A Factorization of Positive Hermitian Matrices

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It is our purpose to give an independent proof of the matrix-analogue of a classical theorem of G. SZEGÖ¹ which runs as follows:

If $F(\theta)$, $0 \leq \theta \leq 2\pi$, is a non-negative function of class L^1 , and if

$$-\infty < \int_0^{2\pi} \log F(\theta) d\theta,$$

then there exists a complex-valued function $G(\theta)$ of class L^2 , the Fourier series of which is of power series type,

$$G(\theta) \sim \sum_0^\infty g_n e^{in\theta},$$

and such that

$$F(\theta) = |G(\theta)|^2$$

for almost all θ .

The generalization of this result, to be proved in this paper, is the following

Theorem. *If $H(\theta) = (h_{jk}(\theta))$ is an Hermitian positive-definite matrix-valued function defined on $0 \leq \theta \leq 2\pi$ such that*

$$h_{jk}(\theta) \text{ belongs to } L^1(0, 2\pi) \quad (j, k = 1, \dots, q)$$

and

$$-\infty < \int_0^{2\pi} \log \det H(\theta) d\theta,$$

then $H(\theta)$ can be factored as

$$H(\theta) = N(\theta)N(\theta)^*,$$

¹ G. SZEGÖ, *Orthogonal Polynomials*, A.M.S. Colloq. Publ., 1939, §10.2.

where $N(\theta) = (n_{ik}(\theta))$ is such that

$$n_{ik}(\theta) \text{ belongs to } L^2(0, 2\pi) \quad (j, k = 1, \dots, q)$$

and each $n_{ik}(\theta)$ has a Fourier series of power series type (without negative frequencies).

Although the essential idea of the present argument derives from the theory of linear least-squares prediction of multiple time series, no use will be made of the theory of stochastic processes in what follows.²

The theorem of SZEGÖ is precisely the special case of the general theorem for a 1×1 matrix, and will constitute the first step in the inductive proof of the general theorem. Therefore the present argument concerns the step from a $q - 1 \times q - 1$ matrix to a $q \times q$ matrix.

1. A generalization of Hadamard's determinant inequality. We shall use the following notation only in this section in order to state a determinantal inequality. For any $q \times q$ matrix $H \equiv (h_{ik})$ with complex coefficients, the symbol

$$H \begin{pmatrix} j_1 j_2 & \cdots & j_p \\ k_1 k_2 & \cdots & k_p \end{pmatrix}$$

stands for the determinant

$$\begin{vmatrix} h_{j_1 k_1} & h_{j_2 k_1} & \cdots & h_{j_p k_1} \\ h_{j_1 k_2} & h_{j_2 k_2} & \cdots & h_{j_p k_2} \\ \vdots & \vdots & & \vdots \\ h_{j_1 k_p} & h_{j_2 k_p} & \cdots & h_{j_p k_p} \end{vmatrix}.$$

If H is a $q \times q$ Hermitian positive-definite matrix, then for $p < q$

$$(1.1) \quad H \begin{pmatrix} 1 & 2 & \cdots & q \\ 1 & 2 & \cdots & q \end{pmatrix} \leq H \begin{pmatrix} 1 & 2 & \cdots & p \\ 1 & 2 & \cdots & p \end{pmatrix} H \begin{pmatrix} p+1 & \cdots & q \\ p+1 & \cdots & q \end{pmatrix}.$$

This implies for an Hermitian positive-definite matrix-valued function $H(\theta) = (h_{ik}(\theta))$, $0 \leq \theta \leq 2\pi$, that

$$\det H(\theta) \leq h_{11}(\theta) \cdots h_{qq}(\theta),$$

² WIENER & MASANI, *Acta Math.* 98 (1957), p. 148, have obtained the same factorization theorem as a corollary of certain facts in the time-domain analysis of stationary multivariate stochastic processes. Their deduction does not show clearly how the integrability of the logarithm of the determinant of the given matrix implies its factorability. A difficulty, mentioned by WIENER & MASANI (*loc. cit.* p. 112) and having its origin in the possibility that the components of the vector process span subspaces "inclined at zero angle," does not in fact arise as shown in the present note.

