Convergence of Solutions of Parabolic Equations to a Steady State

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Introduction. In this paper we discuss the behavior of solutions of second order parabolic equations when the time-variable t goes to infinity. Let L be a linear parabolic operator of the second order whose coefficients are functions of (x, t) (x is n -dimensional). In Part I we consider solutions of the equation $Lu = f(x, t)$ and in Part II we consider solutions of the nonlinear equation $Lu = f(x, t) + g(x, t, u)$. The domain where the behavior of the solutions is studied is a cylinder with its base on $t = 0$ and which is contained in $t > 0$. In Part III we extend some results of Parts I and II to domains whose cross section at $t = \tau$ becomes unbounded as $\tau \rightarrow \infty$.

In §1 we state the following theorems: (a) If $Lu = f(x, t)$, if $f \rightarrow 0$ as $t \rightarrow \infty$ and if $u \rightarrow 0$ on the lateral boundary of the cylinder, then $u \rightarrow 0$ as $t \rightarrow \infty$ inside the cylinder (Theorem 1). (b) If $Lu = f(x, t)$, if f and the coefficients of L tend to limits as $t \rightarrow \infty$ and if u tends to a limit as $t \rightarrow \infty$ on the lateral boundary of the cylinder, then u tends to a limit v inside the cylinder as $t \rightarrow \infty$ and v is a solution of the elliptic equation $(L' + \partial/\partial t)v = f'$ where f' and L' are the limits of f and L (Theorem 2). The proofs of these theorems are given in §2 and §3. In the proofs we make use of the maximum principle [8], [10], of existence theorems for the Dirichlet problem [4], [11], [13] and of the Schauder type interior estimates for parabolic equations [3].

Theorem 2, in the special case where L is the heat operator and the boundary values on the lateral boundary of the cylinder satisfy certain additional conditions, was proved by TYCHONOFF [14] and by W. FULKS [2]. Both methods, however, are of special nature.

In §4 we extend Theorem 1 to equations of the form $Lu = f(x, t) + g(x, t, u)$ where $|g| \leq \eta |u|$ and η is sufficiently small (Theorem 3). In §5 we generalize Theorem 2 to equations of the form $Lu = f(x, t) + g(x, t, u)$ where $\lim g(x, t, u) = g(x, u)$ satisfies the Lipschitz condition with a small coefficient (Theorem 4). The proofs make use of an existence theorem proved in [5].

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Theorem 3, under more restricted conditions on L than in this paper, was already proved by the author in [6]. The method used in that paper is entirely different from the method used here; it makes extensive use of the Green function for L (constructed and studied in [6]) and it does not make use of the *a priori* estimates and of the existence theorems used in this paper. We remark that Theorem 1 can also be proved by using the method of [6], if we put more restrictions on L than those made here.

In Part III we extend Theorems 1 and 3 to non-cylindrical domains whose cross section at $t = \tau$ becomes unbounded as $\tau \rightarrow \infty$.

PART I. LINEAR EQUATIONS

1. Statement of Theorems 1, 2. Let B be a bounded domain in the n -dimensional Euclidean space of the variables $x = (x_1, \dots, x_n)$ and denote by D_τ ($D_\infty = D$) the set of points (x, t) where $x \in B$ and $0 < t < \tau$. Denote by B_τ ($B_0 = B$) the intersection $D \cap \{t = \tau\}$ and by $\dot{D}_\tau$ ($\dot{D}_\infty = \dot{D}$) the topological product of $\dot{B}$ (the boundary of B) with the interval $\{0 \leq t \leq \tau\}$. The complement in a set S_2 of a set S_1 is denoted by $S_2 - S_1$.

Consider the equation

$$(1.1) \quad Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = f(x, t)$$

with coefficients defined in $\bar{D}$ (the closure of D). We shall need the following assumptions:

(A) The coefficients of L are bounded in $\bar{D}$, say by a constant M , and L is uniformly parabolic in $\bar{D}$, that is, there exists a positive constant M' such that, for all (x, t) in $\bar{D}$ and for any real vector $\xi = (\xi_1, \dots, \xi_n)$,

$$(1.2) \quad \sum a_{ij}(x, t) \xi_i \xi_j \geq M' \sum \xi_i^2.$$

(B) The functions

$$(1.3) \quad a_{ij}, \quad b_i, \quad c$$

are Hölder continuous in compact subsets of $\bar{D}$.

(C) The following limits exist (uniformly for $x \in \bar{B}$):

$$\lim_{t \rightarrow \infty} a_{ij}(x, t) = a_{ij}(x), \quad \lim_{t \rightarrow \infty} b_i(x, t) = b_i(x), \quad \lim_{t \rightarrow \infty} c(x, t) = c(x);$$

the functions $a_{ij}(x)$, $b_i(x)$ and $c(x)$ are Hölder continuous (say, exponent λ) in $\bar{B}$.

We now have to make some assumptions on $\dot{B}$, the boundary of B , so that solutions of the Dirichlet problem exist. We shall make use of the following assumption:

(D) To each point x^0 of $\dot{B}$ there exists a sphere S (in the x -space) having x^0 as its only common point with $\bar{B}$.

We shall also make use of the following stronger assumption:

(D') To each point x^0 of $\dot{B}$ there exists a sphere V (in the x -space) with center x^0 such that $V \cap \dot{B}$ can be represented, for some i , in the form

$$(1.4) \quad x_i = \psi(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

and ψ possesses Hölder continuous (exponent λ) x -derivatives of the second order.

We can now state our first result.

Theorem 1. Assume that L satisfies (A), (B) and that $c(x, t) \leq 0$. Assume further that $\dot{B}$ satisfies (D). Let $H(x, t)$ be a continuous function on $\dot{D}$ and let $F(x, t)$ be Hölder continuous in compact subsets of $\bar{D}$. Finally, assume that

$$(1.5) \quad \lim_{t \rightarrow \infty} H(x, t) = 0 \quad \text{uniformly for } x \in \dot{B},$$

$$(1.6) \quad \lim_{t \rightarrow \infty} F(x, t) = 0 \quad \text{uniformly for } x \in \bar{B}.$$

If $w(x, t)$ is a solution of $Lw = F$ in D which is equal to H on $\dot{D}$, then $\lim_{t \rightarrow \infty} w(x, t) = 0$ uniformly for $x \in \bar{B}$.

With the aid of Theorem 1, whose proof is given in §2, we shall proceed in §3 to prove the following theorem.

