On the Law of the Iterated Logarithm

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The Law of the Iterated Logarithm, as proved by KOLMOGOROFF [1], can be stated as follows:

Theorem A. Let $\chi_1, \chi_2, \dots$ be a sequence of independent random variables with mean values zero and dispersions equal to $b_1, b_2, \dots$ respectively. Let

$$S_n = \sum_{\nu=1}^n \chi_\nu, \quad B_n^2 = \sum_{\nu=1}^n b_\nu.$$

Under the conditions

$$(1) \quad B_n^2 \rightarrow \infty, \quad |\chi_n| \leq M_n = o\left(\frac{B_n^2}{\log \log B_n}\right)^{\frac{1}{2}}$$

we have

$$(2) \quad \limsup \frac{S_n}{(2B_n^2 \log \log B_n)^{\frac{1}{2}}} = 1$$

with probability 1.

MARCINKIEWICZ & ZYGMUND [2] have shown the necessity of condition (1) by constructing a sequence of random variables of the form $\{a_{\nu} r_{\nu}(\omega)\}$, where the $r_{\nu}(\omega)$ are the symmetric Bernoulli variables, for which

$$(3) \quad M_n \cong K \left(\frac{B_n^2}{\log \log B_n} \right)^{\frac{1}{2}},$$

$$(4) \quad \limsup \frac{S_n}{(2B_n^2 \log \log B_n)^{\frac{1}{2}}} < 1.$$

They also have shown that for every sequence of the form $\{a_{\nu} r_{\nu}(\omega)\}$ conclusion (4) with $\leq$ instead of $<$ holds, regardless of the behavior of the M_n .*

The object of this paper is to construct a sequence of independent random variables which satisfy (3) and for which

* In this connection, see the remark at the end of the paper.

$$\limsup \frac{S_n}{(2B_n^2 \log \log B_n)^{\frac{1}{2}}} > 1.$$

In view of the remark just made about the $\{r_n\}$, if we want to obtain this inequality, we must use different random variables.

We have the following theorem:

Theorem 1. *Let $\{\chi_n(x)\}$ be a sequence of independent random variables such that $\chi_n(x)$ takes the value $\exp \{v/\log^2 v\}$ with probability $\alpha/2 \log(v e^\alpha)$, the value $-\exp \{v/\log^2 v\}$ with probability $\alpha/2 \log(v e^\alpha)$, and the value zero elsewhere. Then, if α is sufficiently large, we have, using the notation of Theorem A,*

$$(5) \quad M_n \cong K \left(\frac{B_n^2}{\log \log B_n} \right)^{\frac{1}{2}},$$

$$(6) \quad \limsup \frac{S_n}{(2B_n^2 \log \log B_n)^{\frac{1}{2}}} > 1$$

with probability 1.

Remark. The main step in the proof of Theorem A is to show that, for small values of λ ,

$$(7) \quad e^{\frac{1}{2}\lambda^2 B_n^2 (1-\epsilon)} \leq \int e^{\lambda S_n} dx \leq e^{\frac{1}{2}\lambda^2 B_n^2 (1+\epsilon)}$$

where $\epsilon \rightarrow 0$ as $n \rightarrow \infty$. Now suppose that we could show for the sequence of random variables of Theorem 1 that

$$(8) \quad e^{\frac{1}{2}\lambda^2 B_n^2 (1+\eta-\epsilon)} \leq \int e^{\lambda S_n} dx \leq e^{\frac{1}{2}\lambda^2 B_n^2 (1+\eta+\epsilon)}$$

for a fixed $\eta > 0$ and $\epsilon \rightarrow 0$. Substituting $\lambda'^2 = \lambda^2(1+\eta)$ in (8) we see that it is equivalent to

$$(9) \quad e^{\frac{1}{2}\lambda'^2 B_n^2 (1-\delta)} \leq \int e^{\lambda' S_n / \sqrt{1+\eta}} dx \leq e^{\frac{1}{2}\lambda'^2 B_n^2 (1+\delta)}$$

with $\delta \rightarrow 0$. Since from equation (7) we derive (2), we would expect that from (9) we could derive

$$\limsup \frac{S_n}{(2B_n^2 \log \log B_n)^{\frac{1}{2}}} = (1+\eta)^{\frac{1}{2}} > 1.$$

This is the idea of the construction which follows, except that we obtain (8) with η depending on λ and $\epsilon = O(\eta^3)$.

We now proceed with the proof of Theorem 1. In what follows we will drop the subscript n when no confusion will arise.

