and vanishing on $\phi(U - U')$; for example, take the function induced on $\phi(U)$ by a holomorphic function on $\pi_2\phi(U)$ which vanishes on the analytic subvariety containing $\pi_2\phi(U) \cap \pi_2(V_2 - V_2')$. Then $g\phi$ is continuous on U and holomorphic wherever it is non-zero, since the latter set is certainly contained in U' ; but then by the removable singularities theorems [6] the function $g\phi$, hence also the function $f\phi$, is holomorphic on U . Therefore, by Theorem 4, $f\phi$ is holomorphic on V_1 .

A point p of a local analytic space V is called *uniformizable* if there is an open neighborhood U of p with an analytic homeomorphism $\phi: U \rightarrow X$ of U onto an open subset X of the space of n complex variables; and the remaining points are called singular points. The set of all uniformizable points of V forms an open subset V^0 of V , which clearly contains the manifold V' . It follows directly from Lemma 2 that the set V^0 is a complex manifold with a complex structure compatible with that of V' ; and that a homeomorphism $\phi: V_1 \rightarrow V_2$ between two local analytic spaces is an analytic homeomorphism if and only if the restriction $\phi: V_1^0 \rightarrow V_2^0$ is an analytic homeomorphism between the complex manifolds V_1^0 and V_2^0 . Now Theorem 4 holds also when V' is replaced by V^0 , and in that form is intrinsically connected with V as desired.

3. It is convenient to have another characterization of local analytic spaces, one which simplifies the verification that a given space is a local analytic space. A continuous, open, proper mapping π of a separable, connected, Hausdorff space V onto another such space D will be called a *quasi-covering* if there is a closed subset $S \subset D$ such that $D - S$ is arcwise connected, locally arcwise connected, and dense in D , and that the restricted mapping $\pi: V - \pi^{-1}(S) \rightarrow D - S$ is a covering in the usual sense [9]; S will be called the *exceptional set* of the quasi-covering π . A few topological properties of quasi-coverings which will be needed are contained in the following lemmas.

Lemma 3. *If $\pi: V \rightarrow D$ is a quasi-covering, then, for each point $z \in D$, (i) the set $\{p_i(z)\} = \pi^{-1}(z)$ is a finite set of points of V and (ii) if $\{U(p_i(z))\}$ are arbitrary open neighborhoods of the points $\{p_i(z)\}$, there is an open neighborhood $A(z)$ of z such that $\pi^{-1}(A(z)) \subset \bigcup_i U(p_i(z))$.*

Proof. Since π is proper, $V - \pi^{-1}(S)$ must be a finite-sheeted covering space of $D - S$ of, say, m sheets. Let z be any point of D , and select finitely many points $p_i(z) \in \pi^{-1}(z)$ and disjoint open neighborhoods $U(p_i(z))$ around them. By assumption there must be some point $z' \in \bigcap_i \pi(U(p_i(z))) \cap (D - S)$; but

the set $\pi^{-1}(z')$ consists of m points, at least one of which must lie in each $U(p_i(z))$, so that there can be at most m points in $\pi^{-1}(z)$ altogether. Next, to prove assertion (ii), let $\{p_i(z)\} = \pi^{-1}(z)$ and select arbitrary open neighborhoods $U(p_i(z))$ around them; also select a decreasing sequence $A_k(z)$ of neighborhoods of z forming a basis for the open neighborhoods of z . If (ii) were false, there would exist a sequence of points $q_k \in V$ such that $q_k \in \pi^{-1}(A_k(z)) - \bigcup_i U(p_i(z))$; the points $\pi(q_k)$ would converge to z , and since π is proper it would be possible to select a subsequence for which the points q_k would converge to some $q \in V$. But since $\pi(q) = p$, then $q = p_i(z)$ for some index j , and consequently $q_k \in U(p_j(z))$ for all sufficiently large k ; this contradiction completes the proof.

Lemma 4. *If $\pi: V \rightarrow D$ is a continuous, open, proper mapping between two separable Hausdorff spaces, $A \subset D$ is connected, and $B \subset V$ is a connected component of $\pi^{-1}(A)$, then $\pi(B) = A$.*

Proof. Suppose conversely that $\pi(B) \neq A$; then $A \cap \overline{\pi(B)} - \pi(B) \neq \emptyset$, for otherwise $\pi(B) = A \cap \overline{\pi(B)}$ would be both open and closed in A and hence $\pi(B) = A$. Select a point $z \in A \cap \overline{\pi(B)} - \pi(B)$ and a sequence of points $q_k \in B$ such that the points $\pi(q_k)$ converge to z ; since π is proper, there is some subsequence such that the points q_k converge to a point q for which $\pi(q) = z$. But since $q \in \pi^{-1}(A) \cap \bar{B}$ it follows that $q \in B$, hence that $z = \pi(q) \in \pi(B)$, which is a contradiction.