Theorem 2. Assume that L satisfies (A), (B), (C) and that $c(x, t) \leq 0$. Assume further that $\dot{B}$ satisfies (D'). Let $h(x, t)$ be a continuous function on $\dot{D}$ and let $f(x, t)$ be Hölder continuous in compact subsets of $\bar{D}$. Assume, finally, that

$$(1.7) \quad \lim_{t \rightarrow \infty} h(x, t) = h(x) \quad \text{uniformly for } x \in \dot{B}$$

$$(1.8) \quad \lim_{t \rightarrow \infty} f(x, t) = f(x) \quad \text{uniformly for } x \in \bar{B},$$

and that $f(x)$ is Hölder continuous (exponent λ) in $\bar{B}$. If $u(x, t)$ is a solution of $Lu = f$ in D which is equal to h on $\dot{D}$, then

$$(1.9) \quad \lim_{t \rightarrow \infty} u(x, t) = v(x) \quad \text{uniformly for } x \in \bar{B},$$

and $v(x)$ is the solution of

$$(1.10) \quad L_0 v \equiv \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial v}{\partial x_i} + c(x)v = f(x)$$

with the boundary values $h(x)$ on $\dot{B}$.

Remark. The existence of v , under the assumptions of Theorem 2 (note that $h(x)$ is continuous), follows from the Schauder theory [13].

2. Proof of Theorem 1. Let σ be a fixed positive number to be determined later. In $D - D_\sigma$ we write $w = w_0 + w_1$ where w_0 and w_1 are the solutions in $D - D_\sigma$ of the following systems:

$$(2.1) \quad Lw_0 = 0, \quad w_0 = w \text{ on } B_\sigma \text{ and on } \dot{D} - \dot{D}_\sigma,$$

$$(2.2) \quad Lw_1 = F, \quad w_1 = 0 \text{ on } B_\sigma \text{ and on } \dot{D} - \dot{D}_\sigma.$$

(Existence of solutions of the systems (2.1) and (2.2) follows from [4].) We shall estimate w_1 and w_0 with the aid of the maximum principle [10]. We may assume that B contains the origin. Define

$$(2.3) \quad \varphi(x) = 1 - \exp \{ -\alpha |x + A|^2 \}$$

where $x + A = (x_1 + A, \dots, x_n + A)$ and $A = 3R$, R being the diameter of B . If α is sufficiently large then, as one can easily show,

$$(2.4) \quad L\varphi < -\frac{1}{\gamma}$$

where γ is a positive constant depending on R, M, M' . From the definition of φ we obviously have

$$(2.5) \quad 0 < \varphi(x) < 1 \quad \text{for } x \in \bar{B}.$$

We can now apply the maximum principle to the function

$$\tilde{w}_1 = \left(\text{l.u.b.}_{x \in B, t \geq \sigma} |F(x, t)| \right) \gamma \varphi(x) \pm w_1(x, t)$$

which takes non-negative values on B_σ and on $\dot{D} - \dot{D}_\sigma$, and which satisfies $L\tilde{w}_1 \leq 0$. We conclude (on using (2.5)) that

$$(2.6) \quad |w_1(x, t)| \leq \gamma \text{ l.u.b.}_{\xi \in B, \tau \geq \sigma} |F(\xi, \tau)| \quad \text{for } (x, t) \in D - D_\sigma.$$

To estimate w_0 let us first continue the function $H(x, \sigma)$ defined on $\dot{B}$ to a continuous function $H(x)$ defined in $\bar{B}$ and having the same maximum and minimum as $H(x, \sigma)$. We now write $w_0 = w' + w''$ where w' and w'' are solutions of the systems

$$(2.7) \quad Lw' = 0, \quad w' = w(x, \sigma) - H(x) \text{ on } B_\sigma, \quad w' = 0 \text{ on } \dot{D} - \dot{D}_\sigma,$$

$$(2.8) \quad Lw'' = 0, \quad w'' = H(x) \text{ on } B_\sigma, \quad w'' = H(x, t) \text{ on } \dot{D} - \dot{D}_\sigma.$$

(Existence follows from [4]; note that the boundary values are continuous functions.) We proceed to estimate w'' and w' .

In the first place, w'' must attain its positive maximum and its negative minimum on the boundary of $D - D_\sigma$. Hence

$$(2.9) \quad |w''(x, t)| \leq \text{l.u.b.}_{\xi \in \dot{B}, \tau \geq \sigma} |H(\xi, \tau)| \quad \text{for } (x, t) \in D - D_\sigma.$$

To estimate w' , we introduce the function (used already by NARASIMHAN [9; p. 308])

$$(2.10) \quad z(x, t) = (N - e^{\beta x_1})e^{-\delta(t-\sigma)}.$$

It is easy to show that one can choose positive constants N, β, δ such that $z(x, t) \geq \text{const.} > 0$ in $\bar{B}$ (and hence also on $\dot{D} - \dot{D}_\sigma$) and $Lz < 0$ in $D - D_\sigma$. We choose C to be sufficiently large so that

$$Cz(x, \sigma) \geq \text{l.u.b.}_{\xi \in B} |w(\xi, \sigma) - H(\xi)|.$$

Then we can apply the maximum principle to the function $Cz(x, t) \pm w'(x, t)$ in $D - D_\sigma$. We get

$$(2.11) \quad |w'(x, t)| \leq C(N - e^{\beta x_1})e^{-\delta(t-\sigma)} \quad \text{for } (x, t) \in D - D_\sigma.$$

Combining (2.11) with (2.9) and (2.6) and using the definitions $w_0 = w' + w''$, $w = w_0 + w_1$, we get

$$(2.12) \quad |w(x, t)| \leq \gamma \text{l.u.b.}_{\xi \in B, \tau \geq \sigma} |F(\xi, \tau)| + \text{l.u.b.}_{\xi \in \dot{B}, \tau \geq \sigma} |H(\xi, \tau)| + C(N - e^{\beta x_1})e^{-\delta(t-\sigma)}$$

for $(x, t) \in D - D_\sigma$. Now let ϵ be an arbitrary positive number. Using (1.5), (1.6) it follows that we can choose $\sigma = \sigma(\epsilon)$ such that each of the first two terms on the right side of (2.12) is smaller than $\frac{1}{3}\epsilon$. Next we choose T ($T > \sigma$) sufficiently large so that the last term becomes smaller than $\frac{1}{3}\epsilon$ for all $t \geq T$. We conclude that

$$|w(x, t)| < \epsilon \quad \text{for all } x \in \bar{B}, \quad t \geq T$$

and the proof of Theorem 1 is thereby completed.

Added in proof. The assumption $c(x, t) \leq 0$ made in Theorem 1 may sometimes be relaxed. We mention one example. Assume that the elliptic part of L is self-adjoint, that L satisfies (A), (B), (C) and that the first eigenvalue λ_1 of L_0 (in (1.10)) in B is positive. Then the assumption $c(x, t) \leq 0$ is superfluous for the validity of the assertion of Theorem 1.

Indeed, extend the coefficients of L_0 outside of $\bar{B}$ in such a way that they are Hölder continuous and uniformly bounded and let B^* be a domain containing $\bar{B}$ and having the first eigenvalue λ_1^* (of the extension of L_0 in B^*) positive. Let $\varphi^*(x)$ be the corresponding eigenfunction. Then $\varphi^*(x)$ is positive in $\bar{B}$. We now use the comparison function $\varphi^*(x)$ and the lemma of PRODI-WESTPHAL [12], [15] to estimate w_1 in (2.2) and w'' in (2.8). We finally use the function $\varphi^*(x)e^{-\mu t}$ ($0 < \mu < \lambda_1^*$) and the lemma of PRODI-WESTPHAL to estimate w' in (2.7).