HELSON & LOWDENSLAGER, *Acta Math.* 99 (1958), have also proved this theorem by still different methods.

and, hence, if

$$-\infty < \int_0^{2\pi} \log \det H(\theta) d\theta,$$

that

$$-\infty < \int_0^{2\pi} \log h_{aa}(\theta) d\theta,$$

and therefore

$$(1.2) \quad -\infty < \int_0^{2\pi} \log H \begin{pmatrix} 1 & 2 & \cdots & q-1 \\ 1 & 2 & \cdots & q-1 \end{pmatrix} d\theta.$$

(1.2) will be made use of directly.

2. Preliminary reductions. Suppose $H(\theta) = (h_{ik}(\theta))$ is a given Hermitian positive-definite $q \times q$ matrix function defined on $0 \leq \theta \leq 2\pi$ and such that

$$(2.1) \quad h_{ik}(\theta) \text{ belongs to } L^1(0, 2\pi)$$

and

$$(2.2) \quad -\infty < \int_0^{2\pi} \log \det H(\theta) d\theta.$$

Conditions (2.1) and (2.2) imply

$$-\infty < \int_0^{2\pi} \log h_{ii}(\theta) d\theta < \infty, \quad j = 1, \dots, q.$$

Therefore SZEGÖ's theorem implies $h_{ii}(\theta) = |g_i(\theta)|^2$, where g_i belongs to $L^2(0, 2\pi)$ and has Fourier series with only non-negative frequencies. Therefore

$$(2.3) \quad H(\theta) = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & g_q \end{bmatrix} \begin{bmatrix} 1 & h_{12}/g_1\bar{g}_2 & \cdots & h_{1q}/g_1\bar{g}_q \\ h_{21}/\bar{g}_1g_2 & 1 & & \\ \vdots & & \ddots & \\ h_{q1}/\bar{g}_1g_q & \cdots & & 1 \end{bmatrix} \begin{bmatrix} \bar{g}_1 & 0 & \cdots & 0 \\ 0 & \bar{g}_2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \bar{g}_q \end{bmatrix}$$

for almost all θ .³ Writing, for $1 < j, k < q$,

$$\begin{aligned} f_{jk}(\theta) &= h_{jk}(\theta)/(g_j(\theta)\bar{g}_k(\theta)) & \text{if } j < k, \\ f_{jk}(\theta) &= 1 & \text{if } j = k, \\ f_{jk}(\theta) &= \bar{f}_{kj}(\theta) & \text{if } j > k, \end{aligned}$$

it follows that the matrix $F(\theta) = (f_{jk}(\theta))$ is an Hermitian positive-definite

³ We shall usually omit the qualification "almost all θ ."

matrix for almost all θ . Hence every principal minor is non-negative, and in particular $|f_{jk}(\theta)| \leq 1$.

Thus the problem is reduced⁴ to factoring an Hermitian positive matrix function $F(\theta) = (f_{ij}(\theta))$, such that $f_{ii}(\theta) \equiv 1$ and

$$(2.4) \quad -\infty < \int_0^{2\pi} \log \det F(\theta) d\theta,$$

into a product $M(\theta)M(\theta)^*$ where $M(\theta) = (m_{jk}(\theta))$ is such that

$$(2.5) \quad m_{jk}(\theta) \text{ is bounded}$$

and

$$(2.6) \quad m_{jk}(\theta) \text{ has only non-negative frequencies in its Fourier series.}$$

3. The problem further reduced to factoring a simpler matrix. In this section we shall apply the induction assumption according to which for $q > 1$ any $q-1 \times q-1$ matrix satisfying the conditions of the last paragraph of §2 can be factored as described there.

Now let $F(\theta)$ be a $q \times q$ matrix,

$$F(\theta) = \begin{bmatrix} 1 & f_{12}(\theta) & \cdots & f_{1q}(\theta) \\ f_{21}(\theta) & 1 & \cdots & f_{2q}(\theta) \\ \vdots & & & \vdots \\ f_{q1}(\theta) & & \cdots & 1 \end{bmatrix},$$

which is Hermitian and positive-definite and such that (2.4) holds. Put

$$F_0(\theta) = \begin{bmatrix} 1 & f_{12}(\theta) & \cdots & f_{1,q-1}(\theta) \\ f_{21}(\theta) & 1 & \cdots & f_{2,q-1}(\theta) \\ \vdots & & & \vdots \\ f_{q-1,1}(\theta) & & \cdots & 1 \end{bmatrix}.$$