Lemma 1. *There exists a constant K , depending on α , for which*

$$M_n \cong K \left(\frac{B_n^2}{\log \log B_n} \right)^{\frac{1}{2}}.$$

Proof. It is easy to see that

$$\begin{aligned} B_n^2 &= \sum_1^n b_\nu = \sum_1^n \int \chi_\nu^2(x) dx = \sum_1^n \frac{\alpha \exp \{2\nu/\log^2 \nu\}}{\log(\nu e^\alpha)} \\ &\cong \int_1^n \frac{\alpha \exp \{2x/\log^2 x\}}{\log(xe^\alpha)} dx \cong \frac{1}{2} \alpha \exp \{2n/\log^2 n\} \log n. \end{aligned}$$

Therefore

$$\left(\frac{B_n^2}{\log \log B_n} \right)^{\frac{1}{2}} \cong (\tfrac{1}{2}\alpha)^{\frac{1}{2}} \exp \{n/\log^2 n\} = (\tfrac{1}{2}\alpha)^{\frac{1}{2}} M_n.$$

Lemma 2. If $\lambda \exp \{n/\log^2 n\} < 1$ and n is large enough we have

$$(10) \quad e^{\frac{1}{2}\lambda^2 B_n^2 (1+\eta_\lambda - \epsilon)} \leq \int e^{\lambda S_n} dx \leq e^{\frac{1}{2}\lambda^2 B_n^2 (1+\eta_\lambda + \epsilon)}$$

where

$$\eta_\lambda = \lambda^2 \omega + \frac{8}{15} \lambda^4 \omega^2, \quad \omega = \frac{1}{24} \exp \{2n/\log^2 n\}, \quad \epsilon = O(\eta_\lambda^3).$$

Proof. Let $\beta_\nu = \exp \{2\nu/\log^2 \nu\}$, $\gamma_\nu = \alpha/\log(\nu e^\alpha)$. Then

$$\begin{aligned} \int e^{\lambda \chi_\nu(x)} dx &= \int \left(1 + \lambda \chi_\nu(x) + \frac{\lambda^2 \chi_\nu^2(x)}{2!} + \frac{\lambda^3 \chi_\nu^3(x)}{3!} + \dots \right) dx \\ &= 1 + \frac{\lambda^2 b_\nu}{2!} + \gamma_\nu^{-1} \frac{(\lambda^2 b_\nu)^2}{4!} + \gamma_\nu^{-2} \frac{(\lambda^2 b_\nu)^3}{6!} + \dots \\ &= 1 + \frac{\lambda^2 b_\nu}{2} \left[1 + \frac{\gamma_\nu^{-1} \lambda^2 b_\nu}{12} + \frac{(\gamma_\nu^{-1} \lambda^2 b_\nu)^2}{360} + O(\gamma_\nu^{-1} \lambda^2 b_\nu)^3 \right] \\ &= 1 + \frac{\lambda^2 b_\nu}{2} \left[1 + \frac{\lambda^2 \beta_\nu}{12} + \frac{2}{5} \left(\frac{\lambda^2 \beta_\nu}{12} \right)^2 + O(\lambda^2 \beta_\nu)^3 \right] \\ &= 1 + \frac{\lambda^2 b_\nu}{2} [1 + \eta_{\nu, \lambda} + O(\eta_{\nu, \lambda}^3)]. \end{aligned}$$

Using the inequality $\exp \{u(1-u)\} \leq 1+u \leq \exp \{u\}$, $u > 0$, it follows that

$$\begin{aligned} (11) \quad \exp \{ \tfrac{1}{2} \lambda^2 b_\nu (1 + \eta_{\nu, \lambda} - \epsilon)(1 - \tfrac{1}{2} \lambda^2 b_\nu (1 + \eta_{\nu, \lambda})) \} &\leq \int e^{\lambda \chi_\nu(x)} dx \\ &\leq \exp \{ \tfrac{1}{2} \lambda^2 b_\nu (1 + \eta_{\nu, \lambda} + \epsilon) \} \end{aligned}$$

where $\epsilon = O(\lambda^2 \beta_\nu)^3$. Now since $\int e^{\lambda S_n} dx = \prod_1^n \int e^{\lambda \chi_\nu} dx$ and $\lambda^2 b_\nu = o(1)$, it follows from (11) that

$$(12) \quad \exp \left\{ \sum_1^n \frac{1}{2} \lambda^2 b_\nu (1 + \eta_{\nu, \lambda} - \epsilon) \right\} \leq \int e^{\lambda s_n} dx$$

$$\leq \exp \left\{ \sum_1^n \frac{1}{2} \lambda^2 b_\nu (1 + \eta_{\nu, \lambda} + \epsilon) \right\}.$$