We remark that if $\lambda_1 \leq 0$ the above result need not be true. Evidently it is false in the case of coefficients independent of t , as the example of $\varphi(x)e^{-\lambda_1 t}$ shows (φ being the eigenfunction of L_0 in B corresponding to λ_1).

3. Proof of Theorem 2. We first prove Theorem 2 under the following restrictions:

(E) There exists a polynomial $\tilde{h}(x)$ which is equal to $h(x)$ on $\dot{B}$.

Later on we shall show how one can remove this restriction.

Under the assumptions (A), (C), (D'), (E), the solution $v(x)$ of the system

$$(3.1) \quad L_0 v = f(x) \quad \text{in } B, \quad v = h(x) \quad \text{on } \bar{B}$$

satisfies the following inequality (see SCHAUDER [13] and also DOUGLIS & NIRENBERG [1]):

$$(3.2) \quad |v|_{2+\lambda} \leq K(|f|_\lambda + |\tilde{h}|_{2+\lambda})$$

where K depends only on M, M' and on the domain B , and where

$$(3.3) \quad |z|_\lambda = \text{l.u.b. } |z| + \text{l.u.b. } \frac{|z(x) - z(x')|}{|x - x'|^\lambda},$$

$$(3.4) \quad |z|_{m+\lambda} = \sum_{i=0}^m \left| \frac{\partial^i}{\partial x^i} z \right|_\lambda,$$

the summation being taken also on all the i^{th} partial x -derivatives.

Consider now the function $w(x, t) = u(x, t) - v(x)$.

$$(3.5) \quad \begin{aligned} Lw &= L(u - v) = Lu - (L - L_0)v - L_0v \\ &= [f(x, t) - f(x)] - (L - L_0)v \equiv F(x, t). \end{aligned}$$

Using (3.2) and the assumption (C) we conclude that the function $F(x, t)$ (defined by the last identity of (3.5)) satisfies (1.6). On the lateral boundary $\dot{D}$ of D

$$w(x, t) = h(x, t) - h(x) \equiv H(x, t)$$

and $H(x, t)$ satisfies (1.5). Hence we can apply Theorem 1 to the function w . We thus obtain that

$$\lim_{t \rightarrow \infty} u(x, t) = v(x) \quad \text{uniformly for } x \in \bar{B}.$$

It remains to show that the assumption (E) can be removed. Since $h(x)$ is continuous on $\bar{B}$, we can extend it to a continuous function $h'(x)$ in $\bar{B}$. Given $\epsilon > 0$, let $h_\epsilon(x)$ be a polynomial which satisfies

$$(3.6) \quad |h'(x) - h_\epsilon(x)| < \epsilon \quad \text{for } x \in \bar{B}.$$

Denote by $v_\epsilon(x)$ the solution in B of the equation $Lv_\epsilon(x) = f(x)$ which takes the boundary values of $h_\epsilon(x)$ on $\bar{B}$. By the maximum principle for elliptic equations [8] we have

$$(3.7) \quad |v(x) - v_\epsilon(x)| < \epsilon \quad \text{for } x \in \bar{B}.$$

We now observe that if the assumption $\lim_{t \rightarrow \infty} H(x, t) = 0$ (as $t \rightarrow \infty$) made in Theorem 1 is replaced by the assumption $\limsup_{t \rightarrow \infty} |H(x, t)| \leq \eta$ (η a constant), then for any given $\epsilon > 0$ there exists a number $T = T(\epsilon)$ such that

$$|w(x, t)| \leq \eta + \epsilon \quad \text{for } t \geq T \quad \text{and } x \in \bar{B}.$$

(The proof of this statement follows immediately from the proof of Theorem 1.) Applying this remark to the function $w(x, t) = u(x, t) - v_\epsilon(x)$ and noting

by (3.6) that along the lateral boundary $\dot{D}$ of D $\limsup_{t \rightarrow \infty} |w(x, t)| \leq \epsilon$, we conclude that there exists a number $T = T(\epsilon)$ such that

$$(3.8) \quad |u(x, t) - v_\epsilon(x)| \leq \epsilon + \epsilon$$

holds for all $x \in \bar{B}$, $t \geq T$. Combining (3.8) with (3.7) the proof of Theorem 2 is thereby completed.

Remark 1. The assumption (D') made in Theorem 2 was used only in order to assume the validity of (3.2). The inequality (3.2) was used in order to prove that $F(x, t)$ defined in (3.5) tends to zero as $t \rightarrow \infty$ uniformly for $x \in \bar{B}$. However, if the coefficients a_{ij} and b_i of L do not depend on t then

$$(L - L_0)v(x) = (c(x, t) - c(x))v(x) \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly for $x \in \bar{B}$. Hence, in that case, it is enough to replace the assumption (D') by the assumption (D). (Existence of solutions for L_0 and L , under the assumption (D), was proved by NIRENBERG [11] and the author [4] respectively.)

Remark 2. Theorems 1, 2 remain valid for non-cylindrical domains, provided the projections of B_r on the hyperplane $t = 0$ converge to a domain B , and provided the boundaries of D and B are such that the existence theorems and (3.2) are valid. The proofs are very similar to the proofs of Theorems 1, 2.

PART II. NONLINEAR EQUATIONS

4. Theorem 3. In this part we keep the notation of Part I.

Let $f(x, t, u)$ be a function defined in the cylinder

$$(4.1) \quad (x, t) \in \bar{D}, \quad -\infty < u < \infty.$$

We shall make use of the condition

$$(4.2) \quad |f(x, t, u)| \leq \eta |u| \quad \text{for } (x, t) \in \bar{D}, \quad |u| < A.$$

We shall now prove a stability theorem for the solutions of $Lu = f(x, t, u)$.

Theorem 3. Let L satisfy (A), (B) and let $c(x, t) \leq 0$. Let $\dot{B}$ satisfy (D) and let $f(x, t, u)$ be Hölder continuous in compact subsets of the domain (4.1) and of exponent 1 with respect to u . Let $g(x)$, $h(x, t)$ be continuous functions defined on $\bar{B}$ and $\dot{D}$ respectively and assume that $g(x) = h(x, 0)$ on $\dot{B}$. Assume further that

$$(4.3) \quad |h(x, t)| \leq A', \quad |g(x)| \leq A' \quad \text{and} \quad A' < A$$

and that (4.2) holds with η satisfying the inequality $\eta\gamma < \theta/(1 + \theta)$, where γ is the constant of (2.4) and θ is any positive number for which $(1 + \theta)A' < A$. If $u(x, t)$ is a solution of the system

$$(4.4) \quad Lu = f(x, t, u) \quad \text{in } D, \quad u = g \quad \text{on } \bar{B}, \quad u = h \quad \text{on } \dot{D}$$

then

$$(4.5) \quad |u(x, t)| \leq (1 + \theta)A' \quad \text{for all } (x, t) \in D.$$

If, further, η satisfies $2\eta\gamma < \theta/(1 + \theta)$ and if

$$(4.6) \quad \lim_{t \rightarrow \infty} h(x, t) = 0 \quad \text{uniformly for } x \in \bar{B}$$

then $\lim_{t \rightarrow \infty} u(x, t) = 0$ uniformly for $x \in \bar{B}$.

