Spectral Representation of Certain Semi-Groups of Operators

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Summary. In section 1 some definitions and results are given which will be used in the rest of the paper. The main results are proved in sections 4 and 7. In section 2 the characters of a certain class of semi-groups are studied. Section 5 deals with semi-groups of bounded scalar operators. In sections 3 and 6 a series of propositions on hermitian spectral families are given.¹

1. The semi-group algebra. Representations. Let G be a locally compact group, S a locally compact subspace of G and m the restriction of a left invariant Haar measure to S . Suppose

- (1) $st \in S$ if $s \in S, t \in S$,
- (2) $m(U) > 0$ for every non-void set $U \subset S$, open in S .

Let $M(S)$ be the vector space of complex Radon measures defined on S , with compact support; for every measure μ let $S(\mu)$ be its support. For $\mu \in M(S)$, $\nu \in M(S)$ we can define the product $\mu\nu$ by means of the relations

$$\int f(s) d\mu\nu(s) = \iint f(st) d\mu(s) d\nu(t) \quad \text{for } f \in \mathfrak{R}(S);$$

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$\int$ always means $\int_S$. We have $S(\mu\nu) \subset S(\mu)S(\nu)$ for $\mu \in M(S)$, $\nu \in M(S)$ and, when endowed with the multiplication $(\mu, \nu) \rightarrow \mu\nu$, $M(S)$ is an algebra. For every $\mu \in M(S)$ and $a \in S$ we denote by μ_a the measure $\epsilon_a\mu$:

$$\int f(s) d\mu_a(s) = \int f(as) d\mu(s) \quad \text{for } f \in \mathfrak{R}(S).$$

Let $L(S)$ be the set of measures belonging to $M(S)$ and having the form $g.m$, where g is locally integrable. $L(S)$ is a two-sided ideal of $M(S)$. In particular, for any $a \in S$, $\mu_a \in L(S)$ if $\mu \in L(S)$. The mapping $a \rightarrow \mu_a$ of S into $L(S)$, endowed with the topology given by the norm $\|\mu\| = \sup \{|\mu(f)| \mid f \in \mathfrak{R}(S), \|f\| \leq 1\}$, is continuous.

For any compact $K \subset S$ let $L(S, K)$ be the set of measures $\mu \in L(S)$ such that $S(\mu) \subset K$. Endowed with the norm $\mu \rightarrow \|\mu\|$, $L(S, K)$ is a Banach space. On $L(S)$, which is the union of the directed family of subsets $L(S, K)$, let us consider the inductive limit topology of the topologies of the spaces $L(S, K)$; with this topology, $L(S)$ is a barrelled space.² Every continuous linear form $x' \in L(S)'$ can be written in the form

$$x'(\mu) = \int g(s) d\mu(s), \quad \mu \in L(S),$$

where g is a locally bounded³ m -measurable function.

Let $\mathfrak{F}$ be a directed set of m -measurable functions. Suppose that for every compact $K \subset S$ there is a constant $c(K)$ such that, for $f \in \mathfrak{F}$,

$$m(\{s \mid |f(s)| > c(K)\} \cap K) = 0.$$

Using the continuity of the mapping $a \rightarrow \mu_a$ we deduce that if $\mathfrak{F}$ converges weakly to g , then for every $\mu \in L(S)$

$$\lim_{f \in \mathfrak{F}} \int f(ax) d\mu(x) = \int g(ax) d\mu(x),$$

uniformly on every compact $K \subset S$.

Let $\mathfrak{F} = (\mu_i)_{i \in I}$ be a family of bounded Radon measures on a locally compact space Z . We shall say that a complex-valued function f defined on Z is $\mathfrak{F}$ -negligible if $|\mu_i|^*(|f|) = 0$ for every $i \in I$. A subset $A \subset Z$ is $\mathfrak{F}$ -negligible if φ_A , the characteristic function of A , is $\mathfrak{F}$ -negligible. A complex-valued function f defined on Z is $\mathfrak{F}$ -measurable if f is $|\mu_i|$ -measurable for every $i \in I$. A subset $A \subset Z$ is $\mathfrak{F}$ -measurable if φ_A is $\mathfrak{F}$ -measurable. The set of all $\mathfrak{F}$ -measurable subsets of Z is a tribe which will be denoted by $T(\mathfrak{F})$.

Let f be a complex-valued function defined on Z . We shall write $N_\infty(f, \mathfrak{F}) = \inf A_f$, where A_f is the set of numbers $\alpha > 0$ such that $\{z \mid |f(z)| > \alpha\}$ is $\mathfrak{F}$ -negligible. The set of $\mathfrak{F}$ -measurable functions satisfying the inequality

² Barrelled space = *espace tonnellé* ([4], Chap. III, 1).

³ A complex-valued function defined on S is locally bounded if it is bounded on every compact set $K \subset S$.

$N_\infty(f, \mathfrak{F}) < \infty$ will be denoted by $L^\infty(\mathfrak{F})$; $L^\infty(\mathfrak{F})$ is an algebra and $f \rightarrow N_\infty(f, \mathfrak{F})$ is a semi-norm. We shall say that a family $\mathfrak{F} = (\mu_i)_{i \in I}$ of bounded Radon measures on a locally compact space Z is concentrated on $A \subset Z$ if $\mathbb{C}A$ is $\mathfrak{F}$ -negligible.

A *representation* of S is an object $\{X, U_s\}$, where X is a reflexive Banach space and $s \rightarrow U_s$ a mapping of S into the set $\mathfrak{L}(X)$ of linear continuous operators on X such that

- (i) $U_s U_t = U_{st}$ for every $s \in S, t \in S$,
- (ii) $s \rightarrow \langle U_s x, x' \rangle$ is continuous for every $x \in X, x' \in X'$.

Let $\{X, U_s\}$ be a representation of the semi-group S . For every $\mu \in M(S)$ define the operator $U_\mu \in \mathfrak{L}(X)$ as follows:

$$\langle U_\mu x, x' \rangle = \int \langle U_s x, x' \rangle d\mu(s)$$

for $x \in X, x' \in X'$. Then $\mu \rightarrow U_\mu$ is a representation of the algebra $M(S)$ into $\mathfrak{L}(X)$ and for every compact $K \subset S$ there is a constant M_K such that $\|U_\mu\| \leq M_K$ whenever $S(\mu) \subset K$.⁴

2. Characters. A complex-valued function χ defined on S is a character of S if $\chi \neq 0$, χ is continuous and $\chi(ts) = \chi(t)\chi(s)$ for all $s \in S, t \in S$. Let us denote by E the set of all characters of S .