We now consider $\sum_1^n \frac{1}{2} \lambda^2 b_\nu (1 + \eta_{\nu, \lambda} + \epsilon)$. Clearly it is equal to

$$\frac{\lambda^2 B_n^2}{2} \left[1 + \frac{B_n^{-2} \lambda^2}{12} \sum_1^n b_\nu \beta_\nu + \frac{2B_n^{-2}}{5} \left(\frac{\lambda^2}{12} \right)^2 \sum_1^n b_\nu \beta_\nu^2 + AB_n^{-2} \lambda^6 \sum_1^n b_\nu \beta_\nu^3 \right].$$

But

$$\begin{aligned} \sum_1^n b_\nu \beta_\nu &= \sum_1^n \frac{\alpha \exp \{4\nu/\log^2 \nu\}}{\log (\nu e^\alpha)} \cong \alpha \int_1^n \frac{\exp \{4x/\log^2 x\}}{\log (xe^\alpha)} dx \\ &\cong \frac{1}{4} \alpha \log n \exp \{4n/\log^2 n\} \cong \frac{1}{2} B_n^2 \exp \{2n/\log^2 n\}; \\ \sum_1^n b_\nu \beta_\nu^2 &= \sum_1^n \frac{\alpha \exp \{6\nu/\log^2 n\}}{\log (\nu e^\alpha)} \cong \alpha \int_1^n \frac{\exp \{6x/\log^2 x\}}{\log (xe^\alpha)} dx \\ &\cong \frac{1}{6} \alpha \log n \exp \{6n/\log^2 n\} \cong \frac{1}{3} B_n^2 \exp \{4n/\log^2 n\}; \\ \sum_1^n b_\nu \beta_\nu^6 &\cong \frac{1}{4} B_n^2 \exp \{6n/\log^2 n\}. \end{aligned}$$

Now if we take $\omega = \frac{1}{24} \exp \{2n/\log^2 n\}$ we have

$$\sum_1^n \frac{\lambda^2 b_\nu^2}{2} (1 + \eta_{\nu, \lambda} + \epsilon) \cong \frac{\lambda^2 B_n^2}{2} [1 + \lambda^2 \omega + \frac{8}{15} \lambda^4 \omega^2 + O(\lambda^2 \omega)^2]$$

which gives us the conclusion of Lemma 2.

Lemma 3. *If α is large enough then there exists a $\delta > 0$ such that for large enough n*

$$(13) \quad W_n((1 + \delta)(2B_n^2 \log \log B_n)^{\frac{1}{2}}) \geq \frac{1}{\log B_n}$$

where $W_n(y) = |\{x: S_n(x) > y\}|$.

Proof. The reason for taking α large is as follows. It will be seen later that we will need η_λ , for values of λ less than $8(B_n^{-2} \log \log B_n)^{\frac{1}{2}}$, to be so small that certain terms which are $O(\eta_\lambda^2)$ are less than others which are $O(\eta_\lambda)$, etc. Now if λ is less than $8(B_n^{-2} \log \log B_n)^{\frac{1}{2}}$ then

$$\eta_\lambda = \lambda^2 \omega + \frac{8}{15} \lambda^4 \omega^2 \leq \frac{64}{24\alpha} + \frac{8}{15} \left(\frac{64}{24\alpha} \right)^2$$

which is small if α is large.

The general idea of the proof is as follows. We break the integral $\int e^{as} dx$ into six integrals over the real line. That is, we write

$$\begin{aligned}
\int e^{as} dx &= a \int_{-\infty}^{\infty} e^{ay} W(y) dy \\
&= a \left(\int_{-\infty}^0 + \int_0^{aB^2\delta_1} + \int_{aB^2\delta_1}^{aB^2\delta_2} + \int_{aB^2\delta_2}^{aB^2\delta_3} + \int_{aB^2\delta_3}^{8aB^2} + \int_{8aB^2}^{\infty} e^{ay} W(y) dy \right) \\
&= a(J_1 + J_2 + J_3 + J_4 + J_5 + J_6).
\end{aligned}$$

We then show that

$$(14) \quad a(J_1 + J_2 + J_4 + J_5 + J_6) < \frac{1}{2} \int e^{as} dx$$

and therefore it follows from (10) that

$$aJ_3 > \frac{1}{2} e^{\frac{1}{2}a^2B^2(1+\eta_\lambda-\epsilon)}.$$

We then use this last inequality to obtain (13).

