*On Quasi-Conformal Functions of Several Complex Variables**

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Communicated by S. BERGMAN

1. Introduction.

1.1. The well known Riemann mapping theorem asserts that every simply connected plane region with at least two boundary points can be mapped onto the unit disc $\{|z| < 1\}$ conformally, *i.e.*, by a holomorphic transformation. However, a similar situation does not hold in the case of several complex variables. As POINCARÉ [12] showed in 1907, the bicylinder $\{|z_1| < 1, |z_2| < 1\}$ and the hypersphere $\{|z_1|^2 + |z_2|^2 < 1\}$ cannot be mapped onto each other by a one-to-one holomorphic transformation of two complex variables, although these domains have the same structure from the purely topological point of view. In reality, the holomorphic rigidity in the case of several complex variables is far more severe than in the case of one complex variable.

Of course, even in the case of one complex variable, multiply connected domains or domains on Riemann surfaces are not always conformally equivalent to each other, even when they have the same topological structure. For example, the ring domain $\{q < |z| < 1\}$ is a representative domain of the class of doubly connected domains, and two ring domains $\{q < |z| < 1\}$ and $\{q' < |w| < 1\}$ are conformally equivalent to each other if and only if $q = q'$. Hence q is a conformal invariant. For a doubly connected domain D which is mapped conformally onto $\{q < |z| < 1\}$, the quantity

$$(1) \quad \text{mod } D = \frac{1}{2\pi} \log \frac{1}{q}$$

is known as the *modulus* of the domain D .

1.2. The theory of quasi-conformal mappings of one complex variable has recently been extensively developed (for instance, [1], [6], [9], [11], [15], [16]). First I shall give a brief sketch of the main ideas of this theory, because it is necessary for the later discussion.

* This work has been supported by a Fulbright/Smith-Mundt Scholarship and a contract of the National Science Foundation (Grant N.S.F.-G3661) at Stanford University.

A *quasi-conformal function*¹ $f(z)$ of one complex variable $z = x + iy$ in a domain D was originally defined as a smooth function (of class C^1 in x and y), satisfying any one of the following three inequalities at every point of D :

$$(2a) \quad \max_{\theta} \left| \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta \right| \leq KJ,$$

$$(2b) \quad \left| \frac{\partial f}{\partial x} \right|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \leq \left(K + \frac{1}{K} \right) J,$$

$$(2c) \quad \frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}, \quad |\mu| \leq k_0 = \frac{K-1}{K+1} < 1.$$

Here J denotes the Jacobian

$$J = \frac{\partial(\operatorname{Re} f, \operatorname{Im} f)}{\partial(x, y)},$$

$$\frac{\partial f}{\partial z} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right),$$

and K is a constant ≥ 1 , called the *dilatation* of f . It is easy to see that the above three inequalities are equivalent. For brevity, I shall say that f is K -Q.C., if f is a quasi-conformal function with dilatation K .

This definition is somewhat narrow in the light of recent developments. Generalizations have been made in several ways. One is the analytic method, of C. MORREY and L. BERS, in which the smoothness assumption is replaced by the ACT_2 property.² Another is the geometric method of A. PFLUGER, L. V. AHLFORS, and A. MORI, which is based on the distortion of the moduli of quadrilaterals. These two generalizations are, in fact, equivalent as BERS [6] has remarked.

1.3. From a finite Riemann surface³ $\mathfrak{D}$ to another $\mathfrak{D}'$, with the same topological structure as $\mathfrak{D}$, there always exists a one-to-one quasi-conformal mapping with sufficiently large K . Of course, this mapping is not unique. However, TEICHMUELLER [16] and AHLFORS [1] proved the important result that among the various quasi-conformal mappings from $\mathfrak{D}$ to $\mathfrak{D}'$, there is an *extremal one which minimizes the dilatation* K . This extremal mapping is essentially unique under suitable normalization conditions and homotopy type. It is the “natural” mapping from $\mathfrak{D}$ to $\mathfrak{D}'$ in the sense of being as near as possible to an analytic

¹ Sometimes this is called a “pseudo-analytic function” or “pseudo-regular function” (e.g., in [9], [11]). I prefer the terminology “quasi-conformal function” instead of the others, because the latter are used also with a quite different meaning.

² The ACT_2 property means the absolute continuity in TONELLI’s sense and that the partial derivatives (which exist almost everywhere) are square integrable on every compact subset of the domain. For further details, see, e.g., BERS [6].

³ In the sense of SCHIFFER and SPENCER, i.e., a Riemann surface of finite genus whose boundary consists of a finite number of analytic curves.

mapping. The extremal quasi-conformal mapping also plays an essential role in connection with the moduli of Riemann surfaces ([1], [16]).

As an example of such mappings, we mention the case of the ring domain,

$$\{q < |z| < 1\} \rightarrow \{q' < |w| < 1\}.$$

In this case, the extremal quasi-conformal mapping is given by

$$(3) \quad w = e^{i \arg z} |z|^{\log q' / \log q}$$

(or this function multiplied by a constant $e^{i\theta}$). We obtain the extremal quasi-conformal mapping from one doubly connected domain to another by combining (3) with the conformal mappings onto ring domains.

1.4. As was mentioned at the beginning, in the study of analytic functions or analytic transformations in several complex variables it is extremely important to introduce some of the “extended classes” of functions or mappings. Of course, such generalizations may be made in completely different ways for various applications.⁴

The above theory of quasi-conformal mapping of one complex variable seems to suggest the possibility of such a kind of generalization. The main purpose of the present paper is to discuss a class of quasi-conformal functions and mappings of several complex variables (§§4, 5).

However, before discussing them, I shall first give in §2, a brief sketch of recent work on a “quasi-pseudo-conformal” transformation for Reinhardt circular domains by S. BERGMAN [5]. At the end of this paper, it will be proved that this mapping is not in my class of mappings except for very trivial cases. By the way, S. BERGMAN also discusses a different type of “quasi-pseudo-conformal” mapping in a separate paper [4]. As for the latter, I shall give a short remark in §5 on its connection with the present paper.

Probably the most direct generalization of the quasi-conformal mappings of two real variables (*i.e.*, one complex variable) is the quasi-conformal mappings of several *real* variables in §3. I shall give a brief remark on the recent work on this class of mappings due to C. LOEWNER [10] and E. D. CALLENDER [7].