Let $\mathfrak{M}$ be the set of all locally bounded functions r defined on S , such that $r(s) \geq 0$ for $s \in S$. For every $r \in \mathfrak{M}$ let $E(r)$ be the set of all characters $\chi \in E$ satisfying the inequality $|\chi(s)| \leq r(s)$ for all $s \in S$. $E(r)$ can be identified with a weakly bounded part of $L(S)'$; hence $E(r)$ is relatively compact. Suppose now that g is in the weak closure of $E(r)$ and that $g \neq 0$. Let $\mathfrak{F} \subset E(r)$ be a directed set converging weakly to g and let $\mu \in L(S)$ be such that $\int g(s) d\mu(s) \neq 0$. Then

$$\lim_{f \in \mathfrak{F}} \int f(zs) d\mu(s) = \int g(zs) d\mu(s)$$

uniformly on every compact set. But

$$\int f(zs) d\mu(s) = f(z) \int f(s) d\mu(s)$$

for $z \in S$ and $f \in \mathfrak{F}$. It follows that $\mathfrak{F}$ converges uniformly on every compact set to a (continuous) function, necessarily identical (almost everywhere) with g ; we deduce that $g(st) = g(s)g(t)$. Hence we have proved

Proposition 1. *$E(r)$, endowed with the topology of uniform convergence on compact sets, is locally compact, for $r \in \mathfrak{M}$.*

When we consider E as a topological space we shall always suppose it to be endowed with the topology of uniform convergence on compact sets. The follow-

⁴ For the definitions and notations belonging to the theory of vector spaces or to the theory of integration, used without explanation, see [4], [5] and [16].

ing result gives a criterion for local compactness of E (for a subset $A \subset S$, $i(A)$ is the interior of A in S):

Proposition 2. *E is locally compact if there exists a compact subset $V \subset S$ having the property that for every $t \in S$ there is an integer $p > 0$ such that $t^p \in \bigcup_{i=1}^{\infty} i(V^i)$.*

Let $P = \{(n, m) | n, m \text{ integers} > 0\}$ and for each $(n, m) \in P$ denote by $U(n, m)$ the set $\{s | s^n \in i(V^m)\}$; $U(n, m)$ is open and $\bigcup_{(n, m) \in P} U(n, m) = S$. For each integer $p > 0$ let r_p be the function defined on S by the equations $r_p(s) = \inf \{p^{m/n} | U(n, m) \ni s\}$, $s \in S$. Obviously $r_p \in \mathfrak{M}$. Let $E_1(r_p) = \{\chi | |\chi(t)| < p, t \in V\}$. Then $E_1(r_p)$ is an open part of E and also, as is easily seen, an open part of $E(r_p)$, which is locally compact. Hence $E_1(r_p)$ is locally compact. If $\chi \in E$ then there is an integer $p \geq 1$ such that $|\chi(t)| < p$. It follows that $E = \bigcup_{p=1}^{\infty} E_1(r_p)$. Take $\chi \in E$ and let p be such that $\chi \in E_1(r_p)$. Since $E_1(r_p)$ is locally compact there is a compact neighborhood (in $E_1(r_p)$) V of χ , and because $E_1(r_p)$ is open, V is in fact a neighborhood of χ in the topological space E . Hence E is locally compact.

Suppose now that the following conditions are satisfied:

(3) S is endowed with a continuous mapping $s \rightarrow s^+$ of S into S satisfying the relations $(st)^+ = t^+s^+$, $u^{++} = u$ for every $s, t, u \in S$.

(4) $Q^+ = \{s^+ | s \in Q\}$ is locally m -negligible if Q is locally m -negligible.

In this case we denote by A the set of all $\chi \in E$ such that $\chi(t^+) = \overline{\chi(t)}$ for all $t \in S$, and by $A(r)$ the intersection $A \cap E(r)$. A is a closed subset of E and hence A is locally compact if E is; $A(r)$ is also a closed subset of $E(r)$ and so it is locally compact.

Remark. E (and also A) need not be locally compact. For example, take $G = S = R$ where R is the additive group of the real numbers endowed with the discrete topology. To prove that E is not locally compact it is enough to observe that given a finite set $\{x_1, \dots, x_n\}$ of real numbers, there is an $a \in R$ such that for every $b > 0$ there exists $\chi \in E$ satisfying $\chi(x_i) = 1$, $1 \leq i \leq n$, and $\chi(a) = b$.⁵

We shall denote by $\tilde{E}$ the Stone-Čech compactification of E . For every $t \in S$, $\hat{t}$ will represent the continuous function $\chi \rightarrow \chi(t)$, defined on E . To simplify the notation we shall denote by the same letter the restriction of $\hat{t}$ to a space $E(r)$ and also the unique continuous extension (taking values in the Alexandroff compactification of C) to $\tilde{E}$.

We shall prove now the following result which will be needed below:

Proposition 3. *Let μ be a bounded Radon measure on $\tilde{E}$ concentrated on $E(r)$. Then $\mu = 0$ if, for every $t, s, u, v \in S$,*

⁵ For instance, such an $a \in R$ can be constructed as follows: Let R_1 be the smallest divisible ([18], 7-9) subgroup of R containing $\{x_1, \dots, x_n\}$, R_2 the supplement of R_1 ($R_1 \oplus R_2 = R$) and p the projection of R onto R_2 ; we take $a \in R_2$, $a \neq 0$ and define χ by the equation $\chi(s) = \exp \{p(s)(\log b)/a\}$.

$$(5) \quad \int \hat{t}(\chi) d\mu(\chi) = 0, \quad \int \overline{\hat{s}(\chi)} d\mu(\chi) = 0, \quad \int \hat{u}(\chi) \overline{\hat{v}(\chi)} d\mu(\chi) = 0.$$