We take the following values for a , δ_1 , δ_2 , and δ_3 :

$$\begin{aligned}
\delta_1 &= 1 + 2\eta_a - \frac{1}{16}\eta_a, & \delta_2 &= 1 + 2\eta_a + \frac{1}{16}\eta_a, & \delta_3 &= 1 + 4\eta_a, \\
(15) \quad a &= \frac{1 + 10\eta_{\lambda_0}^2}{1 + 2\eta_{\lambda_0} - \frac{1}{16}\eta_{\lambda_0}} (2B^{-2} \log \log B)^{\frac{1}{2}} \quad \text{where } \lambda_0 = (2B^{-2} \log \log B)^{\frac{1}{2}}.
\end{aligned}$$

In the estimates which follow we will mean by $O(\eta^2)$, etc., a fixed polynomial in η^2 with zero constant term, which we do not need to calculate. We will occasionally use the notation $\epsilon + o(1) = \epsilon$.

Estimation of aJ_1 .

$$aJ_1 = a \int_{-\infty}^0 e^{ay} W(y) dy \leq a \int_{-\infty}^0 e^{ay} dy = 1.$$

For n large enough we have $1 < \frac{1}{8} e^{\frac{1}{8}a^2B^2} < \frac{1}{8} \int e^{as} dx$ and hence

$$(16) \quad aJ_1 < \frac{1}{8} \int e^{as} dx.$$

Estimation of aJ_6 .

Since $e^{\lambda y} W(y) \leq \int e^{\lambda s} dx \leq \exp\{\frac{1}{2}\lambda^2 B^2 (1 + \eta_\lambda + \epsilon)\}$ it follows that for $\lambda = 4a$ we have (provided $4a \exp\{n/\log^2 n\} < 1$ which is certainly true if a is large enough) $W(y) \leq \exp\{16a^2 B^2 - 4ay\}$. In the range $8aB^2 \leq y < \infty$ we have $2ay \geq 16a^2 B^2$ and hence $W(y) \leq e^{-2ay}$. Therefore $a \int_{8aB^2}^{\infty} e^{ay} W(y) dy \leq a \int_{8aB^2}^{\infty} e^{-2ay} dy \leq 1$. This gives us

$$(17) \quad aJ_6 \leq \frac{1}{8} \int e^{as} dx.$$

Estimation of $a(J_2 + J_4)$.

Since

$$(18) \quad W(y) \leq \exp\{\frac{1}{2}\lambda^2 B^2 (1 + \eta_\lambda + \epsilon) - \lambda y\}$$

by taking $\lambda = y/(B^2(1 + 2\eta_a)) = y/B^2\gamma$ we have, for $y \leq aB^2\delta_3$,

$$\begin{aligned} \frac{\lambda^2 B^2}{2} (1 + \eta_\lambda + \epsilon) - \lambda y &= \frac{\lambda^2 B^2}{2} \left(1 + \lambda^2 \omega + \frac{8}{15} \lambda^4 \omega^2 + \epsilon \right) - \lambda y \\ &\leq \frac{\lambda^2 B^2}{2} \left(1 + \lambda^2 \omega + \frac{8}{15} \left(\frac{a\delta_3}{\gamma} \right)^4 \omega^2 + \epsilon \right) - \lambda y \\ &= \frac{y^4 \omega}{2B^6 \gamma^4} + \frac{y^2}{2B^2 \gamma^2} \left(1 + \frac{8}{15} a^4 \omega^2 \frac{\delta_3^4}{\gamma^4} + \epsilon - 2\gamma \right). \end{aligned}$$

Therefore

$$(19) \quad a(J_2 + J_4) \leq a \left(\int_0^{aB^2\delta_1} + \int_{aB^2\delta_2}^{aB^2\delta_3} e^{\sigma(y)} dy \right)$$

where

$$g(y) = \frac{y^4 \omega}{2B^6 \gamma^4} + \frac{y^2}{2B^2 \gamma^2} \left(1 + \frac{8}{15} a^4 \omega^2 \frac{\delta_3^4}{\gamma^4} + \epsilon - 2\gamma \right) + ay$$

Now we will show that $g(y)$ has one maximum which is assumed between $aB^2\delta_1$ and $aB^2\delta_2$. If we show this we will then have, from inequality (19),

$$(20) \quad a(J_2 + J_4) \leq 2a^2 B^2 (e^{\sigma(aB^2\delta_1)} + e^{\sigma(aB^2\delta_2)}).$$