Remark. The question of the existence of solutions of the system (4.4) was discussed by the author in [5].

Proof of Theorem 3. We shall use the notation introduced in [3; p. 395]. We shall need the following fact which we state as a lemma.

Lemma. For any compact subset $E = \bar{D}_T$, $|u|_{0,\mu}^E$ exists, where μ is any positive number (smaller than 1).

The proof of the lemma will be given later.

Write $u = w_0 + w_1$ where w_0 and w_1 are the solutions of the systems

$$(4.7) \quad Lw_0 = 0 \quad \text{in } D, \quad w_0 = u \quad \text{on } \bar{B} \quad \text{and on } \bar{D},$$

$$(4.8) \quad Lw_1 = f(x, t, u(x, t)), \quad w_1 = 0 \quad \text{on } \bar{B} \quad \text{and on } \bar{D}.$$

The existence of w_0 follows from [4]. The existence of w_1 also follows from [4] if we note, on using the lemma and the Hölder continuity of $f(x, t, u)$, that the function $\tilde{f}(x, t) = f(x, t, u(x, t))$ satisfies $|\tilde{f}|_{0,\nu}^E < \infty$ for any compact set $E = \bar{D}_T$, ν depending on E .

We proceed to prove (4.5). From the maximum principle we have

$$(4.9) \quad |w_0(x, t)| \leq A' \quad \text{for } (x, t) \in \bar{D}.$$

Comparing $w_1(x, t)$ with $\gamma\varphi(x)$ (see (2.3), (2.4)) and using the maximum principle, we get

$$(4.10) \quad |w_1(x, t)| \leq \gamma \text{ l.u.b.}_{D_T} |f(\xi, \tau, u(\xi, \tau))| \quad \text{in } D_T$$

and T is any positive number (γ is independent of T). Using (4.2) we get

$$(4.11) \quad |w_1(x, t)| \leq \eta\gamma \text{ l.u.b.}_{D_T} |u(\xi, \tau)| \quad \text{for } (x, t) \in D_T$$

provided $|u(\xi, \tau)| \leq A$ in D_T .

Taking the l.u.b. of the left side of (4.11), combining the inequality thus obtained with (4.9) and using the definition $u = w_0 + w_1$, we get

$$(4.12) \quad \text{l.u.b.}_{D_T} |u(x, t)| \leq A' + \eta\gamma \text{ l.u.b.}_{D_T} |u(x, t)|.$$

Since, by assumption, η satisfies

$$(4.13) \quad \eta\gamma < \frac{\theta}{1 + \theta},$$

we conclude from (4.12) that

$$(4.14) \quad \text{l.u.b.}_{D_T} |u(x, t)| \leq (1 + \theta)A'$$

provided $|u(x, t)| \leq A$ in D_T .

The proof of (4.5) can now be completed in the following way: Denote by W the set of numbers T for which $|u(x, t)| \leq A$ in D_T . Since $|u(x, 0)| \leq A'$, the set W is non-empty. It is clearly a closed set. If we show that it is an open set then (4.14) holds for all T and the proof of (4.5) is thus completed.

Let T be such that $|u(x, t)| \leq A$ in D_T . Then (4.11) holds, and the argument given above shows that (4.14) holds in D_T . Since u is continuous, it follows that $|u(x, t)| \leq A$ holds also in $D_{T+\epsilon}$ for $\epsilon > 0$ sufficiently small. We have thus proved that W is an open set.

To prove the last assertion of Theorem 3, we again write $u = w_0 + w_1$, with w_0, w_1 defined as above. If we construct a sequence T_m ($T_m \nearrow \infty$ as $m \nearrow \infty$) such that

$$(4.15) \quad |u(x, t)| \leq \frac{(1 + \theta)A'}{2^m} \quad \text{for } (x, t) \in D - D_{T_m}$$

then the proof is completed.

By Theorem 1 we can choose T_1 such that

$$(4.16) \quad |w_0(x, t)| \leq \frac{A'}{2} \quad \text{for } (x, t) \in D - D_{T_1}$$

Using (4.10), (4.5) and (4.2), we get

$$(4.17) \quad |w_1(x, t)| \leq \eta\gamma(1 + \theta)A' < \frac{\theta A'}{2}$$

where we made use of the assumption

$$(4.18) \quad \eta\gamma < \frac{\theta}{2(1 + \theta)}.$$

(4.17) holds in D_T for all T , and hence it holds in D . Combining (4.17) with (4.16) we obtain

$$(4.19) \quad |u(x, t)| \leq \frac{(1 + \theta)A'}{2} \quad \text{for } (x, t) \in D - D_{T_1}.$$

We can now proceed as before, replacing D by $D - \bar{D}_{T_1}$ and replacing (4.5) by (4.19). We then find T_2 such that

$$|u(x, t)| \leq \frac{(1 + \theta)A'}{2^2} \quad \text{for } (x, t) \in D_{T_2}.$$

Proceeding step by step, the proof of (4.15) follows.

To complete the proof of Theorem 3, it remains to prove the lemma.

Proof of the lemma. Let B^ϵ be a sequence of subdomains converging to B and

let D' be the corresponding cylinders. If we prove, for any D_T^ϵ , that

$$(4.20) \quad |u|_{0,\mu}^{D_T^\epsilon} \leq K$$

for any $\mu < 1$, where K depends on T but not on ϵ , then our proof is completed on letting $\epsilon \rightarrow 0$. Denote, for brevity, $G = D_T^\epsilon$. From [3; Theorem 3] we have

$$(4.21) \quad |u|_{0,2+\nu}^G \leq K' (\text{l.u.b.}_G |u| + |\tilde{f}|_{2,\nu}^G)$$

where ν is sufficiently small (depending on T but not on ϵ) and K' depends only on M, M' (in the assumption (A)) and on the Hölder coefficients of the coefficients of L in G . Here $\tilde{f}(x, t) = f(x, t, u(x, t))$. We have

$$(4.22) \quad |\tilde{f}|_{2,\nu}^G \leq c_1 |\tilde{f}|_{0,\nu}^G \leq c_2 + c_3 |u|_{0,\nu}^G$$

where c_i are constants independent of ϵ . An argument similar to the one used in proving [1; Lemma 1] can be used to show that

$$(4.23) \quad |u|_{0,\nu}^G \leq \frac{1}{2K'c_3} |u|_{0,2+\nu}^G + c_4 \text{l.u.b.}_G |u|$$

where c_4 is an appropriate constant independent of ϵ . Combining (4.23) with (4.22) and substituting in (4.21), we obtain

$$|u|_{0,2+\nu}^G \leq c_5, \quad c_5 \text{ independent of } \epsilon,$$

and our assertion follows.