The aim of the present paper is the consideration of a class of quasi-conformal mappings of several complex variables (in §5). However, I shall first discuss some of the properties of a single quasi-conformal function in §4. I call it a “discwise quasi-conformal function”. The class of these functions is, of course, a kind of “extended class”, but I must remark that it is defined completely locally, independent of the special features of the domains. The main technique

⁴ For example, S. BERGMAN introduced a so-called “extended class of functions” to solve some boundary value problems in order to introduce something like the “harmonic measure” or “Green’s function” in the study of one complex variable. Other kinds of similar functions for the solution of boundary value problems have been introduced recently by D. B. LOWDENSLAGER, H. J. BREMERMAN and others. However, I do not discuss them in detail here, because they are completely different types of generalizations and have no connection with the present paper.

involves the cutting with a holomorphic plane and reducing to the case of one complex variable as was done in the theory of “pluri-subharmonic function” due to P. LELONG and H. J. BREMERMAN.

In the final section, §5, the mapping determined by pairs of the functions discussed in §4 will be analyzed. Unfortunately, this kind of mapping does not appear to be a suitable generalization of holomorphic transformations, as will be seen in the examples in §5, although a single function in §4 has rather interesting properties.

The author wishes to express his gratitude to B. EPSTEIN for his kind suggestions during the preparation of the present paper at Stanford University.

2. Bergman’s “quasi-pseudo-conformal” mapping for Reinhardt circular domains.

2.1. By analogy with the ring domains mentioned in §1, one of the simplest classes of “standard” domains in two complex variables is the class of *Reinhardt circular domains*. A Reinhardt circular domain in (Z_1, Z_2) -space is usually defined to be a domain which admits a group of transformations

$$Z_1^* = e^{i\theta_1} Z_1, \quad Z_2^* = e^{i\theta_2} Z_2$$

onto itself, where θ_1 and θ_2 are real parameters. However, in the present paper, we use this terminology in a more restricted sense. The domain $\mathfrak{R}$ to be considered in [5] is of the following form in (Z_1, Z_2) -space:

$$(4) \quad \mathfrak{R} = \{(Z_1, Z_2); |Z_2| < P(|Z_1|), 0 \leq |Z_1| < a\},$$

and we assume that $P(X)$ is a sufficiently smooth, decreasing, concave function defined on $0 \leq X \leq a$ with the conditions

$$P(a) = 0, \quad P'(0) = 0, \quad P'(X) \searrow -\infty \quad (X \rightarrow a - 0).$$

This condition means that the domain

$$(5) \quad \mathfrak{B} = \{X + iY; -P(|X|) < Y < P(|X|), -a < X < a\}$$

in the (X, Y) -plane is a convex domain with a sufficiently smooth boundary and also is symmetric with respect to both coordinate axes. The domain $\mathfrak{R}$ is given by the rotation of $\mathfrak{B}$ around the $(\operatorname{Re} Z_1, \operatorname{Re} Z_2)$ -plane in the 4-dimensional space.

Now, on account of the symmetry of $\mathfrak{B}$, there exists a conformal mapping $g(x + iy)$ from the unit disc $\{|x + iy| < 1\}$ onto the domain $\mathfrak{B}$ such that the real and the imaginary axes are mapped on the real and the imaginary axes of the (X, Y) -plane, respectively. The mapping

$$(6) \quad \begin{cases} Z_1 = e^{i \arg z_1} \operatorname{Re} g(|z_1| + i |z_2|), \\ Z_2 = e^{i \arg z_2} \operatorname{Im} g(|z_1| + i |z_2|) \end{cases}$$

transforms the unit hypersphere $\{|z_1|^2 + |z_2|^2 < 1\}$ onto the domain $\mathfrak{R}$ in a

one-to-one manner. Combining holomorphic transformations⁵ with (6), we have some kind of standard mapping which maps a domain $\mathfrak{D}$, holomorphically equivalent to a hypersphere, onto a domain $\mathfrak{D}'$, holomorphically equivalent to the domain $\mathfrak{R}$. If we have some distortion theorems on analytic invariants under the mapping (6), we can combine them with the additional holomorphic transformations, and hence we shall have some distortion theorems referring to the standard mapping from $\mathfrak{D}$ onto $\mathfrak{D}'$ given above.

2.2. Along this line, BERGMAN [5] has made an interesting investigation, especially of the distortion of the invariant Kaehlerian metric defined by

$$ds^2 = \sum_{i,k=1}^2 \frac{\partial^2 \log K(z_1, z_2; \bar{z}_1, \bar{z}_2)}{\partial z_i \partial \bar{z}_k} dz_i d\bar{z}_k$$

where K is the Bergman kernel function of the given domain.⁶

3. Quasi-conformal mappings of real variables.

3.1. The inequalities (2a)–(2c) have a very simple geometric meaning. They imply that an infinitesimal circle is mapped onto an infinitesimal ellipse, and that the ratio of the major and the minor axes is bounded uniformly by the constant K . This is the origin of the name “dilatation”. Similarly, we can consider a mapping from a domain in real m -dimensional Euclidean space E^m into another space in such a way that it maps an infinitesimal sphere onto an infinitesimal ellipsoid and that the ratio of the largest and the smallest axes is bounded uniformly by a constant K . For example, a non-degenerate linear mapping has this property. More generally, if a mapping $f: D \rightarrow D'$ is one-to-one and of class C^1 and has a non-vanishing Jacobian even on the boundary of the bounded domain D , then f is of the type in D . I call such mappings “quasi-conformal mappings of real variables”.⁷ Recently, interesting work concerning such mappings has been done by LOEWNER [10] and CALLENDER [7], especially in the case of 3-dimensional Euclidean space. Here I shall indicate in a brief sketch the main ideas of their work.

3.2. One can define the notion of the *modulus* of a doubly connected domain, similar to the notion of modulus in §1, for higher dimensional regions. Here, a doubly connected domain means a domain whose boundary consists of two components. Of course, one cannot do so by means of conformal mappings, as in (1). However, the modulus of a doubly connected plane region has many

⁵ In the terminology of BERGMAN [2]–[5], a “holomorphic mapping” is called a “pseudo-conformal mapping”. In the present paper, I avoid the latter terminology lest it should cause confusion with the “quasi-conformal mapping”. Also, I prefer the word “holomorphic” rather than “analytic” to distinguish it from “real-analytic”.

⁶ As for the details about kernel functions and invariant metrics, see, *e.g.*, [2], [3].

⁷ A mapping $f: D \rightarrow D_1$ of n -dimensional Euclidean space is decomposed into a system of n functions $f_1, \dots, f_n$, which may be considered as a vector field. Now, contrary to the case $n = 2$, the property that f is quasi-conformal in the above sense has no connection with the harmonicity of the vector field $(f_1, \dots, f_n)$ when $n \geq 3$; neither property implies the other.

other geometric characterizations. For instance, one can define it by means of the notion of the “extremal length” introduced by A. BEURLING. There is also another definition in terms of an extremal problem, namely,

$$(7) \quad \begin{aligned} \text{mod } D &= 1/\inf_u \iint_D |\text{grad } u|^2 dx dy, \\ |\text{grad } u| &= \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]^{\frac{1}{2}}, \end{aligned}$$

where $\inf_u$ is taken over the family of all piecewise smooth functions u with boundary value 1 on the one boundary component and 0 on the other boundary component of D , respectively. One can define analogously a notion of “modulus” for a doubly connected domain D in 3-dimensional space as follows:

$$(8) \quad \text{mod } D = 1/\inf_u \iiint_D |\text{grad } u|^3 dx dy dz.$$

(In the paper [10], this is called the “conformal capacity”. Observe that $|\text{grad } u|^2$ in (7) is replaced by $|\text{grad } u|^3$.) As before, $\inf_u$ is taken over the family of all piecewise smooth functions u with boundary values 1 and 0 on the two boundary components of D , respectively.