Differentiating $g(y)$ we obtain

$$g'(y) = \frac{y^3}{B^6 \gamma^4} 2\omega + \frac{y}{B^2 \gamma^2} \left(1 + \frac{8}{15} a^4 \omega^2 \frac{\delta_3^4}{\gamma^4} + \epsilon - 2\gamma \right) + a.$$

Thus

$$g'(aB^2\delta_1) = \frac{a^3 \delta_1^3}{\gamma^4} 2\omega + \frac{a\delta_1}{\gamma^2} \left(1 + \frac{8}{15} a^4 \omega^2 \frac{\delta_3^4}{\gamma^4} + \epsilon - 2\gamma \right) + a.$$

Hence $g'(aB^2\delta_1) > 0$ if and only if

$$(21) \quad \delta_1^3 2a^2 \omega + \delta_1 \gamma^2 \left(\frac{8}{15} a^4 \omega^2 \frac{\delta_3^4}{\gamma^4} + \epsilon \right) + \gamma^4 > \delta_1 \gamma^2 (1 + 4\eta_a).$$

Expanding the left side of (21) we obtain

$$1 + 10\eta + 36\eta^2 - \frac{8}{15}\eta^2 - \frac{6}{16}\eta^2 + O(\eta^3).$$

For the right side we get

$$1 + 10\eta + 36\eta^2 - \frac{1}{16}\eta - \frac{1}{2}\eta^2 + O(\eta^3).$$

Thus $g'(aB^2\delta_1) > 0$ if and only if

$$(22) \quad \left(\frac{8}{15} - \frac{1}{8} \right) \eta^2 + O(\eta^3) < \frac{1}{16} \eta.$$

Similarly $g'(aB^2\delta_2) < 0$ if and only if

$$(23) \quad \frac{1}{16} \eta > -\frac{1}{8} \eta^2 - \frac{8}{15} \eta^2 + O(\eta^3).$$

Clearly for η small enough, (22) and (23) hold, hence $g'(y)$ has a zero between $aB^2\delta_1$ and $aB^2\delta_2$, and hence $g(y)$ has a maximum there. To show that there is no other maximum in $(0, aB^2\delta_3)$ we have only to perform the same elementary computations as above to see that $g'(y) > 0$ for $y \leq aB^2\delta_1$ and $g'(y) < 0$ for $y \geq aB^2\delta_2$. Hence we have established equality (20).

We would like to show

$$(24) \quad a(J_2 + J_4) < \frac{1}{8} \int e^{a^S} dx.$$

In order to establish (24) it is sufficient to show that

$$(25) \quad 8a(J_2 + J_4) < e^{\frac{1}{2}a^2B^2(1+\eta_a-\epsilon)}$$

Since from (20) we have

$$8a(J_2 + J_4) \leq \frac{1}{2} \exp \{g(aB^2\delta_1) + a^2B^2 \cdot o(1)\} + \frac{1}{2} \exp \{g(aB^2\delta_2) + a^2B^2 \cdot o(1)\}$$

it is sufficient to establish the inequalities

$$(26) \quad g(aB^2\delta_1) \leq \frac{1}{2}a^2B^2(1 + \eta_a - \epsilon),$$

$$(27) \quad g(aB^2\delta_2) \leq \frac{1}{2}a^2B^2(1 + \eta_a - \epsilon).$$

Clearly

$$g(aB^2\delta_1) = \frac{1}{2}a^2B^2[a\omega\delta_1^4\gamma^{-4} + \delta_1^2\gamma^{-2}(1 + \frac{8}{15}a^4\omega^2\delta_3^4\gamma^{-4} + \epsilon - 2\gamma) + 2\delta_1].$$

Hence (26) is equivalent to

$$a^2\omega\delta_1^4\gamma^{-4} - \delta_1^2\gamma^{-2}(1 + 4\eta_a - \frac{8}{15}\eta_a^2 - \epsilon) + 2\delta_1 < 1 + \eta_a - \epsilon$$

which is equivalent to

$$(28) \quad a^2\omega\delta_1^4 + \delta_1^2\gamma^2(\frac{8}{15}\eta^2 + \epsilon) + \gamma^4(1 + 3\eta - \frac{1}{8}\eta) < \gamma^2\delta_1^2(1 + 4\eta).$$