Let us denote by $C_\infty(E(r))$ the algebra of continuous complex-valued functions defined on $E(r)$ and vanishing at infinity. For every $\nu \in L(S)$, let f_ν be the function defined on $E(r)$ by the equality $f_\nu(\chi) = \int \chi(s) d\nu(s)$ and denote by $\mathfrak{A}$ the algebra spanned by $f_\nu, \bar{f}_\delta, \nu \in L(S), \delta \in L(S)$. It is easy to see that $\mathfrak{A}$ is dense in $C_\infty(E(r))$. To prove that $\mu = 0$ it is therefore sufficient ($f_\nu f_\delta = f_{\nu\delta}$ if $\nu \in L(S), \delta \in L(S)$) to show that

$$(6) \quad \int_{E(r)} f(\chi) d\mu(\chi) = 0$$

for $f = f_\nu, f = \bar{f}_\delta$, and $f = f_\nu \bar{f}_\gamma$, where $\nu, \delta, \rho, \gamma \in L(S)$. But if we integrate $f = f_\nu$ with respect to μ and use the first equation (5) we obtain (6) for $f = f_\nu$. If we remark that $\bar{f}_\delta(\chi) = \int \overline{\chi(s)} d\bar{\delta}(s)$, integrate $f = \bar{f}_\delta$ with respect to μ and use the second equation (5) we obtain (6) for $f = \bar{f}_\delta$. Finally, to obtain (6) for $f = f_\nu \bar{f}_\gamma$ we remark that

$$f_\nu \bar{f}_\gamma(\chi) = \iint \chi(t) \overline{\chi(s)} d\rho(t) d\bar{\gamma}(s),$$

we integrate this relation with respect to μ and use the third equality (5).

Remark. Obviously the preceding proposition remains true if μ is given on a locally compact space $B, \tilde{E} \supset B \supset E(r)$, and is concentrated on $E(r)$.

3. Hermitian spectral families. Let X be a Hilbert space and Z a locally compact space. A family $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ of bounded Radon measures defined on Z is *hermitian semi-spectral* if

- (H₁) $x \rightarrow \mu_{x,y}$ is linear for all $y \in X$,
- (H₂) $\mu_{x,y} = \bar{\mu}_{y,x}$ for all $x \in X, y \in X$,
- (H₃) there is a constant $M(\mathfrak{F})$ satisfying the inequality $\|\mu_{x,y}\| \leq M(\mathfrak{F}) \|x\| \|y\|$ for all $x \in X, y \in X$.

For every $f \in L^\infty(\mathfrak{F})$ there exists a $U_f \in \mathfrak{L}(X)$ satisfying for $x \in X, y \in X$ the equation

$$(U_f x | y) = \int_Z f(z) d\mu_{x,y}(z).$$

The mapping $f \rightarrow U_f$ is a continuous representation of (the vector space) $L^\infty(\mathfrak{F})$, endowed with the semi-norm $f \rightarrow N_\infty(f, \mathfrak{F})$, into $\mathfrak{L}(X)$ and $U_{\bar{f}} = U_f^*$. We shall say that the family $\mathfrak{F}$ is *hermitian spectral*⁶ if

$$(H_4) \quad g \cdot \mu_{x,y} = \mu_{g \cdot x, y} \text{ for all } g \in L^\infty(\mathfrak{F}), x \in X, y \in X.$$

⁶ There are other equivalent definitions.

Condition (H_4) is satisfied if and only if $U_{f\circ} = U_f U_{\circ}$ for all $f \in L^\infty(\mathfrak{F})$, $g \in L^\infty(\mathfrak{F})$. Let T be a set of bounded absolutely measurable⁷ functions defined on S such that $\|\mu\| < \infty$ and $\int_Z f d\mu = 0$ for $f \in T$ implies $\mu = 0$. If $\mathfrak{F}$ is hermitian semi-spectral and condition (H_4) is satisfied for all $g \in T$, then $\mathfrak{F}$ is hermitian spectral.

If $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ is a hermitian spectral family on Z , then, for each $f \in L^\infty(\mathfrak{F})$, the operator U_f is normal, $\|U_f\| = N_\infty(f, \mathfrak{F})$ and $\|U_f x\|^2 = \int_Z |f(z)|^2 d\mu_{x,x}(z)$ for all $x \in X$.

Let $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ be a hermitian spectral family on Z . If for each $A \in T(\mathfrak{F})$ we write $P(A) = U_{\varphi_A}$, $P(A)$ is a hermitian projection and P is a regular (strongly countably additive) spectral measure; it follows, in particular, that if $\mathfrak{F}$ is hermitian spectral $M(\mathfrak{F})$ can be taken to be ≤ 1 . The regularity of P is to be understood in the following sense: For every $A \in T(\mathfrak{F})$, $P(A) = \sup_{K \in \mathfrak{N}(A)} P(K)$ where $\mathfrak{N}(A)$ is the set of all compact subsets of A . The spectral measure P satisfies the equation $P(Z) = I$ if and only if $\|\mu_{x,x}\| = \|x\|^2$ for every $x \in X$.

4. Semi-groups of bounded normal operators. Let us denote by N_1 the set of all representations $\{X, U_s\}$ of S such that

- (i) X is a Hilbert space,
- (ii) U_s is normal for each $s \in S$,
- (iii) $U_s U_t = U_t U_s$ for all $s \in S, t \in S$.

We shall say that the representation $\{X, U_s\} \in N_1$ satisfies condition (β) if for every non-zero $x \in X$ there is $t(x) \in S$ such that $U_{t(x)} x \neq 0$. For every representation $\{X, U_s\}$ we shall denote by $r = r_{\{X, U_s\}}$ the function defined on S by the equation $r(s) = \|U_s\|$, $s \in S$.

Theorem 1. For each object $\{X, U_s\} \in N_1$ there is one and only one hermitian spectral family $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ on $E(r)$ such that, for all $t \in S, x \in X$,

$$(7) \quad (U_s x \mid y) = \int_{E(r)} \chi(s) d\mu_{x,y}(x).$$

The object $\{X, U_s\}$ satisfies condition (β) if and only if $\|\mu_{x,x}\| = \|x\|^2$ for every $x \in X$.

Let $\mathfrak{A}$ be a von Neumann algebra containing $\{U_s \mid s \in S\}$ and let $f \rightarrow U_f$ be the Gelfand isomorphism of $C(Z)$ (Z is the spectrum of $\mathfrak{A}$) onto $\mathfrak{A}$ ([10], 327–330). We denote by $\mathfrak{F}'' = (\mu''_{x,y})_{x \in X, y \in X}$ the corresponding hermitian spectral family and by $A \rightarrow f(A)$ the reciprocal isomorphism of $f \rightarrow U_f$; to simplify the notation the value of the function $f(A)$ at the point $z \in Z$ will be denoted $f(A, z)$. If $f \in L^\infty(\mathfrak{F})$, we deduce that $U_f \in \mathfrak{A}$ (we use the notation introduced in section 3); hence there is a $g \in C(Z)$ such that $U_f = U_g$ and this implies that $f(z) = g(z)$, except on an $\mathfrak{F}$ -negligible set.