After one introduces the notion of “modulus”, the distortion of the “modulus” (8) under a quasi-conformal mapping of three real variables will be studied. LOEWNER [10] obtained the following important result:

If we have a quasi-conformal mapping f of three real variables in the above sense defined in the whole three-dimensional real space E^3 which maps E^3 into another E^3 in a one-to-one manner, then f must be onto.

This fact corresponds to a generalization of a famous theorem in the plane which asserts that the type of a Riemann surface (hyperbolic or parabolic) is kept invariant under the quasi-conformal mappings of two real variables (*i.e.*, one complex variable) in the usual sense (see KAKUTANI [9] or TEICHMUELLER [15]).

3.3. The investigation of quasi-conformal mappings of real variables in this sense seems to be very interesting and important. However, for the application to holomorphic functions of several complex variables, this class of mappings seems to be less useful than in the case of one complex variable. One reason for this comes from the fact that the space of n complex variables is not merely a $2n$ -dimensional real space but has a more complicated complex structure. A second reason is that not all holomorphic transformations belong to the class of quasi-conformal mappings in the above sense. For instance, the one-to-one holomorphic mapping

$$z_1^* = z_1, \quad z_2^* = (z_2 - 1)^2$$

of the bicylinder $\{|z_1| < 1, |z_2| < 1\}$ is not a quasi-conformal mapping of four real variables, because the ratio of the largest and the smallest axes of the image

of an infinitesimal sphere tends to ∞ if $z_2 \rightarrow 1$. Again, if we fix the constant K , even the linear transformation

$$z_1^* = az_1, \quad z_2^* = bz_2$$

does not belong to the class of quasi-conformal mappings of dilatation K if the ratio $|b/a|$ or $|a/b|$ is sufficiently large.

Accordingly, I prefer to introduce another kind of generalization of quasi-conformal mapping (§5). But before I do so, I begin with the discussion of each component, *i.e.*, a single “quasi-conformal function”, in §4.

4. Discwise quasi-conformal functions.

4.1. Definition 1. A function $f(z_1, \dots, z_n)$ of n complex variables $z_1, \dots, z_n$ defined in a domain D is called a *discwise quasi-conformal function with dilatation* K if

- (i) f is of class C^1 in D , and
- (ii) f is K -Q.C. on each holomorphic plane.

More precisely, the condition (ii) means the following. As far as we have a linear mapping

$$(9) \quad \Lambda(t): z_1 = a_1 t + b_1, \dots, z_n = a_n t + b_n$$

($a_1, b_1, \dots, a_n, b_n$: complex constants, t : complex variable) defined on the unit disc $\{|t| < 1\}$ whose image lies completely in D , the composite function

$$(10) \quad f(t) = f(a_1 t + b_1, \dots, a_n t + b_n)$$

is always a K -Q.C. function in $\{|t| < 1\}$. Here K is a *fixed* constant ≥ 1 , called the *dilatation* of f . For brevity, I say that f is K -D.Q.C.

One can replace the hypothesis of class C^1 by weaker assumptions, but in the present paper I shall consider only smooth K -D.Q.C. functions.

The 1-D.Q.C. function (*i.e.*, when $K = 1$) is nothing but a holomorphic function in D . Hence every holomorphic function is K -D.Q.C. If $n = 1$, this definition gives precisely the usual quasi-conformal function of dilatation K .

Also, if $g(z_1, \dots, z_n)$ is K -D.Q.C. in D and $h(\zeta)$ is a K_1 -Q.C. function of one complex variable ζ defined in a domain on ζ -plane containing the range of g , then the composite function

$$f(z_1, \dots, z_n) = h(g(z_1, \dots, z_n))$$

is KK_1 -D.Q.C. in D .

4.2. Now I shall discuss some of the properties of this class of functions.

Theorem 1. *If f is discwise quasi-conformal in a bounded domain D and continuous also on the boundary ∂D of D , then the maximum principle holds, in the sense*

$$\sup \{|f| ; D\} = \sup \{|f| ; \partial D\}.$$

Here the notation $\sup\{|f|; A\}$ means the supremum of the set $\{|f(z)|; z \in A\}$, i.e., the least upper bound of the values which are taken by f in A . ∂D means the boundary of D .

Also, if $|f|$ assumes the maximum value in $\bar{D} = D \cup \partial D$ at an inner point of D , then f must be a constant.

Proof. Since $\bar{D}$ is compact, $|f|$ takes its maximum value a at a point in $\bar{D}$. We first remark that a non-constant K -Q.C. function g of one complex variable gives rise to an interior mapping in the sense of STOILOW, i.e., an *open* and *light* mapping.⁸ Therefore a K -Q.C. function g satisfies the maximum modulus principle, and if $|g|$ assumes its maximum value at an inner point, then g must be a constant.

Now suppose that there is an inner point $(z_1^0, \dots, z_n^0)$ of D at which $|f|$ takes the maximum value a . I put

$$f(z_1^0, \dots, z_n^0) = \alpha, \quad |\alpha| = a$$

and

$$M = \{(z_1, \dots, z_n); f(z_1, \dots, z_n) = \alpha\}.$$

It is evident that $M \cap D$ is a non-empty relatively closed set in D . But it is also open. To prove this, suppose that a point $\zeta = (\zeta_1, \dots, \zeta_n)$ is in $M \cap D$. For an arbitrary point $z = (z_1, \dots, z_n)$ in a small neighborhood U of ζ , we have a holomorphic plane (9) passing through ζ and z which is determined uniquely. On this plane, f is a K -Q.C. function of one complex variable t and $|f|$ assumes its maximum value at an inner point corresponding to ζ . Therefore f must be a constant on this plane, so that $z \in M \cap D$. This holds for an arbitrary point z in U , which proves that $M \cap D$ is open. Since D is connected, $D \cap M$ must be equal to D itself, which implies that the function f is a constant α .

Therefore, if f is not a constant, $|f|$ cannot assume its maximum value at an inner point of D .