The left side of (28) is equal to

$$\begin{aligned} & (\eta - \frac{8}{15}\eta^2)(1 + 2\eta)^4 + (1 + 2\eta - \frac{1}{16}\eta)^2(1 + 2\eta)^2(\frac{8}{15}\eta^2 + \epsilon) \\ & + (1 + 2\eta)^4(1 + 3\eta - \frac{1}{8}\eta) = 1 + 12\eta - \frac{1}{8}\eta + 55\eta^2 + O(\eta^3). \end{aligned}$$

The right side of (28) is equal to

$$(1 + 2\eta)^2(1 + 2\eta - \frac{1}{16}\eta)(1 + 4\eta) = 1 + 12\eta - \frac{1}{8}\eta + 55\eta^2 + \frac{1}{16}\eta^2 + O(\eta^3).$$

Therefore (26) is true provided

$$(29) \quad \frac{\eta^2}{(16)^2} > O(\eta^3)$$

which is true if η is small enough.

In exactly the same manner we can show that (29) is a sufficient condition for (27) to hold. Thus (29) implies (24).

Estimation of aJ_5 .

Clearly, arguing as above, we have

$$aJ_5 \leq a \int_{aB^2\delta_3}^{8aB^2} e^{h(y)} dy$$

where

$$h(y) = \frac{y^4 \omega}{2B^6 \gamma^4} + \frac{y^2}{2B^2 \gamma^2} \left(1 + \frac{(64)^2}{\gamma^4} a^4 \omega^2 + \epsilon - 2\gamma \right) + ay.$$

Arguing exactly as in the estimation of $J_2 + J_4$ we can show that if $\eta > O(\eta^2)$ then

$$h(y) + \log 64a^2B^2 < \frac{1}{2}a^2B^2(1 + \eta - \epsilon)$$

which implies

$$(30) \quad aJ_5 < \frac{1}{8} \int e^{a^2S} dx.$$

Collecting (16), (17), (24), and (30) we see that we have (14) and hence, as was discussed before,

$$(31) \quad aJ_3 > \frac{1}{2} \exp \{ \frac{1}{2}a^2B^2(1 + \eta - \epsilon) \}$$

This implies

$$a^2B^2\delta_2 \exp \{ a^2B^2\delta_2 \} W(aB^2\delta_1) \geq aJ_3 \geq \frac{1}{2} \exp \{ \frac{1}{2}a^2B^2(1 + \eta - \epsilon) \}$$

which in turn implies

$$(32) \quad W(aB^2\delta_1) \geq \exp \{ \frac{1}{2}a^2B^2(1 + \eta - \epsilon - 2\delta_2) \}.$$

If we can show that

$$(33) \quad aB^2\delta_1 = (1 + \delta)(2B^2 \log \log B)^{\frac{1}{4}},$$

$$(34) \quad \frac{1}{2}a^2B^2(1 + \eta - \epsilon - 2\delta_2) > -\log \log B$$

then it is clear from inspection of (32) that we will have proved our lemma.

Proof of (33).

Using the abbreviations $\eta_{\lambda_0} = \eta_0$, $\eta_a = \eta$, we see that

$$\begin{aligned} aB^2\delta_1 &= \frac{1 + 10\eta_0^2}{1 + 2\eta_0 - \frac{1}{16}\eta_0} (2B^{-2} \log \log B)^{\frac{1}{4}} B^2(1 + 2\eta - \frac{1}{16}\eta) \\ &= (1 + 10\eta_0^2) \left(\frac{1 + 2\eta - \frac{1}{16}\eta}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right) (2B^2 \log \log B)^{\frac{1}{4}}. \end{aligned}$$

Now

$$\eta_0 = (2B^{-2} \log \log B)\omega + \frac{8}{15}(2B^{-2} \log \log B)^2\omega^2 \simeq \frac{1}{24\alpha} + \frac{8}{15} \left(\frac{1}{24\alpha} \right)^2;$$

$$\begin{aligned}
\eta &= a^2\omega + \frac{8}{15} (a^2\omega)^2 = \left(\frac{1 + 10\eta_0^2}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right)^2 \left(\frac{2 \log \log B}{B^2} \right) \omega \\
&\quad + \frac{8}{15} \left(\frac{1 + 10\eta_0^2}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right)^4 \left(\frac{2 \log \log B}{B^2} \right) \omega^2 \\
&\cong \left(\frac{1 + 10\eta_0^2}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right)^2 \frac{1}{24\alpha} + \frac{8}{15} \left(\frac{1 + 10\eta_0^2}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right)^4 \left(\frac{1}{24\alpha} \right)^2 \\
&= \frac{1}{24\alpha} (1 - 4\eta_0 - \frac{1}{8}\eta_0 + O(\eta_0^2)) + \frac{8}{15} \left(\frac{1}{24\alpha} \right)^2 (1 + O(\eta_0)) \\
&= \eta_0 (1 - 4\eta_0 - \frac{1}{8}\eta_0 + O(\eta_0^2)).
\end{aligned}$$