Added in proof. The use of the above lemma is superfluous since instead of constructing w_1 (as we did above) we can simply define $w_1 = u - w_0$. Thus it is enough to assume that $f(x, t, u)$ is continuous in the cylinder (4.1) (instead of Hölder continuous). Also in Theorems 1 and 2 it is enough to assume that $F(x, t), f(x, t)$ are only continuous functions. A similar remark applies to Theorems 4–6. Since the method of proof of the lemma is used in what follows, we included here both the lemma and its proof.

Remark 1. Theorem 3 remains true also for functions $f = f(x, t, u, \text{grad } u)$ satisfying (4.2) and being Hölder continuous in compact subsets of the cylinder

$$(x, t) \in \bar{D}, \quad -\infty < u < \infty, \quad -\infty < \frac{\partial u}{\partial x_i} < \infty \quad (i = 1, \dots, n).$$

The proof is similar to that of Theorem 1.

A similar remark applies to Theorem 4.

Remark 2. Theorem 3 can easily be extended to equations of the form

$$(4.24) \quad Lu = F(x, t) + f(x, t, u)$$

where f satisfies the condition (4.2) and $F(x, t) \rightarrow 0$ as $t \rightarrow 0$ uniformly with respect to $x, x \in \bar{B}$.

5. Theorem 4. Theorem 3, extended as mentioned in the last remark of the previous section, generalizes Theorem 1, which deals with linear equations, to equations with small nonlinear terms. We shall now prove an analogous generalization of Theorem 2.

Theorem 4. Assume that L satisfies (A), (B), (C) and let $c(x, t) \leq 0$. Assume further that $\dot{B}$ satisfies (D'). Let $h(x, t)$ be continuous on $\dot{D}$, let $f(x, t)$ be Hölder continuous in compact subsets of $\bar{D}$ and let $k(x, t, u)$ be Hölder continuous in compact subsets of the cylinder (4.1) and of exponent 1 with respect to u . Assume that (1.7), (1.8) hold with $f(x)$ Hölder continuous in $\bar{B}$, that

$$(5.1) \quad \lim_{t \rightarrow \infty} k(x, t, u) = k(x, u)$$

uniformly for $x \in \bar{B}$, and $|u| < A$, for any A , and that $k(x, u)$ is Hölder continuous in compact subsets of $x \in \bar{B}$, $|u| < \infty$, with exponents 1 with respect to u , and satisfies

$$(5.2) \quad k(x, 0) = 0, \quad |k(x, u') - k(x, u'')| \leq \eta |u' - u''| \quad \text{for all } u', u''.$$

Let $u(x, t)$ be a solution of the system

$$(5.3) \quad \begin{aligned} Lu &= f(x, t) + k(x, t, u) \quad \text{in } D, \\ u &= h(x, t) \quad \text{on } \dot{D}. \end{aligned}$$

If η is sufficiently small (depending on M, M', B and the moduli of continuity of the coefficients $a_{ii}(x)$ of L_0), then

$$(5.4) \quad \lim_{t \rightarrow \infty} u(x, t) = v(x)$$

where $v(x)$ is the solution of the system

$$(5.5) \quad L_0 v = f(x) + k(x, v) \quad \text{in } B, \quad v = h(x) \quad \text{on } \dot{B}.$$

Remark. The existence problem for solutions of (5.3) was discussed in [5].

Proof. We shall first prove that v exists and is uniquely determined. We start with uniqueness. If v_1, v_2 are both solutions of (5.5), then their difference $v = v_1 - v_2$ satisfies

$$(5.6) \quad L_0 v = \tilde{k}, \quad |\tilde{k}| \leq \eta |v|, \quad v = 0 \quad \text{on } \dot{B}.$$

We can consider $v(x)$ as a solution in D of the parabolic equation

$$\left(L_0 - \frac{\partial}{\partial t} \right) v(x) = \tilde{k}.$$

Since v vanishes on the lateral boundary $\dot{D}$ of the cylinder D , we can apply the uniqueness theorem of NARASIMHAN [9] and thus conclude that $\lim_{t \rightarrow \infty} v(x) = 0$ as $t \rightarrow \infty$, for any $x \in \bar{B}$. Hence $v(x) \equiv 0$ and the uniqueness is proved. That $v \equiv 0$ can also be proved directly, by using the maximum principle for elliptic equations [8], [11].

To prove the existence of v , we first remark that if $h(x)$ is a polynomial, then the existence of v was already established in [5] (provided η is small enough (as in the assumptions of the theorem)). Now let $h(x)$ be any continuous function on $\bar{B}$ and extend it to a continuous function $h'(x)$ in $\bar{B}$. Approximate $h'(x)$ uniformly in $\bar{B}$ by polynomials $h_m(x)$. Let $v_m(x)$ be the corresponding solutions. Since

$$(5.7) \quad L_0(v_m - v_p) = k_{mp}, \quad |k_{mp}| \leq \eta |v_m - v_p|, \\ v_m - v_p = h_m - h_p \quad \text{on } \bar{B},$$

we can apply the maximum principle for elliptic equations ([8] or [11]) and get

$$(5.8) \quad \text{l.u.b. } |v_m - v_p| \leq C\eta \text{ l.u.b. } |v_m - v_p| + \text{l.u.b. } |h_m - h_p|$$

where C is a constant depending on M, M' and B . Taking $\eta \leq 1/2C$ we conclude that the sequence $\{v_m\}$ converges uniformly in $\bar{B}$, say, to a function $v(x)$.

It remains to show that $v(x)$ is a solution of $L_0v = f(x) + k(x, v)$. To do that, it is enough to show that

$$(5.9) \quad |v_m|_{0,2+\nu} \leq \tilde{C} \quad (\tilde{C} \text{ independent of } m)$$

for some positive ν ; here we used the notation of [1; p. 508]. For then we can abstract a subsequence of $\{v_m\}$ converging uniformly in closed subsets of B to v , together with the first and second derivatives. The proof of (5.9) can be given similarly to the proof of the lemma in §4, via the Schauder estimates (see [1]); it will therefore be omitted.

