A Variational Problem Relating to Cooling Fins

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1. Introduction. The first part of this paper gives an exact solution of a definite mathematical problem concerning cooling fins. Cooling fins are used to conduct heat away from machines to the ambient medium. For example, circular fins are used on the cylinders of air cooled engines. It is not difficult to see that in order to economize on weight the fin should taper, narrowing in the direction of heat flow. The problem arising may be phrased in this way—find the taper which gives the maximum dissipation of heat for a given weight of fin.

The mathematical formulation of this problem involves a differential equation of Sturm-Liouville type, $Lu = 0$. Here u is the temperature, and the differential operator L depends on the taper. The dissipation is given by the integral of u over the surface of the fin.

A criterion for this optimum problem was proposed by E. SCHMIDT [4]. According to SCHMIDT *the temperature should be a linear function of the distance along the fin*. This relation leads to explicit expressions for the taper of a fin. The same criterion has also been applied to cooling spines [5]. A review of the work of SCHMIDT and others is given in the treatise of JAKOB [6]. It seems that the literature does not contain a proof of the Schmidt criterion. SCHMIDT did advance an intuitive argument, but it is not convincing.

In order to find a rigorous treatment, the problem is here recast in a form suitable for treatment by the calculus of variations. Euler equations are obtained by the formal variational methods. The solution of the Euler equations is extremely simple, namely, that the temperature is a linear function. It is then proved directly that this is a maximizing solution. The Schmidt criterion is thereby justified. Also, explicit expressions are given for the taper and for the maximum dissipation.

The second part of the paper concerns the general problem of finding upper and lower bounds for the solution of the Sturm-Liouville equation $Lu = 0$. Several feasible methods of obtaining bounds are indicated. In particular, these methods could be used to calculate the effect of departure from the optimum taper in cooling fins.

The writer is indebted to Dr. R. L. BRIGHT for calling attention to the optimum problem treated here.

2. The variational problem. We first give an abstract statement of the cooling fin problem. While this statement is somewhat involved, nevertheless the physical interpretation is straightforward. This is seen by turning to the following section, which treats the rectangular fin.

Problem I. Let $q(x)$ be a given positive continuous function defined on the interval $0 \leq x < \infty$. We seek two functions $u(x)$ and $p(x)$ and a positive number b which have the following properties: In the interval $0 \leq x \leq b$ the function u is differentiable and satisfies the Sturm-Liouville differential equation

$$(1) \quad \frac{d}{dx} \left[p(x) \frac{du}{dx} \right] = q(x)u, \quad 0 \leq x \leq b.$$

Also u is to satisfy the two boundary conditions

$$(2) \quad u = 1 \quad \text{at} \quad x = 0$$

and

$$(3) \quad p(x) \frac{du}{dx} = 0 \quad \text{at} \quad x = b.$$

The function $p(x)$ satisfies the conditions

$$(4) \quad p(x) \text{ is continuous}$$

$$(5) \quad p(x) > 0 \quad \text{for} \quad 0 \leq x < b,$$

and

$$(6) \quad \int_0^b p(x) dx = K, \text{ a given positive constant.}$$

Under these conditions we seek the maximum of H defined as

$$(7) \quad H = \int_0^b q(x)u(x) dx.$$

The continuity conditions assumed justify integration of equation (1) to give

$$(8) \quad p \frac{du}{dx} = - \int_x^b qu dx.$$

Here the boundary condition (3) has been employed. By hypothesis (5) equation (8) may be divided by p and integrated to obtain

$$(9) \quad u = - \int_0^x p^{-1} \left[\int_x^b qu dx \right] dx + 1.$$

Here boundary condition (2) has been employed. Clearly, equation (9) is an

integral equation which is equivalent to the differential equation (1) together with the boundary conditions (2) and (3). (It is not difficult to express equation (9) in the standard Fredholm form.)

The solution of the problem can be made to depend on the following integral:

$$(10) \quad E = \int_0^b \left[2qu - qu^2 - p^{-1} \left(\int_x^b qu \, dx \right)^2 \right] dx.$$

Integration by parts and use of (9) gives

$$\int_0^b p^{-1} \left(\int_x^b qu \, dx \right)^2 dx = \int_0^b \left[\int_0^x p^{-1} \left(\int_x^b qu \, dx \right) dx \right] qu \, dx = \int_0^b (u - u^2) q \, dx.$$

Substituting this in (10) shows that

$$(11) \quad E = \int_0^b qu \, dx.$$

In other words, if u satisfies the integral equation, then $E = H$.

Relation (11) suggests that we maximize the integral E , treating u and p as independent variables, not supposing that u and p necessarily satisfy the integral equation (9). Note that the first two terms of the integrand of E can be written as $q - q(1 - u)^2$ and that the third term is never positive. Thus $E \leq \int_0^b q \, dx$, and hence E has a finite supremum.

We first proceed formally, considering a variation of $E(u, p)$ due to a variation of u and p from assumed maximizing values. First only u is varied, so

$$\delta E = 2 \int_0^b \left[q \delta u - qu \delta u - p^{-1} \int_x^b qu \, dx \int_x^b q \delta u \, dx \right] dx.$$

Integration by parts gives

$$\delta E = 2 \int_0^b q \delta u \left[1 - u - \int_0^x p^{-1} \, dx \int_x^b qu \, dx \right] dx.$$

Of course $\delta E = 0$ for any admissible variation δu . It follows that u satisfies the integral equation (9). Now consider a variation of E resulting from a variation in p . Thus

$$\delta E = \int_0^b \delta p \left(p^{-1} \int_x^b qu \, dx \right)^2 dx = 0.$$

Because of relation (6), we also have

$$\delta K = \int_0^b \delta p \, dx = 0.$$

It follows that the integrands of δE and δK must be proportional. Thus

$$\int_x^b qu \, dx = \lambda p$$

where λ is a constant (Lagrange multiplier). Substituting this value of p in the integral equation (9) proves (formally)

$$u = 1 - \lambda x,$$

which is SCHMIDT's criterion. These formal considerations serve as a guide to the rigorous proof which will now be developed.