Therefore

$$\begin{aligned}
aB^2\delta_1 &= (1 + 10\eta_0^2) \left(1 - \frac{18\eta_0^2 - \frac{1}{128}\eta_0^2 + O(\eta_0^3)}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right) (2B^2 \log \log B)^{\frac{1}{2}} \\
&> (1 + 10\eta_0^2) (1 - 8\eta_0^2) (2B^2 \log \log B)^{\frac{1}{2}} > (1 + \eta_0^2) (2B^2 \log \log B)^{\frac{1}{2}}
\end{aligned}$$

for η_0 small enough. This shows (33).

Proof of (34).

Since

$$\frac{a^2 B^2}{2} (1 + \eta - \epsilon - 2\delta_2) = - \left(\frac{1 + 10\eta_0^2}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right)^2 (1 + 3\eta_0 + \frac{1}{8}\eta_0 + \epsilon) \log \log B$$

inequality (34) is equivalent to

$$(35) \quad \left(\frac{1 + 10\eta_0^2}{1 + 2\eta_0 - \frac{1}{16}\eta_0} \right)^2 (1 + 3\eta + \frac{1}{8}\eta + \epsilon) < 1.$$

But the left side of (35) is $(1 - \frac{3}{4}\eta_0 + O(\eta_0^2))$ which is less than 1 for η_0 small enough.

Lemma 4. *If α and n are large enough, then*

$$W(2B_n^2 \log \log B_n)^{\frac{1}{2}} \leq \frac{1}{(\log B_n)^2}.$$

Proof. From Lemma 2 it follows that $e^{\lambda y} W(y) \leq \exp\{\lambda^2 B^2\}$ for $\lambda \exp\{n/\log^2 n\} < \frac{1}{4}$. Therefore

$$(36) \quad W(y) \leq \exp\{\lambda^2 B^2 - \lambda y\}$$

Taking

$$y = 2(2B^2 \log \log B)^{\frac{1}{2}}, \quad \lambda = (2B^{-2} \log \log B)^{\frac{1}{2}}$$

(we note that $\lambda \exp\{n/\log^2 n\} \cong 2/\alpha < \frac{1}{4}$ for $\alpha > 8$) we obtain, from (36),

$$W(2(2B^2 \log \log B)^{\frac{1}{2}}) \leq \exp \{2 \log \log B - 4 \log \log B\} = \frac{1}{(\log B)^2}.$$

We turn now to the proof of the theorem.

We first choose $\theta > 1$, to be determined later, and choose a sequence $\{n_k\}$ of integers such that n_k is the least integer m for which $B_m^2 > \theta^k$. Now since

$$B_{n_{k+1}}^2 / B_{n_k}^2 \rightarrow 1$$

it is clear that, for any fixed $\gamma > 0$ and for all k large enough, $\theta^k < B_{n_k}^2 < (1 + \gamma)\theta^k$.

Let

$$S'_{n_k} = S_{n_k} - S_{n_{k-1}}$$

and let

$$\begin{aligned} U_k &= \{x: S_{n_k}(x) > (1 + \delta)(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}\}, \\ V_k &= \{x: |S_{n_{k-1}}(x)| > 2(2B_{n_{k-1}}^2 \log \log B_{n_{k-1}})^{\frac{1}{2}}\}, \\ W_k &= \{x: S'_{n_k}(x) > (1 + \frac{3}{4}\delta)(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}\}. \end{aligned}$$

Clearly

$$\begin{aligned} \frac{S'_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} &\geq \frac{S_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} \\ &\quad - \frac{|S_{n_{k-1}}(x)|}{(2B_{n_{k-1}}^2 \log \log B_{n_{k-1}})^{\frac{1}{2}}} \left(\frac{2B_{n_{k-1}}^2 \log \log B_{n_{k-1}}}{2B_{n_k}^2 \log \log B_{n_k}} \right)^{\frac{1}{2}}. \end{aligned}$$

Now for k large enough

$$\left(\frac{B_{n_{k-1}}^2 \log \log B_{n_{k-1}}}{B_{n_k}^2 \log \log B_{n_k}} \right)^{\frac{1}{2}} \leq \left(\frac{4}{\theta} \right)^{\frac{1}{2}}$$

and hence if $\theta^{\frac{1}{2}} > 16/\delta$ we have, for $x \in U_k - V_k$,

$$\frac{S'_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} > (1 + \delta) - 2 \left(\frac{4}{\theta} \right)^{\frac{1}{2}} > 1 + \frac{3}{4}\delta.$$