Remark 1. For some domains, we have a proper subset Δ of the whole boundary of D such that $\sup\{f; D\} = \sup\{f; \Delta\}$ holds for all K -D.Q.C. functions f continuous on $\bar{D}$. For instance, if D is the bicylinder $\{|z_1| < 1, |z_2| < 1\}$, we may take

$$\Delta = \{|z_1| = 1, |z_2| = 1\},$$

i.e., we can choose Δ to be the "Bergman distinguished boundary surface" of the bicylinder. More generally, Δ can be taken as the "Bergman distinguished

⁸ Here "open" means that open sets are mapped onto open sets, and "light" means that the inverse image of a point is a discrete set. As STOILOW has remarked, an interior mapping in this sense is nothing but a combination of two transformations, the first of which is a homeomorphism and the second a holomorphic mapping (see MORR [11], p. 58).

If we start from the analytic definition of quasi-conformal function (as in BERS [6]), an intensive analysis is necessary in order to prove that a (non-schlicht) K -Q.C. mapping has this property.

boundary surface" S of D whenever D is bounded by a finite number of analytic surfaces $\Gamma_1, \dots, \Gamma_m$ whose mutual intersections give rise to the distinguished boundary surface S . However, it is not clear whether Δ can always be taken to be the *Bergman-Šilov boundary* of D . (Every bounded domain D has a uniquely determined smallest set S in its boundary on which holomorphic functions assume their maximum values. Precisely speaking, $\sup\{f; D\} = \sup\{f; S\}$ holds for every function f holomorphic in D and continuous on the boundary of D , and this property no longer holds for any set S' smaller than S . This set S is the *Bergman-Šilov boundary*⁹ of D .)

In the above examples, we can choose Δ as the Bergman-Šilov boundary, which is also equal to the "Bergman distinguished boundary surface". In the consideration of K -D.Q.C. functions, the difficulty comes from the fact that the totality of K -D.Q.C. functions is neither a ring nor a linear space as mentioned later.

4.3 Theorem 2. *A K -D.Q.C. function f is K -Q.C. on every holomorphic surface. More precisely, as far as we have a holomorphic mapping*

$$(11) \quad \Phi(t): z_1 = \varphi_1(t), \dots, z_n = \varphi_n(t) \quad (t: \text{complex variable})$$

defined on the unit disc $\{|t| < 1\}$ whose image lies completely in D , the composite function

$$(12) \quad \hat{f}(t) = f(\varphi_1(t), \dots, \varphi_n(t))$$

is a K -Q.C. function in $\{|t| < 1\}$.

Corollary. *The notion of K -D.Q.C. function is invariant under the holomorphic transformation in the following sense. Let $f(z_1, \dots, z_n)$ be K -D.Q.C. in D and let*

$$(13) \quad z_1 = \varphi_1(w_1, \dots, w_m), \dots, z_n = \varphi_n(w_1, \dots, w_m)$$

be a holomorphic transformation from a domain B in $(w_1, \dots, w_m)$ -space¹⁰ into D . Then the composite function

$$(14) \quad F(w_1, \dots, w_m) = f(\varphi_1(w_1, \dots, w_m), \dots, \varphi_n(w_1, \dots, w_m))$$

is again K -D.Q.C. in B .

Proof of Theorem 2. If all $\varphi_i(t)$ in (11) are constants, the assertion is trivial. Therefore we may assume that at least one of $\varphi_i(t)$ is not constant. Hence we may assume that the set

⁹ Such sets S were discussed by BERGMAN in some special cases under the name of "maximum surface" (e.g., in *Jahrb. D.M.V.*, 38 (1929), p. 26). Later he also discussed some aspects of the geometric structure of S for domains with sufficiently smooth boundaries (for example, in his paper in *Amer. Jour. of Math.*, 65 (1943), pp. 679–700). ŠILOV [14] first proved the existence and the uniqueness of such a set S for a completely arbitrary domain, in connection with his theory of normed rings, and in a rather abstract way.

¹⁰ Here m need not be equal to n .

$$E = \{t; \varphi'_1(t) = \cdots = \varphi'_n(t) = 0, |t| < 1\}$$

is discrete in $\{|t| < 1\}$. Now suppose that t_0 is not in E . We have a linear mapping from another unit disc $\{|\tau| < 1\}$ into D by

$$\Lambda(\tau): z_k = \lambda_k(\tau) = c\varphi'_k(t_0)\tau + \varphi_k(t_0) \quad (k = 1, \cdots, n)$$

where c is a sufficiently small positive constant. By Definition 1, the function $\tilde{f}(\tau) = f(\lambda_1(\tau), \cdots, \lambda_n(\tau))$ is a K -Q.C. function in $\{|\tau| < 1\}$. Hence, by (2c), we have

$$(15) \quad \left(\frac{\partial \tilde{f}}{\partial \tau}\right)_{\tau=0} = \mu \left(\frac{\partial \tilde{f}}{\partial \tau}\right)_{\tau=0}, \quad |\mu| \leq k_0 = \frac{K-1}{K+1} < 1.$$

On the other hand, the relations

$$\left(\frac{\partial \tilde{f}}{\partial \tau}\right)_{\tau=0} = \sum_{k=1}^n \frac{\partial f}{\partial z_k} \frac{\partial \lambda_k}{\partial \tau} = c \sum_{k=1}^n \frac{\partial f}{\partial z_k} \varphi'_k(t_0) = c \left(\frac{\partial f}{\partial t}\right)_{t=t_0},$$

and similarly

$$\left(\frac{\partial \tilde{f}}{\partial \bar{\tau}}\right)_{\tau=0} = c \left(\frac{\partial \hat{f}}{\partial \bar{t}}\right)_{t=t_0}$$

hold, where $\hat{f}$ is the function defined by (12). Therefore the function $\hat{f}(t)$ satisfies the relation (2c) at $t = t_0$ according to (15), and this property holds at every point t_0 outside E .

Now, since E is discrete in $\{|t| < 1\}$ and $\hat{f}(t)$ is of class C^1 everywhere, the set E is a removable singularity set for such a smooth K -Q.C. function $\hat{f}$ (see, e.g., MORI [11], p. 66). Hence $\hat{f}(t)$ is K -Q.C. in $\{|t| < 1\}$ without exception, which proves our theorem.

Remark 2. According to Theorem 2, one can define a K -D.Q.C. function as a smooth function which is K -Q.C. on every holomorphic surface. This definition is valid not only in a domain D , but also for an arbitrary set, in particular on an analytic subset in the $(z_1, \cdots, z_n)$ -space. If $K = 1$, this latter definition coincides with the definition of *D-holomorphic function* which I have discussed in another paper [8].