⁴ It may be useful to recall the

Lemma. *If $\phi(\theta)$ belongs to $L(0, 2\pi)$ and $\psi(\theta)$ is bounded and both have Fourier series without negative frequencies then their product has a Fourier series without negative frequencies.*

According to FEJÉR and LEBESGUE there exist polynomials P_N of degree $\leq N$ in $e^{i\theta}$ such that, for almost all θ , $P_N(e^{i\theta}) \rightarrow \psi(\theta)$ boundedly. Therefore, by dominated convergence,

$$\int_0^{2\pi} \phi \cdot P_N \cdot e^{in\theta} d\theta \rightarrow \int_0^{2\pi} \phi \cdot \psi \cdot e^{in\theta} d\theta.$$

But, for $n = 1, 2, \dots$, for all N , the left hand side is null, since ϕ is of power series type. Hence

$$\int_0^{2\pi} \phi \cdot \psi \cdot e^{in\theta} d\theta = 0, \quad n = 1, 2, \dots,$$

as required.

By the generalized Hadamard inequality of §1, (1.2),

$$(3.1) \quad -\infty < \int_0^{2\pi} \log \det F_o(\theta) d\theta,$$

and hence, by our inductive hypothesis,

$$(3.2) \quad F_o(\theta) = M_o(\theta) \cdot M_o(\theta)^*,$$

where $M_o(\theta) = (\mu_{jk}(\theta))$ is such that $\mu_{jk}(\theta)$ is bounded and has only non-negative frequencies appearing in its Fourier series. Consider now the matrix equation

$$F(\theta) = \begin{bmatrix} \mu_{11} & \cdots & \mu_{1,q-1} & 0 \\ \mu_{21} & \cdots & \mu_{2,q-1} & 0 \\ \vdots & & \vdots & \vdots \\ \mu_{q-1,1} & \cdots & \mu_{q-1,q-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 & \gamma_1 \\ 0 & 1 & \cdots & 0 & \gamma_2 \\ \vdots & & & & \vdots \\ 0 & \cdots & 1 & \gamma_{q-1} \\ \bar{\gamma}_1 & \cdots & \bar{\gamma}_{q-1} & 1 \end{bmatrix} \begin{bmatrix} \bar{\mu}_{11} & \cdots & \bar{\mu}_{q-1,1} & 0 \\ \vdots & & \vdots & \vdots \\ \bar{\mu}_{1,q-1} & \cdots & \bar{\mu}_{q-1,q-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

to determine $\gamma_1(\theta), \dots, \gamma_{q-1}(\theta)$. This is equivalent to (3.2) together with the following linear system for the γ_k 's:

$$\sum_{k=1}^{q-1} \mu_{jk} \gamma_k = f_{ja}, \quad j = 1, \dots, q-1.$$

Since $\det M_o(\theta) = (\det F_o(\theta))^{\frac{1}{2}}$ and since $F_o(\theta)$ satisfies (3.1) this system has a unique solution $\gamma_1(\theta), \dots, \gamma_{q-1}(\theta)$ for almost all θ . Writing the above matrix equation as $QCQ^* = F$ we have that $C = Q^{-1}F(Q^{-1})^*$. But F is positive. Hence C is positive. Hence

$$|\gamma_j(\theta)| \leq 1, \quad j = 1, \dots, q-1.$$

Therefore the problem is further reduced to factoring the Hermitian positive matrix

$$(3.3) \quad \Gamma(\theta) = \begin{bmatrix} 1 & 0 & \cdots & 0 & \gamma_1 \\ 0 & 1 & & & \\ \vdots & & & \vdots & \vdots \\ 0 & & & 1 & \gamma_{q-1} \\ \bar{\gamma}_1 & \cdots & \bar{\gamma}_{q-1} & 1 \end{bmatrix}$$

where

$$\det \Gamma(\theta) = 1 - \sum_{k=1}^{q-1} |\gamma_k(\theta)|^2$$

satisfies the condition

$$(3.4) \quad -\infty < \int_0^{2\pi} \log \det \Gamma(\theta) d\theta.$$

That $\det \Gamma(\theta)$ has this property follows because every matrix that has been introduced as a factor of the original $H(\theta)$ by our reductions has a determinant whose absolute value has this property.