⁷ A complex-valued function defined on a locally compact space Z is absolutely measurable if it is measurable for every Radon measure given on Z .

(a) Let^s $Z(1) = \bigcup_{s \in S} \{z | f(U_s, z) \neq 0\}$, $Z(2) = \bigcup_{\mu \in L(S)} \{z | f(U_\mu, z) \neq 0\}$ and $Z(3) = Z(1) \cap Z(2)$. We shall prove now that $Z(1) \cap \mathbf{C}Z(3)$ is $\mathfrak{F}''$ -negligible. Let $A \in \mathfrak{A}$ be such that $f(A, z) = \varphi_{Z(1) \cap \mathbf{C}Z(3)}(z)$ for $z \notin N$ where N is $\mathfrak{F}''$ -negligible. Let $\mu \in L(S)$ and let $g = f(A)f(U_\mu) = f(AU_\mu)$. It follows that $g(z) = 0$ for $z \notin Z$, whence $g = 0$. Hence $AU_\mu = 0$ for every $\mu \in L(S)$, and then, using the integral representation of the operators U_μ , we obtain $AU_s = 0$ for every $s \in S$. If $Z(1) \cap \mathbf{C}Z(3)$ is not $\mathfrak{F}''$ -negligible, there is a $z \in Z(1) \cap \mathbf{C}Z(3)$ for which $f(A, z) \neq 0$; since there is an $s \in S$ such that $f(U_s, z) \neq 0$ it now follows that $f(AU_s, z) \neq 0$, and this contradicts $AU_s = 0$. Hence $Z(1) \cap \mathbf{C}Z(3)$ is $\mathfrak{F}''$ -negligible.

For all $x \in X$, $y \in X$ denote by $\mu'_{x,y}$ the restriction of $\mu''_{x,y}$ to the open set $Z(3)$. Then $\mathfrak{F}' = (\mu'_{x,y})_{x \in X, y \in X}$ is a spectral family and we have, for $s \in S$, $x \in X$, $y \in X$,

$$(U_s x | y) = \int_{Z(3)} f(U_s, z) d\mu'_{x,y}(z).$$

(b) For each $z \in Z(3)$ define χ_z by the relation $\chi_z(s) = f(U_s, z)$; then $\chi_z \neq 0$, $\chi_z(ts) = \chi_z(t)\chi_z(s)$ for $t, s \in S$ and $|\chi_z(s)| \leq \|U_s\| = r(s)$ for $s \in S$.

We remark now that for every $\mu \in L(S)$ the mapping $(s, z) \rightarrow f(U_s U_\mu, z)$ defined on $S \times Z(3)$ is continuous. In fact, we have, for $(s, z) \in S \times Z(3)$ and $(s_0, z_0) \in S \times Z(3)$,

$$\begin{aligned} |f(U_s U_\mu, z) - f(U_{s_0} U_\mu, z_0)| &\leq |f(U_{\mu_s}, z) - f(U_{\mu_{s_0}}, z)| \\ &+ |f(U_{\mu_{s_0}}, z) - f(U_{\mu_{s_0}}, z_0)| \leq \|U_{\mu_s} - U_{\mu_{s_0}}\| + |f(U_{\mu_{s_0}}, z) - f(U_{\mu_{s_0}}, z_0)|, \end{aligned}$$

from which inequality the assertion follows. Take $(z_0, s_0) \in S \times Z(3)$ and $\mu \in L(S)$ such that $f(U_\mu, z_0) \neq 0$. Since $\chi_z(s) = f(U_s U_\mu, z)/f(U_\mu, z)$, we deduce that the mapping $(s, z) \rightarrow \chi_z(s)$ of $S \times Z(3)$ into the complex numbers is continuous. It follows that $\chi_z \in E(r)$ for all $z \in Z(3)$ and that the mapping $z \rightarrow \chi_z$ of $Z(3)$ into $E(r)$ is continuous.

(c) Now for all $x \in X$, $y \in X$ let $\mu_{x,y}$ be the measure defined on $E(r)$ by means of the relations

$$\mu_{x,y}(f) = \int_{Z(3)} f(\chi_z) d\mu'_{x,y}(z) \quad \text{for } f \in \mathfrak{R}(E(r)).$$

Then $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ is a spectral family on $E(r)$. Let $f(\chi) = \chi(t)$, so that $f(\chi_z) = \chi_z(t)$, and hence

$$\int_{E(r)} \chi(t) d\mu_{x,y}(x) = \int_{Z(3)} f(U_t, z) d\mu'_{x,y}(z) = (U_t x | y).$$

(d) If $P(Z) \neq I$ there is an $x \neq 0$ such that $P(Z)x = 0$. Then obviously $U_s x = 0$ for all $s \in S$. Hence if condition (β) is satisfied, $\|\mu_{x,x}\| = \|x\|^2$ for every $x \in X$. Conversely, suppose that, for each $x \in X$, $\|\mu_{x,x}\| = \|x\|^2$ and let $y \in Y$ be a non-zero element. Then we cannot have $\int_{E(r)} \chi(t) d\mu_{y,x}(\chi) = 0$ for all $x \in X$, $t \in S$,

^s It is obvious that $U_\mu \in \mathfrak{A}$ for each $\mu \in M(S)$.

since this would imply (by property (H_4)) that $\int_{E(r)} \chi(t) f(\chi) d\mu_{y,v}(\chi) = 0$ for all $f \in L^\infty(\mathfrak{F})$, which, by proposition 3, contradicts the relation $\|\mu_{y,v}\| = \|y\|^2 \neq 0$.

(e) It remains to prove the uniqueness assertion. Let us suppose that $\mathfrak{U} = (\nu_{x,v})_{x \in X, v \in X}$ is a hermitian spectral family on $E(r)$ which satisfies (7) for $t \in S$, $x \in X$, $y \in X$. Using the properties of spectral families we deduce that, for all $x \in X$, $y \in X$,

$$\int_{E(r)} f(\chi) d\mu_{x,v}(\chi) = \int_{E(r)} f(\chi) d\nu_{x,v}(\chi)$$

if $f = \hat{t}$, $f = \bar{\hat{s}}$ or $f = \hat{u}\hat{v}$. It follows then, from proposition 3, that $\mathfrak{U} = \mathfrak{F}$.⁹

Remarks. 1. The method used to prove the preceding theorem is a variant of a method due to R. S. PHILLIPS [21] and A. E. NUSSBAUM [20]. 2. If S has the properties (3), (4) and the representation $\{X, U_s\}$ is such that $U_{s+} = U_s^*$ for $s \in S$ then the support of every measure $\mu_{x,v}$ is contained in $A(r)$. 3. In a somewhat weaker form theorem 1 has been stated without proof in [17].