Remark 3. In connection with the corollary to Theorem 2, it is in general *false* that the function (14) is K -D.Q.C. if $f(z_1, \cdots, z_n)$ is holomorphic and the functions (13) are K -D.Q.C. This statement is true if $n = 1$, but is no longer valid if $n \geq 2$. In fact, the *sum* of two K -D.Q.C. functions is not always K -D.Q.C. For example, it is easy to see that both the functions $2z_1 + 2z_2 + \bar{z}_1 + \bar{z}_2$ and $-z_1 - z_2$ are K -D.Q.C. But the sum

$$(16) \quad z_1 + z_2 + \bar{z}_1 + \bar{z}_2 = 2(\operatorname{Re} z_1 + \operatorname{Re} z_2)$$

is not K -D.Q.C. because, as was remarked before, a quasi-conformal function of one complex variable must give an open mapping, so that it cannot take only real values, unless it is a constant. On the holomorphic plane $z_2 = \text{const.}$, the function (16) is a non-constant function of one complex variable z_1 which assumes only real values. Hence it is not a quasi-conformal function of z_1 , so that (16) does not satisfy the condition of Definition 1.

4.4. Next I am going to give a differential equation which a K -D.Q.C. function must satisfy.

Definition 2. I say that a point $(z_1^0, \dots, z_n^0)$ is *ordinary* with respect to a complex-valued smooth function f if at least one of the derivatives $\partial f / \partial z_1, \dots, \partial f / \partial z_n$ does not vanish at the point $(z_1^0, \dots, z_n^0)$.

Theorem 3. At every ordinary point, a K -D.Q.C. function f satisfies the differential equations

$$(17) \quad \frac{\partial \bar{f}}{\partial z_j} = \kappa \frac{\partial f}{\partial z_j}, \quad |\kappa| \leq k_0 = \frac{K-1}{K+1} < 1 \quad (j = 1, 2, \dots, n).$$

Corollary. A K -D.Q.C. function f must satisfy the following system of differential equations identically

$$(18) \quad \frac{\partial f}{\partial z_j} \frac{\partial \bar{f}}{\partial z_k} = \frac{\partial f}{\partial z_k} \frac{\partial \bar{f}}{\partial z_j} \quad (j, k = 1, 2, \dots, n).$$

Writing (18) in real form, we have the relations

$$\frac{\partial(u, v)}{\partial(x_j, x_k)} = \frac{\partial(u, v)}{\partial(y_j, y_k)}, \quad \frac{\partial(u, v)}{\partial(x_j, y_k)} = \frac{\partial(u, v)}{\partial(x_k, y_j)} \quad (j, k = 1, 2, \dots, n),$$

where we put $f = u + iv$, $z_j = x_j + iy_j$ ($j = 1, \dots, n$).

Proof of Theorem 3. We may assume, by changing the coordinates if necessary, that $z_1^0 = \dots = z_n^0 = 0$ and $(\partial f / \partial z_1)_0 \neq 0$. Hereafter, until the end of the proof, $(\)_0$ means the value of the derivative at the origin. The linear mapping

$$z_1 = \alpha_1 t, \dots, z_n = \alpha_n t$$

maps the unit disc $\{|t| < 1\}$ into D , provided that

$$(19) \quad |\alpha_1| < \delta, \dots, |\alpha_n| < \delta,$$

where δ is a sufficiently small positive constant. By Definition 1, the function $\tilde{f}(t) = f(\alpha_1 t, \dots, \alpha_n t)$ is K -Q.C. in $\{|t| < 1\}$, so that for every set of constants satisfying (19), the relation

$$(20) \quad \sum_{i=1}^n \bar{\alpha}_i \left(\frac{\partial f}{\partial \bar{z}_i} \right)_0 = \left(\frac{\partial \tilde{f}}{\partial \bar{t}} \right)_0 = \mu \left(\frac{\partial \tilde{f}}{\partial t} \right)_0 = \mu \sum_{i=1}^n \alpha_i \left(\frac{\partial f}{\partial z_i} \right)_0$$

follows from (2c). Here the coefficient $\mu = \mu(\alpha_1, \dots, \alpha_n)$ depends upon the

parameters, but it always satisfies the condition

$$|\mu| \leq k_0 = \frac{K-1}{K+1} < 1,$$

i.e., μ is uniformly bounded by k_0 .

For each $j = 2, \dots, n$, we put

$$S_i(\zeta) = \frac{(\partial \bar{f} / \partial z_1)_0 \zeta + (\partial \bar{f} / \partial z_i)_0}{(\partial f / \partial z_1)_0 \zeta + (\partial f / \partial z_i)_0}$$

where ζ is a complex variable on whole complex sphere $\{|\zeta| \leq \infty\}$. We are going to prove that $S_i(\zeta)$ is a *constant*. In fact, for every ζ , we can choose the n numbers $\alpha_1, \dots, \alpha_n$ in (19) in such a way that the conditions

$$(21) \quad \alpha_2 = \dots = \alpha_{j-1} = \alpha_{j+1} = \dots = \alpha_n = 0, \quad \zeta = \alpha_1 / \alpha_j$$

are satisfied. $\zeta = \infty$ corresponds to $\alpha_i = 0$. Inserting (21) into (20), we obtain the boundedness of $S_i(\zeta)$, namely,

$$\begin{aligned} |S_i(\zeta)| &= \left| \frac{(\partial \bar{f} / \partial \bar{z}_1)_0 \bar{\zeta} + (\partial \bar{f} / \partial \bar{z}_i)_0}{(\partial f / \partial z_1)_0 \zeta + (\partial f / \partial z_i)_0} \right| = \left| \frac{(\partial \bar{f} / \partial \bar{z}_1)_0 \bar{\alpha}_1 + (\partial \bar{f} / \partial \bar{z}_i)_0 \bar{\alpha}_j}{(\partial f / \partial z_1)_0 \alpha_1 + (\partial f / \partial z_i)_0 \alpha_j} \right| \\ &= |\mu(\alpha_1, 0, \dots, 0, \alpha_j, 0, \dots, 0)| \leq k_0. \end{aligned}$$

Hence $S_i(\zeta)$ reduces to a constant κ_i , which implies the relations

$$\left(\frac{\partial \bar{f}}{\partial z_1} \right)_0 = \kappa_i \left(\frac{\partial \bar{f}}{\partial z_1} \right)_0, \quad \left(\frac{\partial \bar{f}}{\partial z_j} \right)_0 = \kappa_i \left(\frac{\partial \bar{f}}{\partial z_j} \right)_0.$$

Since we have assumed $(\partial f / \partial z_1)_0 \neq 0$, the value of κ_i is the same for each $j = 2, \dots, n$. Denoting this common value by κ , we have the relation (17). (18) follows immediately from (17) at every ordinary point, but it is trivially true at any non-ordinary point such as $0 = 0$.

4.5. The relation (17) or (18) has a very significant geometric meaning. For simplicity, I shall discuss it only in the case $n = 2$.