4. The Hilbert space $\mathfrak{L}_r^2$. Since the matrix $\Gamma(\theta)$ of (3.3) is Hermitian and non-negative, an inner product can be introduced into the set of (equivalence classes of) all vectors

$$(4.1) \quad \mathbf{f}(\theta) = (f_1(\theta), \dots, f_q(\theta)),$$

where the $f_i(\theta)$ are complex-valued, Lebesgue measurable functions on $0 \leq \theta \leq 2\pi$, the $\mathbf{f}(\theta)$'s being such that

$$(4.2) \quad \int_0^{2\pi} \mathbf{f}(\theta) \Gamma(\theta) \mathbf{f}(\theta)^* d\theta < \infty.$$

The inner product for such $\mathbf{f}, \mathbf{g}$ is by definition

$$(\mathbf{f}, \mathbf{g})_r = \int_0^{2\pi} \mathbf{f}(\theta) \Gamma(\theta) \mathbf{g}(\theta)^* d\theta.$$

In this way we obtain a separable pre-Hilbert space, whose completion will be denoted $\mathfrak{L}_r^2$.

It will be convenient to have a certain lower bound for the norm (4.2). For fixed θ let U be a unitary matrix diagonalizing Γ :

$$U^* \Gamma U = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_q \end{bmatrix}.$$

Then, putting $\mathbf{y} = \mathbf{x}U$,

$$\begin{aligned} \mathbf{x} \Gamma \mathbf{x}^* &= \mathbf{y} \Lambda \mathbf{y}^* = \lambda_1 |y_1|^2 + \dots + \lambda_q |y_q|^2 \\ &= \frac{\lambda_1 \lambda_2 \dots \lambda_q}{\lambda_2 \dots \lambda_q} |y_1|^2 + \dots + \frac{\lambda_1 \lambda_2 \dots \lambda_q}{\lambda_1 \lambda_2 \dots \lambda_{q-1}} |y_q|^2 \\ &\geq \frac{\det \Gamma}{\sigma_{q-1}} (|y_1|^2 + \dots + |y_q|^2), \end{aligned}$$

where

$$\sigma_{q-1} = (\lambda_2 \lambda_3 \dots \lambda_q) + \dots + (\lambda_1 \lambda_2 \dots \lambda_{q-1})$$

is the sum of all $q-1 \times q-1$ principal minors of Λ , and therefore of Γ . Hence, from the explicit form of Γ , for all θ , $\sigma_{q-1} \leq q$. It follows that

$$\mathbf{x} \Gamma \mathbf{x}^* \geq \frac{\det \Gamma}{q} \cdot \sum_1^q |x_i|^2$$

and

$$(4.3) \quad ||\mathbf{x}||_{\Gamma}^2 \geq \frac{1}{q} \int_0^{2\pi} \sum_1^q |x_j|^2 \det \Gamma \, d\theta.^5$$

5. Existence of a non-null innovation space. For non-negative integers n let Σ_n denote the closed linear span in $\mathfrak{L}_{\Gamma}^2$ of the set of vectors

$$\{e^{in_1\theta}\phi_1, \dots, e^{in_q\theta}\phi_q \mid n_j \geq n, \quad j = 1, \dots, q\},$$

where ϕ_j has the constant function 1 in the j^{th} place and 0's elsewhere.

Lemma 5.1. *Each element $\phi_1, \dots, \phi_q$ has a non-null projection orthogonal to Σ_1 .*

Put $\mathbf{P} = (P_1, \dots, P_q)$, where P_i 's are polynomials in $e^{i\theta}$ without constant term. Such $\mathbf{P}$'s are dense in Σ_1 . On the other hand, according to the lower bound (4.3),

$$||\phi_j - \mathbf{P}||^2 \geq \frac{1}{q} \int_0^{2\pi} |1 - P_j|^2 \det \Gamma \, d\theta \geq \frac{1}{q} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \log \det \Gamma \, d\theta \right\} > 0,$$

where we have, in succession, taken a smaller lower bound, used a classical result⁶ of SZEGÖ on one-sided approximation and used the hypothesis that $\log \det \Gamma$ belongs to L^1 . Therefore ϕ_j does not belong to Σ_1 , $j = 1, \dots, q$. *Q.e.d.*

Let $\psi'_1, \dots, \psi'_q$ be the projections of $\phi_1, \dots, \phi_q$ into the orthogonal complement of Σ_1 . The vectors $\psi'_1, \dots, \psi'_q$ can be orthonormalized into r orthogonal unit vectors in $\mathfrak{L}_{\Gamma}^2$, to be denoted henceforth as $\psi_1, \dots, \psi_r$, where $0 < r \leq q$. Actually, $r = q$, as will be shown below.