Thus $W_k \supset U_k - V_k$. By Lemmas 3 and 4 we have, where the A_i 's are constants,

$$\begin{aligned} \sum_{k=1}^{\infty} |W_k| &\geq \sum_{k=1}^{\infty} |U_k| - \sum_{k=1}^{\infty} |V_k| \geq A_1 \sum_1^{\infty} \frac{1}{\log B_{n_k}} - A_2 \sum_1^{\infty} \frac{1}{(\log B_{n_k})^2} \\ &\geq A_3 \left(\sum_1^{\infty} \frac{1}{k \log \theta} - \sum_1^{\infty} \frac{1}{k^2 \log^2 \theta} \right) = \infty. \end{aligned}$$

Now the $S'_{n_k}(x)$ are independent and hence so are the sets W_k . Thus $\sum |W_k| = \infty$ implies that almost every point x is in an infinite number of the sets W_k . This means that

$$(37) \quad \limsup \frac{S'_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} \geq 1 + \frac{3}{4}\delta$$

with probability 1.

If k_0 is any fixed integer and $x \notin \bigcup_{k=k_0}^{\infty} V_k$ then, for all $k > k_0$,

$$\frac{|S_{n_{k-1}}(x)|}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} < \frac{\delta}{4}$$

and hence

$$\begin{aligned} \frac{S_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} &\geq \frac{S'_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} - \frac{|S_{n_{k-1}}(x)|}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} \\ &\geq \frac{S'_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} - \frac{\delta}{4}. \end{aligned}$$

This, together with (37), means that for almost every x in the complement of $\bigcup_{k=k_0}^{\infty} V_k$ we have

$$(38) \quad \limsup \frac{S_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} \geq \limsup \frac{S'_{n_k}(x)}{(2B_{n_k}^2 \log \log B_{n_k})^{\frac{1}{2}}} - \frac{\delta}{4} \geq 1 + \frac{\delta}{2}.$$

Since $\sum_1^{\infty} |V_k| < \infty$, almost every x is in the complement of $\bigcup_{k_0}^{\infty} V_k$ for some integer $k_0(x)$. Hence (38) is satisfied with probability 1 and we have proved our theorem.

Remark. This fact led MARCINKIEWICZ to the following observation which apparently has never been published and which, at the suggestion of Professor ZYGMUND, I reproduce here. Let $\chi_1(t), \chi_2(t), \dots, \chi_\nu(t), \dots$ be any sequence of independent and *symmetric* random variables. Then, with probability 1, we have

$$(i) \quad \limsup \frac{S_n(t)}{(2\sigma_n^2(t) \log \log \sigma_n(t))^{\frac{1}{2}}} \leq 1$$

where $S_n(t) = \chi_1(t) + \dots + \chi_n(t)$ and $\sigma_n^2(t) = \chi_1^2(t) + \dots + \chi_n^2(t)$. For the proof we consider the series $\sum \chi_\nu(t)r_\nu(\omega)$. By the result of MARCINKIEWICZ & ZYGMUND we have, for almost every t and some ω_0 (actually for almost every ω),

$$(ii) \quad \limsup \frac{\sum_1^n \chi_\nu(t)r_\nu(\omega_0)}{(2\sigma_n^2(t) \log \log \sigma_n(t))^{\frac{1}{2}}} \leq 1.$$

It is enough to observe, the variables χ_ν being symmetric, that $\{\chi_\nu(t)\}$ and $\{\chi_\nu(t)r_\nu(\omega_0)\}$ have the same joint distribution functions and hence (ii) implies (i).

It is natural to ask if anything resembling (i) is true for non-symmetric random variables. It is, however, immediately evident that in this case it is possible to construct a sequence for which

$$(iii) \quad \limsup \frac{S_n(t)}{(2\sigma_n^2(t) \log \log \sigma_n(t))^{\frac{1}{2}}} = \infty.$$

For example, we take a sequence of independent sets $\{E_\nu\}$ for which $\sum_1^\infty |CE_\nu| < \infty$. Then if

$$\chi_\nu(t) = 1, \quad t \notin E_\nu; \quad \chi_\nu(t) = -\frac{|E_\nu|}{|CE_\nu|}, \quad t \in CE_\nu,$$

it is clear that (iii) holds.

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