Theorem 4. Let $f(z_1, z_2)$ be a K -D.Q.C. function of two complex variables z_1, z_2 in a domain D . At every ordinary point (z_1^0, z_2^0) , the set given by the equation

$$(22) \quad f(z_1, z_2) = f(z_1^0, z_2^0) = \text{const.}$$

is a 2-dimensional holomorphic surface.

Proof. We may again assume without loss of generality that $\partial f / \partial z_1 \neq 0$ at the point (z_1^0, z_2^0) . From (17), we obtain

$$\frac{\partial(f, \bar{f})}{\partial(z_1, \bar{z}_1)} = (1 - |\kappa|^2) \left| \frac{\partial f}{\partial z_1} \right|^2 \neq 0,$$

so that we can solve the equation (22) in the form

$$(23) \quad z_1 = g(z_2), \quad \bar{z}_1 = \bar{g}(z_2)^{11}$$

¹¹ At present, we do not know whether $g(z_2)$ is holomorphic in z_2 or not. More precisely, it will be better to write $z_1 = g(z_2, \bar{z}_2)$.

in some neighborhood of z_2^0 . Differentiating the identity

$$f(g(z_2), z_2) = \text{const.}$$

with respect to z_2 and $\bar{z}_2$, we have

$$(24a) \quad \frac{\partial f}{\partial z_1} \frac{\partial g}{\partial z_2} + \frac{\partial f}{\partial \bar{z}_1} \frac{\partial \bar{g}}{\partial z_2} + \frac{\partial f}{\partial z_2} = 0,$$

$$(24b) \quad \frac{\partial \bar{f}}{\partial z_1} \frac{\partial g}{\partial z_2} + \frac{\partial \bar{f}}{\partial \bar{z}_1} \frac{\partial \bar{g}}{\partial z_2} + \frac{\partial \bar{f}}{\partial z_2} = 0.$$

Multiplying (24a) by $-\kappa$ and adding it to (24b), we have by (17)

$$0 = \left(\frac{\partial \bar{f}}{\partial \bar{z}_1} - \kappa \frac{\partial f}{\partial \bar{z}_1} \right) \frac{\partial \bar{g}}{\partial z_2} = (1 - |\kappa|^2) \left(\frac{\partial \bar{f}}{\partial z_1} \right) \frac{\partial \bar{g}}{\partial z_2}.$$

Since $|\kappa| \leq k_0 < 1$, and $\partial f / \partial z_1 \neq 0$, we have $\partial \bar{g} / \partial z_2 = 0$. This last equation is precisely the system of Cauchy-Riemann equations for $g(z_2)$ with respect to z_2 when separated into real and imaginary parts. Hence (22) is a holomorphic surface defined by the holomorphic relation (23) in a neighborhood of the point (z_1^0, z_2^0) .

Remark 4. In the case $n = 2$, the equation (18) means, in general, that the 2-dimensional tangential plane of the level-surface (22) is always a holomorphic plane. As SEVERI has remarked ([13], p. 54), this property is a necessary and sufficient condition that (22) be a holomorphic surface.

4.6. Finally, I discuss the converse, starting from the system of differential equations (17). In the present paper, I do not discuss the existence of a solution of (17).

Theorem 5. Suppose κ is a given continuous function of $2n$ real variables $x_1, y_1, \dots, x_n, y_n$ ($z_j = x_j + iy_j$, $j = 1, \dots, n$) in a domain D . We assume that $|\kappa|$ is bounded by a constant $k_0 < 1$. If $f(z_1, \dots, z_n)$ is a solution of class C^1 of the system of differential equations (17) in D , then f is a K -D.Q.C. function in D . Here the dilatation K is given by

$$K = \frac{1 + k_0}{1 - k_0}, \quad |\kappa| \leq k_0 < 1.$$

Proof. Let (9) be a linear mapping from the unit disc $\{|t| < 1\}$ into D . Then the composite function (10) satisfies the condition

$$\left(\frac{\partial \bar{f}}{\partial \bar{t}} \right) = \frac{\partial \bar{f}}{\partial t} = \sum_{j=1}^n a_j \frac{\partial \bar{f}}{\partial z_j} = \kappa \sum_{j=1}^n a_j \frac{\partial f}{\partial z_j} = \kappa \frac{\partial f}{\partial t},$$

$$|\kappa| \leq k_0 = \frac{K - 1}{K + 1} < 1,$$

which implies that $\bar{f}(t)$ is K -Q.C. in $\{|t| < 1\}$ (see (2c)). This proves our theorem.

Definition 3. The function κ given in (17) is called the *characteristic function* of the K -D.Q.C. function f .

Remark 5. As already observed in Remark 3, the sum $f_1 + f_2$ of two K -D.Q.C. functions is not always K -D.Q.C. If f_1 and f_2 have the *same* characteristic function κ , Theorem 5 asserts that the sum $f_1 + f_2$ is again K -D.Q.C. However, the collection of all K -D.Q.C. functions with a fixed characteristic function κ is not a linear space over the *complex* field, though it is a linear space over the *real* field.

5. Mapping by a pair of K -D.Q.C. functions.

5.1. In this last section, I restrict myself, for simplicity, to the case $n = 2$. The object of this section is to discuss one-to-one mappings f given by a pair of K -D.Q.C. functions f_1, f_2 , namely,

$$(25) \quad f : (z_1, z_2) \rightarrow (w_1, w_2), \quad w_1 = f_1(z_1, z_2), \quad w_2 = f_2(z_1, z_2).$$

I call this, for brevity, a D.Q.C. mapping.

By direct computation, one can easily obtain

Theorem 6. Let κ_1 and κ_2 be the characteristic functions of f_1 and f_2 , respectively. Then the Jacobian

$$(26) \quad J = \frac{\partial(\operatorname{Re} f_1, \operatorname{Im} f_1, \operatorname{Re} f_2, \operatorname{Im} f_2)}{\partial(x_1, y_1, x_2, y_2)} = \frac{\partial(f_1, \bar{f}_1, f_2, \bar{f}_2)}{\partial(z_1, \bar{z}_1, z_2, \bar{z}_2)}$$

is given by

$$(27) \quad J = (1 - |\kappa_1|^2)(1 - |\kappa_2|^2)|A|^2$$

where

$$(28) \quad A = \frac{\partial(f_1, f_2)}{\partial(z_1, z_2)}.$$

Also, the derivatives of the inverse mapping

$$(29) \quad z_1 = g_1(w_1, w_2), \quad z_2 = g_2(w_1, w_2)$$

are given by the following expressions:

$$(30) \quad \begin{aligned} \frac{\partial g_1}{\partial w_1} &= \frac{1}{(1 - |\kappa_1|^2)A} \frac{\partial f_2}{\partial z_2}, & \frac{\partial g_1}{\partial w_2} &= \frac{-\bar{\kappa}_1}{(1 - |\kappa_1|^2)A} \frac{\partial f_2}{\partial z_2}, \\ \frac{\partial g_1}{\partial w_2} &= \frac{-1}{(1 - |\kappa_2|^2)A} \frac{\partial f_1}{\partial z_2}, & \frac{\partial g_1}{\partial w_1} &= \frac{\bar{\kappa}_2}{(1 - |\kappa_2|^2)A} \frac{\partial f_1}{\partial z_2}, \\ \frac{\partial g_2}{\partial w_1} &= \frac{-1}{(1 - |\kappa_1|^2)A} \frac{\partial f_2}{\partial z_1}, & \frac{\partial g_2}{\partial w_2} &= \frac{\bar{\kappa}_1}{(1 - |\kappa_1|^2)A} \frac{\partial f_2}{\partial z_1}, \\ \frac{\partial g_2}{\partial w_2} &= \frac{1}{(1 - |\kappa_2|^2)A} \frac{\partial f_1}{\partial z_1}, & \frac{\partial g_2}{\partial w_1} &= \frac{-\bar{\kappa}_2}{(1 - |\kappa_2|^2)A} \frac{\partial f_1}{\partial z_1}. \end{aligned}$$

Corollary 1. *The D.Q.C. mapping (25) satisfies the inequality*

$$|A|^2 = \left| \frac{\partial(f_1, f_2)}{\partial(z_1, z_2)} \right|^2 \leq \frac{(K+1)^4}{16K^2} J,$$

where J is the Jacobian (26).

Corollary 2. *If the mapping (25) is one-to-one, we have $J \neq 0$, hence every point is ordinary in the sense of Definition 2 with respect to an arbitrary linear combination of f_1 and f_2 . Similarly, every point in the range of (25) is ordinary with respect to an arbitrary linear combination of g_1 and g_2 , where g_1 and g_2 are the functions determined by the inverse mapping (29).*

Corollary 3. *The inverse mapping (29) of a schlicht D.Q.C. mapping satisfies the relations*

$$(31) \quad \frac{\partial g_j}{\partial \bar{w}_1} = -\bar{\kappa}_1 \frac{\partial g_j}{\partial w_1}, \quad \frac{\partial g_j}{\partial \bar{w}_2} = -\bar{\kappa}_2 \frac{\partial g_j}{\partial w_2} \quad (j = 1, 2).$$

Hence the inverse mapping (29) also satisfies the relations

$$(32) \quad \left| \frac{\partial(g_1, g_2)}{\partial(w_1, w_2)} \right|^2 = \frac{\tilde{J}}{(1 - |\kappa_1|^2)(1 - |\kappa_2|^2)}$$

and

$$(33) \quad \frac{\partial(g_1, g_2)}{\partial(u_1, u_2)} = (1 - \bar{\kappa}_1)(1 - \bar{\kappa}_2) \frac{\partial(g_1, g_2)}{\partial(w_1, w_2)},$$

where $w_j = u_j + iv_j$ ($j = 1, 2$) and

$$\tilde{J} = \frac{\partial(\operatorname{Re} g_1, \operatorname{Im} g_1, \operatorname{Re} g_2, \operatorname{Im} g_2)}{\partial(u_1, v_1, u_2, v_2)} = \frac{1}{J}.$$

Remark 6. The relations (32) and (33) lead to the inequality

$$(34) \quad \left| \frac{\partial(g_1, g_2)}{\partial(u_1, u_2)} \right|^2 \leq K \tilde{J}$$

for the mapping (29). The quantity $\partial(g_1, g_2)/\partial(u_1, u_2)$ appearing on the left hand side of (34) is the so-called “B-area element”¹² of the real 2-dimensional surface given by the image of the plane $v_1 = \text{const.}$, $v_2 = \text{const.}$, under the mapping (29). A mapping satisfying an inequality similar to (34) is discussed in a forthcoming paper of BERGMAN [4].

5.2. Theorem 7. *Let (25) be a one-to-one D.Q.C. mapping and let f_1 and f_2 have the same characteristic function κ . If the inverse mapping (29) is again D.Q.C., then f_1 and f_2 satisfy the conditions*

$$(35) \quad \frac{\partial \bar{f}_1}{\partial z_j} \frac{\partial f_2}{\partial \bar{z}_j} = \frac{\partial \bar{f}_2}{\partial z_j} \frac{\partial f_1}{\partial \bar{z}_j} = \text{real} \quad (j = 1, 2).$$

¹² The notion of “B-area” is discussed, for example, in BERGMAN [2], p. 9 and [3], p. 51. As far as I know, this notion was first introduced by S. BERGMAN in 1934.

Proof. If the inverse mapping (29) is again D.Q.C., we have by Theorem 3

$$(36) \quad \frac{\partial \bar{g}_1}{\partial w_j} = \mu_1 \frac{\partial g_1}{\partial w_j}, \quad \frac{\partial \bar{g}_2}{\partial w_j} = \mu_2 \frac{\partial g_2}{\partial w_j}, \quad |\mu_j| \leq k_0 \quad (j = 1, 2).$$

For simplicity, let us put

$$(37) \quad \frac{\partial g_j}{\partial w_k} = g_{jk} \quad (j, k = 1, 2).$$

The relations (30) and (36) easily yield

$$(38) \quad \begin{aligned} \mu_1 g_{11} &= -\kappa \bar{g}_{11}, & \mu_1 g_{12} &= -\kappa \bar{g}_{12}, \\ \mu_2 g_{21} &= -\kappa \bar{g}_{21}, & \mu_2 g_{22} &= -\kappa \bar{g}_{22}. \end{aligned}$$

Now, if $\kappa = 0$, the relation (35) holds trivially, since both sides in (35) are equal to 0. Hence we may assume $\kappa \neq 0$. Under this assumption, we have $\mu_1 \neq 0$ and $\mu_2 \neq 0$, for otherwise (38) leads to a relation contradicting the properties in Corollary 2 of Theorem 6. Hence from (38) we have

$$g_{11} \bar{g}_{12} = \bar{g}_{11} g_{12}, \quad g_{21} \bar{g}_{22} = \bar{g}_{21} g_{22}.$$

Multiplying by $|\kappa|^2$ and using the relations (17), (30), and (37), we have our assertion.

5.3. Remark 7. These results indicate the highly restricted character of D.Q.C. mappings. In fact, D.Q.C. mappings have a number of unfavorable properties, so that they do not constitute a suitable generalization of holomorphic transformations.

First, D.Q.C. mappings have even no intrinsic content, since a linear combination of two K -D.Q.C. functions is not necessarily such a function (see Remark 3). Hence the D.Q.C. property of such a mapping is not, in general, preserved by a change of coordinates in the (w_1, w_2) -space. Secondly, the set of all one-to-one D.Q.C. mappings does *not* form a group. Neither the composition of two D.Q.C. mappings nor the inverse is necessarily such a mapping (see Remark 3 and Theorem 7). This is again in strong contrast to the case $n = 1$.