6. Emptiness of the infinitely remote future. In order to obtain Fourier expansions for $\phi_1, \dots, \phi_q$ in $\mathfrak{L}_{\Gamma}^2$ in terms of $e^{in_1\theta}\psi_1, \dots, e^{in_r\theta}\psi_r$, where $n_1, \dots, n_r$ run independently over all non-negative integers, the following lemma is essential.

Lemma 6.1. *If S_N denotes the closed linear span in $\mathfrak{L}_{\Gamma}^2$ of the vectors*

$$\{e^{in_1\theta}\phi_1, \dots, e^{in_q\theta}\phi_q \mid n_j \geq N, \quad j = 1, \dots, q\},$$

then

$$\bigcap_{N>0} S_N = (0).$$

Suppose $\mathbf{f}$ belongs to $\bigcap_{N>0} S_N$. It is permissible to take $\mathbf{f}$ to be of the form (4.1) $\mathbf{f} = (f_1, \dots, f_q)$. According to SZEGÖ's theorem, which can be applied in view of (3.4),

$$(6.1) \quad \det \Gamma = \gamma_+ \bar{\gamma}_+,$$

⁵ We owe this inequality to Professor P. R. MASANI of Bombay whom we should like to thank for several critical discussions on this paper, which resulted in essential improvements.

⁶ G. SZEGÖ, *loc. cit.*¹, pp. 293-294.

where γ_+ belongs to L^2 and has a Fourier series of power series type. We recall that $\gamma^+(\theta) \neq 0$ for almost all θ (Lebesgue measure). By (4.3) and (6.1) for any $\mathbf{g} = (g_1, \dots, g_a)$ belonging to $\mathfrak{L}_1^2$,

$$\|\mathbf{g}\|_1^2 \geq \frac{1}{q} \int_0^{2\pi} \sum_1^a |g_i \gamma_+|^2 d\theta.$$

Therefore $g_i \gamma_+$ belongs to L^2 . Let us put P_{iN}^k for polynomials in $e^{iN\theta}$, $e^{i(N+1)\theta}$, $\dots$, and $\mathbf{P}_N^k = (P_{1N}^k, \dots, P_{aN}^k)$, these quantities being such that

$$(6.2) \quad \lim_{k \rightarrow \infty} \|\mathbf{f} - \mathbf{P}_N^k\| = 0, \quad \text{for all } N.$$

On the other hand,

$$\|\mathbf{f} - \mathbf{P}_N^k\|^2 \geq \sum_1^a \int_0^{2\pi} |f_i \gamma_+ - P_{iN}^k \gamma_+|^2 d\theta.$$

For N sufficiently large this lower bound is bounded away from zero uniformly in k (by Parseval's identity) unless $f_1 = 0, \dots, f_a = 0$.

7. Certain one-sided developments. The orthogonal unit vectors $\psi_1, \dots, \psi_r$ defined at the end of §5 generate a set $\mathbf{B}$ of vectors $\{e^{in_1\theta}\psi_1, \dots, e^{in_r\theta}\psi_r\}$, where $n_1, n_2, \dots, n_r$ run independently over all non-negative integers. The set $\mathbf{B}$ is evidently orthonormal in $\mathfrak{L}_1^2$. It is also a basis for Σ_0 . For if $\mathbf{f}$ belongs to Σ_0 , the difference

$$\mathbf{f} - \sum_{\psi \in \mathbf{B}} (\mathbf{f}, \psi)_\Gamma \psi$$

belongs to $\mathbf{S}_N$ for $N = 1, 2, \dots$. By Lemma 6.1 this difference must be null. It follows at once that we can obtain Fourier expansions for ϕ_i that are convergent in $\mathfrak{L}_1^2$ to ϕ_i of the form

$$\phi_i = \sum_{\psi \in \mathbf{B}} (\phi_i, \psi)_\Gamma \psi, \quad j = 1, \dots, q,$$

or, more explicitly,

$$\phi_i = \psi_1 \sum_{n_1=0}^{\infty} a_{in_1}^{(1)} e^{in_1\theta} + \dots + \psi_r \sum_{n_r=0}^{\infty} a_{in_r}^{(r)} e^{in_r\theta},$$