Having proved the existence of $v(x)$, we now proceed to prove the desired convergence. We first prove it in case $h(x)$ is a polynomial. In that case we have

$$(5.10) \quad |v|_{2+\mu} \leq K_1(|f|_\mu + |k|_\mu + |h|_{2+\mu}) \leq K_2 + K_3\eta |v|_\mu$$

where μ ($0 < \mu < 1$) and K_i are constants. Either by using the method of proof of [1; Lemma 3] or by taking η sufficiently small, we obtain

$$|v|_\mu \leq \frac{1}{2K_3\eta} |v|_{2+\mu} + C' \text{ l.u.b. } |v| \quad (C' \text{ constant}).$$

Hence

$$|v|_{2+\mu} \leq K_4.$$

Denoting $w(x, t) = u(x, t) - v(x)$ we have

$$(5.11) \quad Lw = Lu - (L - L_0)v - L_0v \\ = [f(x, t) - f(x)] + [k(x, t, u) - k(x, v)] - (L - L_0)v.$$

We write

$$(5.12) \quad k(x, t, u) - k(x, v) = [k(x, t, u) - k(x, u)] + [k(x, u) - k(x, v)]$$

and note that

$$(5.13) \quad |k(x, u) - k(x, v)| \leq \eta |w|.$$

Hence the right side of (5.11) can be written as $F(x, t) + \tilde{f}$, where $\tilde{f} = k(x, u) - k(x, v)$ satisfies (5.13) and $\lim_{t \rightarrow \infty} F(x, t) = 0$ uniformly with respect to $x \in \bar{B}$.

As in the proof of the lemma we can show that $|u|_{0, \mu}^B < \infty$ for any compact subset $E = \bar{D}_\tau$. Hence the same is true for the $\tilde{f}$ defined previously. We can now use the proof of Theorem 3 (see also the last remark of §4) and thus conclude that

$$\lim_{t \rightarrow \infty} w(x, t) = \lim_{t \rightarrow \infty} [u(x, t) - v(x)] = 0$$

uniformly for $x \in \bar{B}$.

It remains to consider the case where $h(x)$ is merely a continuous function. We can approximate $h(x)$ by polynomials $h_m(x)$. Denoting

$$\epsilon_m = \text{l.u.b. } |h - h_m|$$

and following the proof of Theorem 3, we can show that

$$(5.14) \quad \limsup_{t \rightarrow \infty} |u(x, t) - v_m(x)| \leq C_0 \epsilon_m \quad (C_0 \text{ constant})$$

uniformly for $x \in \bar{B}$, where v_m is the solution of (5.5) when we replace h by h_m . On the other hand, if η is small, we obtain, on using the maximum principle (see (5.8)), that

$$(5.15) \quad \text{l.u.b.}_B |v_m(x) - v(x)| \leq C' \epsilon_m \quad (C' \text{ constant}).$$

Combining (5.15) with (5.14), our theorem is thereby proved for any continuous function $h(x)$.

Remark. Both theorems 3 and 4 can be extended to non-cylindrical domains provided $B_\tau = D \cap \{t = T\}$ tends to a domain B and provided the boundaries of D and B are sufficiently smooth (to secure the existence of solutions).

PART III. ARBITRARY DOMAINS

6. Generalization of Theorem 1. As was remarked previously, Theorems 1–4 hold also for non-cylindrical domains D , provided the domains $B_\tau = D \cap \{t = T\}$ tend to a bounded domain. In this part, we shall not make even that assumption and generalize Theorems 1 and 3. We denote by C (and not by $\bar{D}$) the lateral boundary of D and otherwise keep the notations of Parts I, II.

We shall make use of the following assumptions:

(D₁) D lies in a paraboloid P_β defined by

$$(6.1) \quad P_\beta : |x|^2 \leq \beta t \quad (\beta > 0).$$

(D₂) To each point (x^0, t^0) on C there exists a sphere S with center (x', t') such that $x^0 \neq x'$ and such that S and $\bar{D}$ have (x^0, t^0) as their only common point.

(B') $\liminf \sum x_i b_i(x, t) \geq 0$ as $|x| + t \rightarrow \infty$, uniformly in D .

Theorem 5. Assume that L satisfies (A), (B), (B') and that $c(x, t) \leq 0$. Assume further that (D₁) and (D₂) hold. Let $H(x, t)$ be a continuous function on C and let $F(x, t)$ be Hölder continuous in compact subsets of $\bar{D}$. Finally assume that

$$(6.2) \quad \lim_{t \rightarrow \infty} H(x, t) = 0 \quad \text{uniformly for } (x, t) \in C,$$

$$(6.3) \quad |F(x, t)| \leq \frac{K}{t^{1+\epsilon}} \quad \text{in } \bar{D}, \quad \text{where } \epsilon > 0, \quad K > 0.$$

If $w(x, t)$ is a solution of $Lw = F$ in D which is equal to $H(x, t)$ on C , then $\lim_{t \rightarrow \infty} w(x, t) = 0$ uniformly for $(x, t) \in D$.

Proof. We can decompose w in D_σ in the form $w = \bar{w} + w''$, where w'' is defined by (2.8) and $\bar{w} = w' + w_1$, where w' and w_1 are defined by (2.7) and (2.2) respectively. Here we make use of an existence theorem proved in [4].

The estimation of w'' remains the same as in the proof of Theorem 1. It remains to estimate $\bar{w}$. Clearly, the functions φ and z used in the proof of Theorem 1 cannot be used here. We shall use the following function

$$(6.4) \quad \psi(x, t) = \frac{1}{t^\epsilon} \exp \left\{ -\frac{\bar{H}|x|^2}{t} \right\}$$

introduced by the author in [7]. We have

$$(6.5) \quad L\psi \leq \psi \left\{ \sum a_{ii} \frac{4\bar{H}^2 x_i x_i}{t^2} - \sum a_{ii} \frac{2\bar{H}}{t} - \sum x_i b_i \frac{2\bar{H}}{t} + \frac{\epsilon}{t} - \frac{\bar{H}|x|^2}{t^2} \right\}.$$

Taking $\bar{H}$ to be sufficiently small, then taking (as we may) ϵ to be smaller than $\bar{H} \inf \sum a_{ii}$ and noting (by (B')) that $-\sum x_i b_i \leq \eta$ where $\eta > 0$ can be taken arbitrarily small, we conclude from (6.5) that

$$(6.6) \quad L\psi + \frac{K'}{t} \psi < 0 \quad \text{in } D_\sigma$$

where K' is an appropriate positive constant and σ is supposed to be sufficiently large.

We now observe that in P_β

$$(6.7) \quad \psi(x, t) \geq \frac{1}{t^\epsilon} \exp \{ -\bar{H}\beta \}.$$

Hence we can take A to be a positive constant such that

$$(6.8) \quad A\psi \geq |\bar{w}| \quad \text{on } D \cap \{t = \sigma\}$$

and such that

$$(6.9) \quad L(A\psi) + \frac{K}{t^{1+\epsilon}} < 0 \quad \text{in } D \cap \{\sigma < t < \infty\}.$$

Using (6.3) we have

$$L\bar{w} + \frac{K}{t^{1+\epsilon}} \geq 0;$$

hence $L(A\psi - \bar{w}) < 0$. Using the maximum principle we get

$$(6.10) \quad \bar{w}(x, t) \leq A\psi(x, t) \quad \text{for } t > \sigma.$$

Since $-\bar{w}$ satisfies $L(-\bar{w}) = -F(x, t)$ we also have

$$L(-\bar{w}) + \frac{K}{t^{1+\epsilon}} \geq 0;$$

hence by the maximum principle $-\bar{w} \leq A\psi$. Combining this result with (6.10) we get

$$(6.11) \quad |\bar{w}(x, t)| \leq \frac{A}{t^\epsilon} \exp \left\{ -\frac{H|x|^2}{t} \right\} \quad \text{for } t > \sigma.$$

Since the right side of (6.11) tends to zero as $t \rightarrow \infty$, we can easily complete the proof of Theorem 5.