5.4. Finally, I shall give a remark showing that the class of D.Q.C. mappings is very narrow. The Bergman mapping (6) in §2 is *not* a D.Q.C. mapping unless the domain $\mathfrak{R}$ is a hypersphere and the mapping degenerates into a trivial transformation. Here I state the following result which is slightly wider than the above assertion.

Theorem 8. Let $\mathfrak{R}$ be a Reinhardt circular domain of type $(\pm)$ (see §2) and consider a mapping of type (6) from the hypersphere $\{|z_1|^2 + |z_2|^2 < 1\}$ onto $\mathfrak{R}$. Here we do not assume that the function $g(x + iy) = u(x, y) + iv(x, y)$ which maps the unit disc $\{|x + iy| < 1\}$ onto the domain (5) in a one-to-one manner is holomorphic, but we assume merely the following properties of g :

- (i) g is of class C^1 .
- (ii) $g(0) = 0$ and g maps the x -axis and y -axis into the u -axis and v -axis, respectively.
- (iii) There exists a non-trivial relation

$$(39) \quad F\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) = 0.$$

Precisely speaking, there exists a function (not necessarily continuous) $F(\xi, \eta)$ of two complex variables ξ and η satisfying the condition that, for every fixed value ξ_0 (or η_0), the set $\{\eta; F(\xi_0, \eta) = 0\}$ (or the set $\{\xi; F(\xi, \eta_0) = 0\}$) is discrete, and that the relation (39) holds identically.¹³ Under these assumptions, the mapping (6) is a D.Q.C. mapping if and only if the domain $\Re$ is an ellipsoid

$$\frac{|Z_1|^2}{a^2} + \frac{|Z_2|^2}{b^2} < 1 \quad (a, b : \text{real constants})$$

and the mapping (6) degenerates into the simple linear mapping

$$(40) \quad Z_1 = az_1, Z_2 = bz_2.$$

Proof. It is trivial that (40) is a D.Q.C. mapping. Conversely, suppose that the mapping (6) is D.Q.C. in the hypersphere. According to the definition, the function $Z_2(z_1, z_2)$ must be a K -D.Q.C. function. Hence, by Definition 1, it gives rise to a quasi-conformal function of z_1 on each holomorphic plane given by $z_2 = c$ (c : constant; $0 < |c| < 1$). On this plane, the function Z_2 is

$$Z_2 = e^{i \arg c} v(|z_1| + i |c|), \quad v = \text{Im } g,$$

so that the argument of Z_2 is constant. As was remarked before, a non-constant quasi-conformal function gives rise to an inner transformation. Hence Z_2 must be a constant on the plane $z_2 = c$. This is also true when $c = 0$ by the assumption (ii). This implies

$$(41a) \quad \frac{\partial v}{\partial x} = 0.$$

Similarly, we have the relation

$$(41b) \quad \frac{\partial u}{\partial y} = 0,$$

where $u = \text{Re } g$. The hypothesis (39) yields a non-trivial relation

$$F\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}\right) = 0.$$

¹³ This condition is certainly fulfilled if $g(x + iy)$ is holomorphic in $z = x + iy$ as in §2, namely, by the Cauchy-Riemann equation $\partial g/\partial x + i\partial g/\partial y = 0$. More generally, (39) is satisfied if $g(x + iy)$ is a quasi-conformal function such that the function μ in (2c) is a constant.

Also (41a) and (41b) imply that $\partial u/\partial x$ and $\partial v/\partial y$ are functions only of x and y , respectively. Therefore we have

$$\frac{\partial u}{\partial x} = a = \text{const.}, \quad \frac{\partial v}{\partial y} = b = \text{const.}$$

Since $g(0) = 0$, we have $u = ax, v = by$, which proves that the mapping is equal to (40).

REFERENCES

- [1] L. V. AHLFORS, On quasi-conformal mappings, *Journal d'Analyse Mathématique*, **3** (1954), pp. 1–58, 207–208.
- [2] S. BERGMAN, *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques*, Mémorial des Sci. Math., 106 (1947), Paris.
- [3] S. BERGMAN, *Sur la fonction-noyau d'un domaine et ses applications dans la théorie des transformations pseudo-conformes*, Mémorial des Sci. Math., 108 (1948), Paris.
- [4] S. BERGMAN, A class of pseudo-conformal and quasi-pseudo-conformal mappings, to appear in *Math. Annalen*.
- [5] S. BERGMAN, A class of quasi-pseudo-conformal transformations in the theory of functions of two complex variables, *Journal of Mathematics and Mechanics* **7** (1958), pp. 937–956.
- [6] L. BERS, On a theorem of Mori and on the definition of quasi-conformality, *Trans. Amer. Math. Soc.*, **84** (1957), pp. 78–84.
- [7] E. D. CALLENDER, N -dimensional quasi-conformal mappings, *Stanford University Thesis*, to appear.
- [8] S. HITOTUMATU, Note on the holomorphy on an analytic subset, *Journal of Fac. of Sci., University of Tokyo, Sec. I*, **7** (1958), pp. 605–613.
- [9] S. KAKUTANI, Applications of the theory of pseudo-regular functions to the type-problem of Riemann surfaces, *Jap. Journal of Math.*, **13** (1937), pp. 375–392.
- [10] C. LOEWNER, On conformal capacity in space, To appear in *Proc. Amer. Math. Soc.*
- [11] A. MORI, On quasi-conformality and pseudo-analyticity, *Trans. Amer. Math. Soc.*, **84** (1957), pp. 56–77.
- [12] H. POINCARÉ, Les fonctions analytiques de deux variables et la représentation conforme, *Rend. di Circolo Mat. di Palermo*, **23** (1907), pp. 185–220.
- [13] F. SEVERI, *Lectures on the theory of analytic functions of several variables*, University of Tokyo (in Japanese), translated into Japanese by S. IYANAGA (1936), Tokyo.
- [14] G. E. ŠILOV, Extensions of maximal ideals, Appendix IV of *Commutative Normed Ring* by GEL'FAND-RAĬKOV-ŠILOV, *Uspekhi Mat. Nauk, N. S.*, **2** (12) (1946), pp. 48–146; *Amer. Math. Soc. Translations, Ser. 2*, **5** (1957), pp. 115–220.
- [15] O. TEICHMUELLER, Eine Anwendung quasikonformer Abbildungen auf das Typenproblem, *Deutsche Mathematik*, **2** (1937), pp. 321–327.
- [16] O. TEICHMUELLER, Extremale quasi-konforme Abbildungen und quadratische Differentiale, *Berichte d. Preuss. Akad. d. Wiss., Mathemat.-Naturwiss. Kl.*, **22** (1940).

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