A stronger uniqueness assertion than the one given in theorem 1 is given below:

Proposition 4. Let $\mathfrak{U} = (\nu_{x,v})_{x \in X, v \in X}$ be a hermitian spectral family on $\tilde{E}$, concentrated on E , which satisfies for all $x \in X$, $y \in X$ and $s \in S$ the relations (7). Then $\mathfrak{F} = \mathfrak{U}$.

It is enough to show that $\mathfrak{U}$ is concentrated on $E(r)$. For every $t \in S$, let $V(t) = \{\chi | |\chi(t)| > r(t)\}$. The $V(t)$ are open and $\bigcup_{t \in S} V(t) \supset B = \mathbb{C}E(r)$. If $\mathfrak{F}$ is not concentrated on $E(r)$, then, for at least one $t \in S$, $V(t) \cap B$ is not $\mathfrak{F}$ -negligible. We deduce that $\|U_t\| = N_\infty(\hat{t}, \mathfrak{F}) > r(t)$. Since this inequality leads to a contradiction the proof is complete.

5. Semi-groups of bounded scalar operators. Let X be a reflexive Banach space and Z a locally compact space. A family $\mathfrak{F} = (\mu_{x,v})_{x \in X, v \in X}$ of bounded Radon measures defined on Z is *semi-spectral* if

(S₁) the mapping $(x, x') \rightarrow \mu_{x,x'}$ is bilinear,

(S₂) there is a constant $M(\mathfrak{F})$ satisfying the inequality $\|\mu_{x,v}\| \leq M(\mathfrak{F}) \|x\| \|v\|$ for all $x \in X$, $x' \in X'$.

For every $f \in L^\infty(\mathfrak{F})$ there exists a $U_f \in \mathcal{L}(X)$ satisfying for $x \in X$, $x' \in X'$ the equation

$$\langle U_f x, x' \rangle = \int_Z f(z) d\mu_{x,x'}(z).$$

The mapping $f \rightarrow U_f$ is a continuous representation of (the vector space) $L^\infty(\mathfrak{F})$,

⁹ Some of the results given in [1], [13], [14], [19], [20], [22] and [23] are particular cases of theorem 2 (see also remark 2). In the case $S = (0, \infty)$ an exhaustive study of the semi-groups of bounded hermitian operators or bounded normal operators can be found in [15], Chap. XXII and [23], Chap. XI.

endowed with the semi-norm $f \rightarrow N_\infty(f, \mathfrak{F})$, into $\mathcal{L}(X)$. We shall say that the family F is *spectral* if

(S₃) $g \cdot \mu_{x, x'} = \mu_{U_g x, x'}$ for all $g \in L^\infty(\mathfrak{F})$, $x \in X$, $x' \in X'$ ([9], 176–177).

Condition (S₃) is satisfied if and only if $U_f U_g = U_{fg}$ for all $f \in L^\infty(\mathfrak{F})$, $g \in L^\infty(\mathfrak{F})$. Let $\mathfrak{F} = (\mu_{x, x'})_{x \in X, x' \in X'}$ be a spectral family on a locally compact space Z . For every $A \in T(\mathfrak{F})$ define $P(A) = U_{\varphi_A}$. Then P is a (strongly countably additive) spectral measure on $T(\mathfrak{F})$, and $P(Z) = I$ if and only if for every non-zero $x' \in X'$ there is an $x \in X$ such that $\mu_{x, x'} \neq 0$.

Let X be a reflexive space. A sub-algebra $\mathcal{A} \subset \mathcal{L}(X)$ will be called a (D)-algebra if

- (i) $\mathcal{A} \ni I$,
- (ii) there is an algebraic and topological isomorphism $f \rightarrow U_f$ of $\mathcal{A}$ onto $C(Z)$, where Z is a compact space,
- (iii) $\mathcal{A}$ is weakly closed.

In every (D)-algebra $\mathcal{A}$ we can introduce an involution, writing $A^* = U_{\bar{f}}$ if $A = U_f$; the introduced involution does not depend on the isomorphism $f \rightarrow U_f$ or on $C(Z)$.

If $\mathcal{A}$ is a (D)-algebra every $T \in \mathcal{A}$ is a scalar operator ([11], 341) and the ranges of the spectral measures corresponding to the operators $T \in \mathcal{A}$ are contained in a bounded Boolean algebra of projections. Conversely, if $\mathfrak{D}$ is a set of commuting scalar operators and if the ranges of the corresponding spectral measures are contained in a bounded Boolean algebra, then a result due to W. G. BADE (see, for instance, [2]) implies that there is a (D)-algebra containing $\mathfrak{D}$.

Let us denote by S_1 the set of all representations of S having the property that $\{U_s | s \in S\}$ is contained in a (D)-algebra. We shall say that the representation $\{X, U_s\}$ of S satisfies condition (β) if for every non-zero $x' \in X'$ there is an $s(x') \in S$ such that ${}^t U_{s(x')} x' \neq 0$. For every representation $\{X, U_s\}$ we shall denote by $r = r_{(X, U_s)}$ the function defined on S by the equation $r(s) = \|U_s\|$, $s \in S$.