where

$$a_{i\nu}^{(k)} = (\phi_i, \psi_k e^{i\nu\theta})_\Gamma.$$

Put

$$M_{ik}(\theta) \sim \sum_{\nu=0}^{\infty} a_{i\nu}^{(k)} e^{i\nu\theta}, \quad j = 1, \dots, q, \quad k = 1, \dots, r,$$

so that M_{ik} belongs to $L^2(0, 2\pi)$, and

$$\phi_i = M_{i1}\psi_1 + \dots + M_{ir}\psi_r, \quad j = 1, \dots, q.$$

Hence, for $1 \leq j, k \leq q$,

$$\begin{aligned}
 (\phi_k e^{in\theta}, \phi_i)_\Gamma &= \left(\sum_J M_{kJ} e^{in\theta} \psi_J, \sum_L M_{iL} \psi_L \right)_\Gamma = \sum_{J,L} (M_{kJ} e^{in\theta} \psi_J, M_{iL} \psi_L)_\Gamma \\
 &= \sum_{J,L} \sum_{\nu=0}^{\infty} \sum_{\mu=0}^{\infty} a_{k\nu}^{(J)} \overline{a_{i\mu}^{(L)}} (e^{i(\nu+n)\theta} \psi_J, e^{i\mu\theta} \psi_L)_\Gamma \\
 &= \sum_{J,L} \sum_{\mu=\max(0,n)}^{\infty} a_{k,\mu-n}^{(J)} \overline{a_{i\mu}^{(L)}} (\psi_J, \psi_L)_\Gamma \\
 &= \sum_{J=1}^r \sum_{\mu=\max(0,n)}^{\infty} a_{k,\mu-n}^{(J)} \overline{a_{i\mu}^{(J)}} \\
 &= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \sum_{J=1}^r M_{iJ}(\theta) \overline{M_{kJ}(\theta)} d\theta, \quad n = 0, \pm 1, \pm 2, \dots
 \end{aligned}$$

On the other hand, $(\phi_k e^{in\theta}, \phi_i)$ can also be expressed in terms of the coefficients of the matrix $\Gamma(\theta)$:

$$\begin{aligned}
 (\phi_k e^{in\theta}, \phi_i)_\Gamma &= \int_0^{2\pi} e^{in\theta} \delta_{ik} d\theta \quad \text{if} \quad 1 \leq j, k \leq q-1, \\
 (\phi_k e^{in\theta}, \phi_i)_\Gamma &= \int_0^{2\pi} e^{in\theta} \overline{\gamma_i(\theta)} d\theta \quad \text{if} \quad 1 \leq j \leq q-1, \\
 (\phi_k e^{in\theta}, \phi_q)_\Gamma &= \int_0^{2\pi} e^{in\theta} \gamma_k(\theta) d\theta \quad \text{if} \quad 1 \leq k \leq q-1, \\
 (\phi_q e^{in\theta}, \phi_q)_\Gamma &= \int_0^{2\pi} e^{in\theta} d\theta, \quad n = 0, \pm 1, \dots
 \end{aligned}$$

By the uniqueness of Fourier coefficients in $L^1(0, 2\pi)$,

$$(7.1) \quad \begin{pmatrix} & & \gamma_1 \\ & & \vdots \\ I_{q-1} & & \vdots \\ & & \gamma_{q-1} \\ \tilde{\gamma}_1 & \cdots & \tilde{\gamma}_{q-1} & 1 \end{pmatrix} = (M_{ik}(\theta)(M_{ik}(\theta))^*,$$

where I_{q-1} stands for the $q-1$ dimensional unit matrix. That $M_{ik}(\theta)$ is bounded follows from the diagonal equations

$$\sum_{j=1}^r |M_{ij}(\theta)|^2 = 1, \quad j = 1, \dots, q.$$

The matrix $(M_{ik}(\theta))$ has q rows and r ($\leq q$) columns. Suppose $r < q$. Then if $(M_{ik}(\theta))$ is augmented by $q-r$ columns of zeros on the right and $(M_{ik}(\theta))^*$ is augmented by $q-r$ rows of zeros below, the product $(M_{ik}(\theta)(M_{ik}(\theta))^*)$ remains

unchanged. But the determinant of the augmented $(M_{ik}(\theta))$ is identically zero, whereas by (7.1)

$$|\det M_{ik}|^2 = \det \Gamma(\theta).$$

This contradiction proves that $r = q$, and completes the proof of the existence of a factorization of the required sort.

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