Lemma 5. *A quasi-covering $\pi: V \rightarrow D$ is o -disciplined for some mapping $o: V \rightarrow (\text{integers})$ such that $o(p) = 1$ whenever $\pi(p)$ is not in the exceptional set of the quasi-covering.*

Proof. Select an arbitrary point $p \in V$ and let $z = \pi(p)$. By Lemma 3 there is an arcwise connected open neighborhood $A(z)$ of z sufficiently small that the connected component $U(p)$ of $\pi^{-1}(A(z))$ which contains p contains no other points of the set $\pi^{-1}(z)$; consequently $\pi: U(p) \cap (V - \pi^{-1}(S)) \rightarrow D$ is a covering over some subset of D in the ordinary sense of, say, $o(p)$ sheets. If $A'(z) \subset A(z)$ is an arcwise connected subneighborhood, Lemma 4 shows that each connected component of $\pi^{-1}(A'(z)) \cap U(p)$ must contain p , hence that $\pi^{-1}(A'(z)) \cap U(p)$ is connected; an immediate consequence of this is that the number of sheets $o(p)$ is independent of the choice of $A(z)$, and hence is a well-defined mapping of V into the integers. For any point $z' \in A(z)$ let $\{p_i(z')\} = \pi^{-1}(z') \cap U(p)$; by Lemma 3 there is an arcwise connected subneighborhood $A(z') \subset A(z)$ sufficiently small so that the points $p_i(z')$ are contained in separate connected components $U(p_i(z'))$ of $\pi^{-1}(A(z')) \cap U(p)$. Each $U(p_i(z')) \cap (V - \pi^{-1}(S))$ is an $o(p_i(z'))$ -sheeted covering; and since there are altogether $o(p)$ sheets, each of which must lie in some $U(p_i(z'))$ by the same argument as above, it follows that $\sum_i o(p_i(z')) = o(p)$, hence that π is o -disciplined. The remainder is obvious.

Theorem 5. *A space is a local analytic space if and only if it is a quasi-covering of a bounded domain in the space of n complex variables whose exceptional set lies in an analytic subvariety of that domain.*

Proof. If $\pi: V \rightarrow D$ is a local analytic space then it is obviously a quasi-covering whose exceptional set $\pi(V - V')$ lies in an analytic subvariety. Conversely any such quasi-covering is clearly a local analytic space, as an immediate consequence of Lemma 5.

Remarks. It is clear that the analytic spaces with singularities, as defined by BEHNKE & STEIN [1], BOCHNER & MARTIN [3], CARTAN [7], and others, are such quasi-coverings; hence the maximum modulus theorem and the convergence theorems of Section 1 are established for all such spaces, while the results of Section 2 show that the definitions of holomorphic functions on such spaces are generally the same as considered here.

4. A Hausdorff space V is called an *analytic space* if it has a covering by open sets U_i , each of which admits a homeomorphism $\phi_i: U_i \rightarrow X_i$ onto a local analytic space X_i such that for each non-vacuous intersection $U_i \cap U_k$ the mapping $\phi_k \phi_i^{-1}: \phi_i(U_i \cap U_k) \rightarrow \phi_k(U_i \cap U_k)$ is an analytic homeomorphism. Note that the set of points of V which correspond to uniformizable points in the local analytic spaces X_i forms a well defined complex manifold $V^0 \subset V$; and, as a further consequence of the results of Section 2, a homeomorphism $\phi: V_1 \rightarrow V_2$ between two analytic spaces is an analytic homeomorphism if and only if the restriction $\phi: V_1^0 \rightarrow V_2^0$ is an analytic homeomorphism between the complex manifolds V_1^0 and V_2^0 in the usual sense.

Suppose that V is a compact analytic space; topologically it is a compact metric space, with metric $\rho'(p_1, p_2)$, say. Let G be the group of all analytic homeomorphisms of V onto itself, the action of a transformation $t \in G$ on a point $p \in V$ being denoted by $t(p)$. The group G is a metric space with the metric $\rho(t_1, t_2) = \sup_{p \in V} \rho'(t_1(p), t_2(p))$; in the topology so defined G is a topological group of transformations of the space V .

Lemma 6. *The group of analytic homeomorphisms of a compact analytic space onto itself is locally compact.*

Proof. Select finite open coverings $\{U_i\}, \{V_i\}$ of the analytic space, such that $\bar{U}_i \subset V_i$ for each i and that each V_i admits a homeomorphism $\phi_i: V_i \rightarrow X_i$ onto a local analytic space X_i ; and select a neighborhood $\bar{W}$ of the identity in G sufficiently small that $t(U_i) \subset V_i$ for all i and for all $t \in \bar{W}$. If $t_k \in \bar{W}$ is any sequence of transformations, then $\phi_i t_k \phi_i^{-1}$ is a sequence of analytic mappings of $\phi_i(U_i)$ into X_i ; hence by Theorem 3 there is some subsequence for which the mappings converge uniformly on compact subsets to a limit mapping t which is also analytic; but then $t \in \bar{W}$, so that $\bar{W}$ is compact.

Theorem 6. *The group of analytic homeomorphisms of a compact analytic space onto itself is a Lie group.*

Proof. The group of such mappings is locally compact by Lemma 6, and the restrictions are an effective group of analytic homeomorphisms of the

complex manifold V^0 onto itself in the usual sense; hence, as is well known, the group must be a Lie group [4, 5, 8].

Remark. A straightforward and hence uninteresting modification of the proof of the Bochner-Montgomery theorem shows that this group is indeed a complex Lie group [5].

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