Theorem 2. *For each object $\{X, U_s\} \in S_1$ there is one and only one spectral family $\mathfrak{F} = (\mu_{x, x'})_{x \in X, x' \in X'}$ on $E(r)$ such that for all $t \in S$, $x \in X$, $x' \in X'$*

$$(7') \quad \langle U_s x, x' \rangle = \int_{E(r)} \chi(s) d\mu_{x, x'}(\chi).$$

The object $\{X, U_s\}$ satisfies condition (β) if and only if for every non-zero $x' \in X'$ there is an $x \in X$ such that $\mu_{x, x'} \neq 0$. If S satisfies conditions (3) and (4) and if $U_{s+} = U_s^$ for every $s \in S$ then $\mathfrak{F}$ is concentrated on $A(r)$.*

The theorem can be proved by the same method as theorem 1. Formula (7') can also be deduced from theorem 1 using the following result, which will be stated without proof: For every (D)-algebra $\mathcal{A}$ there is a von Neumann algebra $\mathcal{J}$ and an algebraic isomorphism φ of $\mathcal{A}$ onto $\mathcal{J}$ such that

- (i) there is a constant M such that $\|\varphi(A)\| \leq \|A\| \leq M\|\varphi(A)\|$ for every $A \in \mathfrak{A}$,
- (ii) the restriction of φ to bounded sets (for the uniform norm) is weakly bicontinuous,
- (iii) for every bounded Borel measurable function h , defined on C , and $A \in \mathfrak{A}$ we have $\varphi(h(A)) = h(\varphi(A))$.

We finally remark that proposition 4 remains valid for representations $\{X, U_s\} \in S_1$, if in the statement the words "hermitian spectral" are replaced by "spectral".

6. Direct sums of hermitian spectral families. Let X be a Hilbert space and $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ a hermitian spectral family on a locally compact space Z . For every $\mathfrak{F}$ -measurable function f defined on Z let $D(f, \mathfrak{F})$ be the set of all $x \in X$ for which $|f|^2$ is $\mu_{x,x}$ -integrable; $D(f, \mathfrak{F})$ is a linear space dense in X and there is a unique normal operator U_f having domain $D(f, \mathfrak{F})$ and satisfying

$$(U_f x | y) = \int_Z f(z) d\mu_{x,y}(z)$$

for every $x \in X, y \in X$.

Let X be a Hilbert space,¹⁰ $(X_n)_{n \in J}$ a family of orthogonal closed subspaces such that $X = \bigoplus_{n \in J} X_n$, and for each $n \in J$ let P_n be the projection of X onto X_n . For every $n \in J$ let $\mathfrak{F}_n = (\mu_{x,y}^n)_{x \in X_n, y \in X_n}$ be a spectral family on a locally compact space $\hat{P}$. Since

$$\sum_{n \in J} \|\mu_{P_n x, P_n y}^n\| \leq \sum_{n \in J} \|P_n x\| \|P_n y\| \leq \|x\| \|y\|$$

for all $x \in X, y \in X$, it follows that the family $(\mu_{P_n x, P_n y}^n)_{n \in J}$ is (absolutely) summable (with respect to the norm $\mu \rightarrow \|\mu\|$). Let $\mu_{x,y} = \sum_{n \in J} \mu_{P_n x, P_n y}^n$; $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ will be called the direct sum of the family $(\mathfrak{F}_n)_{n \in J}$. With the above notation we can now state

Proposition 5. (1) $\mathfrak{F}$ is a spectral family; (2) f is $\mathfrak{F}$ -measurable if and only if f is $\mathfrak{F}_n$ -measurable for each $n \in J$; (3) $x \in D(f, \mathfrak{F})$ if and only if $x \in D(f, \mathfrak{F}_n)$ for each $n \in J$ and $(\int_Z |f(z)|^2 d\mu_{x,x}^n(z))_{n \in J}$ is summable; (4) for $x \in D(f, \mathfrak{F})$ and $y \in X$

$$(8) \quad \int_Z f(z) d\mu_{x,y}(z) = \sum_{n \in J} \int_Z f(z) d\mu_{P_n x, P_n y}^n(z).$$

The fact that $\mathfrak{F}$ is hermitian semi-spectral follows by direct computation. For every $n \in J$ and $g \in \mathfrak{R}(Z)$ denote by U_g and $U_{g,n}$ respectively the operators on X and X_n corresponding to $\mathfrak{F}$ and $\mathfrak{F}_n$. Then

¹⁰ J is not necessarily denumerable.

$$\begin{aligned}
(P_n U_{\sigma} x \mid y) &= (U_{\sigma} x \mid P_n y) = \sum_{i \in J} \int_Z g(z) d\mu_{P_i x, P_i P_n y}^i(z) \\
&= \int_Z g(z) d\mu_{P_n x, P_n y}^n(z) = (U_{\sigma, n} P_n x \mid P_n y) = (U_{\sigma, n} P_n x \mid y)
\end{aligned}$$

(because $U_{\sigma, n} P_n x \in X_n$), and hence $P_n U_{\sigma} x = U_{\sigma, n} P_n x$ for all $x \in X$. We deduce that

$$g \cdot \mu_{x, y} = \sum_{n \in J} g \cdot \mu_{P_n x, P_n y}^n = \sum_{n \in J} \mu_{U_{\sigma, n} P_n x, P_n y}^n = \sum_{n \in J} \mu_{P_n U_{\sigma} x, P_n y}^n = \mu_{U_{\sigma} x, y},$$

and hence that $\mathfrak{F}$ is a spectral family. The assertion (2) follows from proposition 6, [5], Chap. V, 28, and the assertion (3) from the corollary to proposition 4, [5], Chap. V, 24–25. To prove (4) we remark first that using (3) and proposition 5, [5], Chap. V, 26–27, we deduce, for every $x \in D(f, \mathfrak{F})$ and subset $L \subset J$,

$$(9) \quad \int_Z |f(z)|^2 d\mu_{\sum_{n \in L} P_n x, \sum_{n \in L} P_n x}(z) = \sum_{n \in L} \int_Z |f(z)|^2 d\mu_{P_n x, P_n x}(z).$$

Since we have, for every $x \in D(f, \mathfrak{F})$, $y \in X$ and finite part $I \subset J$,

$$\int_Z f(z) d\mu_{x, y}(z) = \sum_{n \in I} \int_Z f(z) d\mu_{P_n x, P_n y}(z) + \int_Z f(z) d\mu_{A, B}(z),$$

where $A = \sum_{n \in I} P_n x$, $B = \sum_{n \in I} P_n y$, and since

$$\left| \int_Z f(z) d\mu_{A, B}(z) \right| \leq \|y\| \left(\sum_{n \in I} \int_Z |f(z)|^2 d\mu_{P_n x, P_n x}(z) \right)^{\frac{1}{2}},$$

we deduce, using (9) and (3), that the family $(\int_Z f(z) d\mu_{P_n x, P_n y}(z))_{n \in J}$ is summable and that the equation (8) is satisfied.