Remark. Consider the following example: $n = 1$, D is a rectangle, with base $-l \leq x \leq l$, $t = \tau$, contained in P_β , such that l is of order of magnitude $\tau^{\frac{1}{2}}$. The function $u(x, t) = \sin \alpha(x + l)$ satisfies

$$u_{xx} - u_t = -\alpha^2 \sin \alpha(x + l) \equiv F(x, t)$$

and it vanishes on $x = \pm l$, $t > \tau$, provided $\alpha = \frac{1}{2}\pi/l$. Since $u(x, t)$ does not tend to zero inside D when $t \rightarrow \infty$, the assertion of Theorem 5 does not hold. In this example $F(x, t)$ is of order of magnitude $1/\tau$. If we assume that $F(x, t)$ is decreasing in t , and if we want to obtain a stability theorem with F independent of the specific domain D and of the specific P_β , then we conclude that it is not enough to assume that $F(x, t)$ is of order of magnitude $1/t$; F should be of smaller order of magnitude. In that sense, Theorem 5 is sharp.

7. Generalization of Theorem 3. We shall now generalize Theorem 3.

Theorem 6. Assume that L satisfies (A), (B), (B') and that $c(x, t) \leq 0$. Assume further that (D₁), (D₂) hold. Let $H(x, t)$ be a continuous function on C and let $f(x, t, u)$ be Hölder continuous in compact subsets of the cylinder (4.1). Finally assume that (6.2) holds and that

$$(7.1) \quad |f(x, t, u)| \leq \frac{K|u|}{t^{1+\epsilon}} \quad \text{in the cylinder (4.1), for some } \epsilon > 0, \quad K > 0.$$

If $u(x, t)$ is a solution of $Lu = f$ in D which is equal to $H(x, t)$ on C , then $\lim_{t \rightarrow \infty} u(x, t) = 0$ uniformly for $(x, t) \in D$.

Proof. The proof will be based on the proof of Theorem 5 rather than on that of Theorem 3. Using an existence theorem of [4] and the lemma proved

in §4, we can write $u = \bar{u} + w''$ where w'' is a solution of (2.8). Thus, if σ becomes large then w'' becomes small. Now $\bar{u}$ satisfies

$$(7.2) \quad L\bar{u} = f(x, t, u)$$

and

$$(7.3) \quad |f(x, t, u)| \leq \frac{K|u|}{t^{1+\epsilon}} \leq \frac{K|\bar{u}|}{t^{1+\epsilon}} + \frac{K|w''|}{t^{1+\epsilon}} \leq \frac{K|\bar{u}|}{t^{1+\epsilon}} + \frac{K_0}{t^{1+\epsilon}}$$

where K_0 is a constant.

Again we use the function $\psi(x, t)$. It satisfies (see (6.6))

$$L\psi + \frac{K'}{2t}\psi + \frac{K''}{t^{1+\epsilon}} < 0 \quad \text{for some } K'' > 0.$$

If we take A sufficiently large and note that if σ is large then $K'/2t > K/t^{1+\epsilon}$, we obtain

$$(7.4) \quad L(A\psi) + \frac{K(A\psi)}{t^{1+\epsilon}} + \frac{K_0}{t^{1+\epsilon}} < 0.$$

We can take A such that also

$$A\psi > |\bar{u}| \quad \text{on } D \cap \{t = \sigma\}.$$

We now apply the lemma of PRODI-WESTPHAL [12], [15] (see also [9]) and thus conclude that $\bar{u} < A\psi$ in $D \cap \{t > \sigma\}$. Similarly, $-\bar{u} < A\psi$. We thus have

$$(7.5) \quad |\bar{u}(x, t)| \leq \frac{A}{t^\epsilon} \exp \left\{ -\frac{\bar{H}|x|^2}{t} \right\}.$$

If we now take σ sufficiently large so that $|w''(x, t)| < \frac{1}{2}\delta$ (for any given $\delta > 0$) for $t > \sigma$, and then take $T > \sigma$ such that $|\bar{u}(x, t)| < \frac{1}{2}\delta$ for $t > T$, then we obtain

$$|u(x, t)| < \delta \quad \text{for } t > T$$

and the proof is thereby completed.

Remark 1. The following modification of Theorem 6 holds:

Theorem 6'. Assume that L satisfies (A), (B') and that $c(x, t) \leq 0$. Assume further that (D₁) holds. Let $H(x, t)$ be a continuous function on C and let $f(x, t, u)$ be continuous in compact subsets of the cylinder (4.1). Finally, assume that for some $\epsilon > 0$

$$(7.6) \quad \lim_{t \rightarrow \infty} \frac{H(x, t)}{t^\epsilon} = 0 \quad \text{uniformly for } (x, t) \in C$$

and that

$$(7.7) \quad |f(x, t, u)| \leq \frac{\eta|u|}{t} \quad \text{in the cylinder} \quad (4.1)$$

where η is sufficiently small (depending on M, M'). If $u(x, t)$ is a solution of $Lu = f(x, t, u)$ in D and if $u = H$ on C , then $\lim_{t \rightarrow \infty} u(x, t) = 0$ uniformly for $(x, t) \in D$.

The proof of Theorem 6' does not make use of any existence theorems (note that (B) is not assumed). First we may assume that ϵ is sufficiently small. We next take α sufficiently large, and then prove that

$$L\psi < -\frac{\eta\psi}{t} \quad \text{in } D \cap \{t > \sigma\}$$

and that $A\psi > |u|$ on the boundary of $D \cap \{t > \sigma\}$ if A is sufficiently large (here we make use of (7.6)). We then use the lemma of PRODI-WESTPHAL and thus conclude that

$$|u(x, t)| \leq \frac{A}{t^\epsilon} \exp \left\{ -\frac{H|x|^2}{t} \right\} \quad \text{for } t > \sigma,$$

and the proof is completed.

One can easily find that it is enough to assume

$$(7.8) \quad \eta < \frac{nM'}{2A'} \quad \text{where } A' = \sup \sum a_{ii}(x, t)\xi_i\xi_i \quad ((x, t) \in D, |\xi| = 1).$$

Remark 2. The remark that follows Theorem 5 can also be applied to Theorems 6, 6' to show that they are sharp theorems in a certain sense. We also remark that if D is the half space $t > 0$ then Theorem 6' is false as the following counter-example shows:

$$u(x, t) = t^\alpha, \quad Lu = -\frac{\alpha u}{t} \equiv f, \quad |f| = \frac{\alpha|u|}{t}.$$

α can be taken arbitrarily small, $u = 0$ on the boundary of D , but $u \rightarrow \infty$ along any interior direction of D issuing from the origin.