Proposition 6. Let $\mathfrak{F} = (\mu_{x, y})_{x \in X, y \in X}$ be a spectral family on a locally compact subspace Z and $(P_n)_{n \in J}$ a family of projections such that $\sum_{n \in J} P_n = I$ and $\mu_{P_n x, y} = \mu_{x, P_n y}$ for $n \in J$ and $x \in X, y \in X$. For each $n \in J$ let $X_n = P_n(X)$; then $\mathfrak{F}_n = (\mu_{x, y})_{x \in X_n, y \in X_n}$ is a spectral family and $\mathfrak{F}$ is the direct sum of $(\mathfrak{F}_n)_{n \in J}$.

We leave the proof to the reader.¹¹

7. Semi-groups of unbounded normal operators. Let X be a Hilbert space and let us denote with N_2 the set of all objects $\{X, U_s\}$ where $s \rightarrow U_s$ is a mapping of S into the set of normal operators on X , not necessarily bounded, such that

- (10) $U_s U_t \supset U_{st}$ for every $s \in S, t \in S$,
- (11) U_s and U_t commute for every $s \in S, t \in S$,
- (12) $s \rightarrow (U_s x \mid y)$ is continuous for $x \in \bigcap_{s \in S} D(U_s)$ and $y \in X$,
- (13) $\bigcap_{s \in S} D(U_s) = \bigcap_{s \in S_0} D(U_s)$ for some denumerable subset S_0 of S .

A theorem concerning the spectral representation of objects $\{X, U_s\} \in N_2$ will be given below. First, we shall make the following remarks:

¹¹ See also C. IONESCU TULCEA, *Hilbert spaces* (Published in Roumanian), Bucharest, 1956.

If $U_s, s \in S$, are hermitian operators then condition $\{(10), (11)\}$ is equivalent to (10) [7]. Condition $\{(10), (11)\}$ is implied by the condition $U_s U_t = U_t U_s = U_{st} = U_{ts}$ for every $s \in S, t \in S$ [8]. Condition (13) is implied by $\{(11), (N)\}$ where

(N) There is a denumerable subset $S_0 \subset S$ such that $S_0 \cap St \neq \emptyset$, for all $t \in S$.

In fact, let $t \in S$ and $s \in S_0 \cap St$; then there is a $t' \in S$ such that $s = t't$. It follows that $U_s = U_{t't} \subset U_{t'} U_t$; this implies $D(U_t) \supset D(U_s)$, that is, $\bigcap_{s \in S} D(U_s) = \bigcap_{s \in S_0} D(U_s)$. Condition (N) is essentially due to A. E. NUSSBAUM (*Amer. Math. Soc., Notices*, February 1958, Abstract 543-5). Condition (N) is implied by the following:

(14) The interior of $S, \overset{0}{S}$, is non void.

(15) There is a denumerable part $S_0 \subset S$ such that $\overset{0}{S}_0 = S$.

In fact, in this case $St \supset \overset{0}{S}t$, which is open and non-void, and since $\overset{0}{S}_0 = S$ it follows that $S_0 \cap St \neq \emptyset$.

Proposition 7. Let $\mathfrak{F}$ be a family of commuting normal operators, not necessarily bounded. Suppose that there is a denumerable part $\mathfrak{F}_0 \subset \mathfrak{F}$ such that $\bigcap_{A \in \mathfrak{F}_0} D(A) = \bigcap_{A \in \mathfrak{F}_0} D(A)$. Then there is an orthogonal family of hermitian projections $(P_n)_{n \in J}$ such that (1) $\sum_{n \in J} P_n = I$; (2) $P_n(X) \subset D(A)$ for every $n \in J$ and $A \in \mathfrak{F}$; (3) $AP_n \supset P_n A$ for all $n \in J$ and $A \in \mathfrak{F}$; (4) $x \in D(A)$ if and only if the family $(AP_n x)_{n \in J}$ is summable. For $x \in D(A)$, $Ax = \sum_{n \in J} AP_n x$.¹²

For every $A \in \mathfrak{F}$ let E^A be the spectral measure corresponding to A and let $\mathfrak{A}$ be the von Neumann algebra spanned by $\{E^A(\omega)|A \in \mathfrak{F}, \omega \text{ a Borel set}\}$. Denote by $\mathfrak{B}$ the set $\{(I + AA^*)^{-1} | A \in \mathfrak{F}_0\}$, by Z the spectrum of $\mathfrak{A}$ and by $f \rightarrow U_f$ the Gelfand isomorphism of $C(Z)$ onto $\mathfrak{A}$ ([10], 327-330). Let $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ be the hermitian spectral family corresponding to the isomorphism $f \rightarrow U_f$ and P be the spectral measure corresponding to $\mathfrak{F}$. Denote by $A \rightarrow f_A$ the isomorphism reciprocal to $f \rightarrow U_f$. Let us remark that if $P(M) = 0$ then there is a family (not necessarily denumerable) of disjoint compact subsets of $\mathbb{CM}$, $(F_n)_{n \in J}$, such that $\sum_{n \in J} P(F_n) = I$ ([10], lemme 2).

Let $M = \bigcup_{B \in \mathfrak{B}} f_B^{-1}(0)$. Since every $B \in \mathfrak{B}$ is one to one, $P(f_B^{-1}(0)) = 0$ for $B \in \mathfrak{B}$, and hence $P(M) = 0$. Choose a family $(F_n)_{n \in J}$ of disjoint compact subsets of the set $\mathbb{CM}$ such that $\sum_{n \in J} P(F_n) = I$. Let us define $P_n = P(F_n)$, for $n \in J$; then $(P_n)_{n \in J}$ is a family of orthogonal hermitian projections satisfying (1). Because $\inf_{z \in F_n} |f_B(z)| > 0$ for all $n \in J$ and $B \in \mathfrak{B}$, we can define g_B^n by the relations $g_B^n(z) = 0$ if $z \notin F_n$ and $g_B^n(z) = 1/f_B(z)$ if $z \in F_n$ and consider the operator $U_{B,n} = \int z g_B^n(z) dP(z)$. Obviously $U_{B,n} \in \mathfrak{A}$ for $n \in J$ and $B \in \mathfrak{B}$. Also $A(I + AA^*)^{-1} \in \mathfrak{A}$ for every $A \in \mathfrak{F}$.

For $A \in \mathfrak{F}$ and $n \in J$, the P_n commute with E^A ; hence $AP_n \supset P_n A$. If $A \in \mathfrak{F}_0$, and if we write $B = (I + AA^*)^{-1}$, then for every $n \in J$

¹² For the sake of completeness we give here a direct proof of this proposition.

$$AP_n = A(BU_{B,n}) = (AB)U_{B,n} \in \mathcal{Q}.$$

We deduce that $P_n(X) \subset D(A)$, that is, that we have $P_n(X) \subset \bigcap_{A \in \mathfrak{F}_n} D(A) = \bigcap_{A \in \mathfrak{F}} D(A)$. Since (4) is obvious the proof is complete.

Theorem 3. *For every object $\{X, U_s\} \in N_2$ there is one and only one hermitian spectral family $\mathfrak{F} = (\mu_{x,y})_{x \in X, y \in X}$ on $\tilde{E}$, concentrated on E , and such that*

$$(14) \quad (U_s x | y) = \int_{\tilde{E}} \chi(s) d\mu_{x,y}(\chi),$$

for all $s \in S$, $x \in D(U_s)$ and $y \in X$. $\mathfrak{F}$ satisfies $\|\mu_{x,x}\| = \|x\|^2$ for all $x \in X$ if and only if, for each non-zero $x \in \bigcap_{s \in S} D(U_s)$, there is an $s(x) \in S$ such that $U_{s(x)} x \neq 0$.

Let $\mathfrak{F} = \{U_s | s \in S\}$ and $(P_n)_{n \in J}$ be the family of hermitian projections constructed in proposition 7. If for each $n \in J$ we denote by $U_{s,n}$ the mapping of $X_n = P_n(X)$ into X_n defined by means of the relations $U_{s,n}x = U_s P_n x$, then $\{X_n, U_{s,n}\} \in N_1$. Hence there is on $\tilde{E}$ a hermitian spectral family $\mathfrak{F}_n = (\mu_{x,y}^n)_{x \in X_n, y \in X_n}$ concentrated on E such that, for $s \in S$, $x \in X_n$, $y \in X_n$,

$$(U_{s,n}x | y) = \int_{\tilde{E}} \chi(s) d\mu_{x,y}^n(\chi).$$

Let $\mathfrak{F}$ be the direct sum of the family $(\mathfrak{F}_n)_{n \in J}$. Now, for $n \in J$, $s \in S$ and $x \in X$,

$$\|U_{s,n}x\|^2 = \int_{\tilde{E}} |\chi(s)|^2 d\mu_{P_n x, P_n x}^n(\chi).$$

We deduce, using proposition 7, (4), and theorem 4, that for each $x \in D(U_s)$ the function $|\hat{s}|^2$ is $\mu_{x,x}$ -integrable and

$$(U_s x | y) = \int_{\tilde{E}} \chi(s) d\mu_{x,y}(\chi)$$

for all $x \in D(U_s)$ and $y \in X$.

It remains to prove the uniqueness of $\mathfrak{F}$. Let $\mathfrak{V} = (\nu_{x,y})_{x \in X, y \in X}$ be a hermitian spectral family on $\tilde{E}$, concentrated on E and satisfying the relations (14). Let us show first that, for $n \in J$, $x \in X$, $y \in X$, we have $(\delta) \nu_{P_n x, y} = \nu_{x, P_n y}$. Denote by P the spectral measure corresponding to $\mathfrak{V}$. Let F be the topological sum of $\tilde{E}$ and of a point $\{\omega\}$ ([2], 70–71), and for $x \in X$, $y \in X$ and $f \in \mathfrak{R}(F)$ write $\hat{\nu}_{x,y} = \nu_{x,y}(f_{\tilde{E}}) + f(\omega)((I - P(\tilde{E}))x | y)$. $\hat{\mathfrak{V}} = (\hat{\nu}_{x,y})_{x \in X, y \in X}$ is a hermitian spectral family on F and we have

$$(U_s x | y) = \int_{\tilde{E}} \chi(s) d\nu_{x,y}(\chi) = \int_F f_s(\chi) d\hat{\nu}_{x,y}(\chi),$$

for $s \in S$, $x \in D(U_s)$ and $y \in X$ ($f_s(\chi) = \chi(s)$ if $\chi \in \tilde{E}$ and $f_s(\chi) = 0$). But then

$$((I + U_s U_s^*)^{-1} x | y) = \int_F (1 + |f_s(x)|^2)^{-1} d\hat{\nu}_{x,y}(x)$$

for all $s \in S$, $x \in X$, $y \in X$. If $\hat{P}$ is the spectral measure corresponding to $\hat{\mathfrak{V}}$, it

follows that $(I + U_s U_s^*)^{-1} \hat{P}(K) = \hat{P}(K)(I + U_s U_s^*)^{-1}$ for every compact $K \subset F$ and $t \in S$. In the same manner we prove that $\hat{P}(K)$ commutes with $U_s(I + U_s U_s^*)^{-1}$. We deduce $A \hat{P}(K) = \hat{P}(K)A$ for every compact $K \subset F$ and $A \in A_0$ (see proposition 7) and hence $P_n \hat{P}(K) = \hat{P}(K)P_n$ for every compact $K \subset F$ and $n \in J$. But then $\hat{p}_{P_n x, y} = \hat{p}_{x, P_n y}$ for all $n \in J$, $x \in X$, $y \in X$ and this implies (δ) . We deduce then (proposition 6) that $\mathfrak{U}$ is the direct sum of $(\mathfrak{U}_n)_{n \in J}$ where, for each $n \in J$, $\mathfrak{U}_n = (\nu_{x, y})_{x \in X_n, y \in X_n}$. Since, for $n \in J$, $s \in S$, $x \in X_n$ and $y \in X_n$,

$$(U_{s, n} x \mid y) = \int_{\tilde{E}} \chi(s) d\mu_{x, y}^n(\chi) = \int_{\tilde{E}} \chi(s) d\nu_{x, y}(\chi),$$

it follows that $\mathfrak{F}_n = \mathfrak{U}_n$ for each $n \in J$ and hence $\mathfrak{F} = \mathfrak{U}$. The last assertion in the statement being obvious, the proof is complete.¹³

Remark. If S satisfies conditions (3) and (4) and $U_{s+} = U_s^*$ for all $s \in S$, then $\mathfrak{F}$ is concentrated on A .¹⁴

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¹³ The representation formulas given in [6], [8], [12] are particular cases of the preceding results. See also A. E. NUSSBAUM (*Amer. Math. Soc., Notices*, February 1958, Abstract 543–5). In a weaker form and without proof theorem 3 has been stated in [17].

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