Remark 3. Theorem 6' remains true if $|f| \leq \eta|u|/t$ is replaced by $|f| \leq \eta|u|^\nu/t$ and $\nu > 1$. One can also extend Theorems 6, 6' to functions f of the form $f = f(x, t, u, \text{grad } u)$.

8. Another stability theorem. In this section we shall prove a stability theorem for more general domains than those in §§6 and 7 which roughly states the following: If the boundary values are uniformly small, then the solution is uniformly small; the equation being $Lu = f(x, t, u)$.

We shall make use of the following assumption:

(D') D lies in the paraboloid

$$(8.1) \quad |x|^2 \leq \beta(t+1)^\mu \quad \text{for some } \mu \geq 1.$$

For simplicity we shall assume also the following:

(B'') $b_i(x, t) \equiv 0$ in D .

Theorem 7. Assume that L satisfies (A), (B'') and that $c(x, t) \leq 0$. Assume further that (D₁') holds. Let $f(x, t, u)$ be a continuous function in compact subsets of the cylinder (4.1) and suppose that

$$(8.2) \quad |f(x, t, u)| \leq \frac{\eta|u|}{(t+1)^\mu}$$

where

$$(8.3) \quad \eta < \frac{nM'}{2A'} \mu \quad (\text{see (7.8)}).$$

If $u(x, t)$ satisfies the equation $Lu = f(x, t, u)$ and if $|u| \leq \delta$ ($\delta \geq 0$) on the boundary $B + C$ of D , then $|u(x, t)| \leq K\delta$ in D , where

$$(8.4) \quad K = \exp \{ \beta\mu/4A' \}.$$

Proof. Consider the function

$$(8.5) \quad \theta(x, t) = \exp \left\{ -\frac{\bar{H}|x|^2}{(t+1)^\mu} \right\}.$$

It satisfies the inequality

$$L\theta \leq \theta \left\{ \sum a_{ii} \frac{4\bar{H}^2 x_i x_i}{(t+1)^{2\mu}} - \sum a_{ii} \frac{2\bar{H}}{(t+1)^\mu} - \frac{\mu\bar{H}|x|^2}{(t+1)^{\mu+1}} \right\}.$$

The sum of the first sum and the last term in the braces is non-positive if

$$4\bar{H}A' \leq \mu(t+1)^{\mu-1} \quad (t > 0),$$

which is clearly satisfied on taking

$$(8.6) \quad \bar{H} = \frac{\mu}{4A'}.$$

We conclude on using (8.3) and the positivity of $\theta(x, t)$ that

$$(8.7) \quad L\theta + \eta \frac{\theta}{(t+1)^\mu} < 0.$$

Using (8.1) and (8.5) we get

$$(8.8) \quad A_1 = \exp \{ -\bar{H}\beta \} \leq \theta(x, t) < 1.$$

We now compare the function $(\delta + \epsilon)\theta(x, t)/A_1$ ($\epsilon > 0$) with the function $\pm u(x, t)$ which satisfies

$$L(\pm u) + \eta \frac{|u|}{(t+1)^\mu} \geq 0.$$

Applying in D the lemma of PRODI-WESTPHAL we conclude that

$$|u(x, t)| \leq (\delta + \epsilon)\theta(x, t)/A_1.$$

Taking $\epsilon \rightarrow 0$ and using the definition of A_1 in (8.8) and the definition of $\bar{H}$ in (8.6), we get $|u(x, t)| \leq \delta K$, where K is given by (8.4).

Remark 1. Theorem 7 can be extended to the case where

$$-\sum x_i b_i \leq \rho \sum a_{ii} \quad \text{in } D, \quad \text{and} \quad \rho < 1.$$

Remark 2. If Theorem 7 holds with $K = 1$, then we have obtained a maximum principle. In this respect Theorem 7 may be considered as an extension of the maximum principle. It would be interesting to see to what extent one can generalize theorems which are known for linear equations to nonlinear equations $Lu = f(x, t, u)$ where f is "small" in some sense. We shall give here one example. Denote

$$m(R) = \max_{|X|=R} u(x, t) \quad \text{where} \quad |X| = (|x|^2 + |t|)^{\frac{1}{2}}.$$

Theorem 8. *Assumptions: D lies in the half space $t \geq t_0 > 0$ and there exists T , $0 < T \leq \infty$, such that $C \cap \{t = \tau\}$ is a non-empty compact set for all $\tau < T$ and D lies in $0 < t < T$; the coefficients of L are bounded in compact subsets of $\bar{D}$ and*

$$(8.9) \quad \text{l.u.b.}_D \sum a_{ii}(x, t) \leq A_0 < \infty,$$

$$(8.10) \quad \text{l.u.b.}_D \sum x_i b_i(x, t) \leq B_0 < \infty;$$

L is of parabolic type and uniformly so in compact subsets of $\bar{D}$. Finally, $f(x, t, u)$ is defined in the cylinder (4.1) and for some $\delta < 1$

$$(8.11) \quad |f(x, t, u)| \leq \frac{\delta |u|}{|X|^2}.$$

Assertion: If $Lu \geq f(x, t, u)$ in D and $u \leq 0$ on the boundary of D , and if

$$(8.12) \quad \liminf_{R \rightarrow \infty} \frac{m(R)}{R^2} = 0$$

then $u(x, t) \leq 0$ in D .

Remark. Theorem 8 is a Phragmén-Lindelöf type theorem; the linear case ($f \equiv 0$) was discussed in [7].

Proof. The function

$$(8.13) \quad v_R = \frac{|x|^2 + Kt}{R^2} \quad (K > 1)$$

satisfies the differential inequalities

$$Lv_R \leq \frac{2 \sum a_{ii} + 2 \sum x_i b_i - K}{R^2} < \frac{2A_0 + 2B_0 - K}{|x|^2 + Kt} v_R;$$

here we used (8.9), (8.10). Since $|x|^2 + Kt \leq K(|x|^2 + t)$, we obtain, on taking K to be sufficiently large,

$$(8.14) \quad Lv_R + \frac{\delta}{|X|^2} v_R < 0.$$

By (8.11), u satisfies

$$(8.15) \quad Lu + \frac{\delta}{|X|^2} |u| \geq 0.$$

Introducing the function

$$\mu(R) = \max \{1, 1 + m(R)\},$$

we observe that $\mu(R)v_R > u$ on the boundary of $D \cap \{|x|^2 + t < R\}$. Because of our assumption on C , we can apply the lemma of PRODI-WESTPHAL. We thus get

$$u(x, t) \leq \frac{|x|^2 + Kt}{R^2} \max \{1, 1 + m(R)\} \quad \text{if} \quad |x|^2 + t \leq R^2.$$

Fixing (x, t) , taking $R \rightarrow \infty$ and using (8.12), our proof is completed.

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