It follows from the thermodynamic interpretation of the problem that u is non-negative. This can be demonstrated mathematically on the basis of equation (8). Thus suppose $u < 0$ at $x = b$, so, from (8), $du/dx > 0$ in the left neighborhood of $x = b$. Hence u is an increasing function and therefore never positive. This is in contradiction with the boundary condition (2). A similar argument applies if u has a negative minimum at a point $x = b'$ inside the interval $(0, b)$. Then equation (8) holds with b' replacing b .

Presumably Problem I is equivalent to

Problem II. Let $q(x)$ be a given positive continuous function defined on the interval $0 \leq x < \infty$. We seek two functions $u(x)$ and $p(x)$ which have the following properties: The functions $u(x)$ and $p(x)$ are non-negative and piecewise continuous. Also

$$(12) \quad \int_0^\infty p(x) dx = K, \text{ a given positive constant.}$$

Under these conditions we seek the maximum of $E(u, p)$ defined as

$$(13) \quad E(u, p) = \int_0^\infty \left[2qu - qu^2 - p^{-1} \left(\int_x^\infty qu dx \right)^2 \right] dx.$$

A solution of Problem II will now be developed: Let u and p be any functions which satisfy the conditions and which render the integrals appearing in (13) convergent. Define $p_0(x)$ by

$$(14) \quad \lambda p_0(x) = \int_x^\infty qu dx.$$

Here λ is a constant chosen so that p_0 satisfies (12). Thus

$$(15) \quad \lambda K = \int_0^\infty \int_x^\infty qu dx = \int_0^\infty x qu dx.$$

To show that the integrals here are convergent, we apply the Schwarz inequality as follows

$$\left(\int_0^\infty \int_x^\infty qu dx \right)^2 \leq \int_0^\infty p dx \int_0^\infty p^{-1} \left(\int_x^\infty qu dx \right)^2 dx < \infty.$$

On the other hand, by (14)

$$\left(\int_0^\infty \int_x^\infty qu dx \right)^2 = \int_0^\infty p_0 dx \int_0^\infty p_0^{-1} \left(\int_x^\infty qu dx \right)^2 dx.$$

The last two relations and (12) give

$$\int_0^\infty p_0^{-1} \left(\int_x^\infty qu \, dx \right)^2 dx \leq \int_0^\infty p^{-1} \left(\int_x^\infty qu \, dx \right)^2 dx.$$

Hence it follows that E is not decreased by replacing p by p_0 .

Substituting p_0 in E gives $E(u, p_0) = F(u)$ where

$$(16) \quad F(u) = \int_0^\infty (2qu - qu^2) \, dx - K^{-1} \left[\int_0^\infty xqu \, dx \right]^2.$$

The problem is reduced to maximizing the expression (16). Let U be another function and let $\delta F = F(U) - F(u)$ and $v = U - u$. Then substitution in (16) gives

$$(17a) \quad \begin{aligned} \delta F = 2 \int_0^\infty q \left[1 - u - xK^{-1} \int_0^\infty xqu \, dx \right] v \, dx \\ - \int_0^\infty qv^2 \, dx - K^{-1} \left(\int_0^\infty xqv \, dx \right)^2. \end{aligned}$$

Deleting the last two terms, we have

$$(17) \quad \delta F \leq 2 \int_0^\infty q \left[1 - u - xK^{-1} \int_0^\infty xqu \, dx \right] v \, dx.$$

A solution is obtained by supposing the expression in the square brackets vanishes in an interval. This yields the following result.

Theorem 1. *A solution of Problem I is given by*

$$(18) \quad u(x) = 1 - x/b,$$

$$(19) \quad p(x) = \int_x^b (b - x)q(x) \, dx,$$

$$(20) \quad H = \int_0^b (1 - x/b)q(x) \, dx,$$

where b is the unique solution of the equation

$$(21) \quad K = \int_0^b \int_0^t xq(x) \, dx \, dt.$$

Proof. First it is clear that the expression on the right side of (21) is an unbounded monotone increasing function of b . Consequently equation (21) has a unique solution $b > 0$. The double integral on the right side of (21) can be reduced to a single integral. Thus

$$(22) \quad K = \int_0^b (b - x)xq(x) \, dx = b \int_0^b xqu \, dx.$$

Let $u(x)$ be defined by (18) for $0 \leq x \leq b$, and let $u(x) = 0$ for $x > b$. The

function u so defined is substituted in (17). By (22) it follows that the expression in square brackets vanishes for $0 \leq x \leq b$. Thus (17) becomes

$$(23) \quad \delta F \leq 2 \int_b^\infty q[1 - x/b]v \, dx.$$

For $x \geq b$ we have $v = U - u = U \geq 0$. Thus the integrand in (23) is never positive, so $\delta F \leq 0$. Thus u furnishes a maximum to (16).

Comparing (15) and (22), we see $\lambda = 1/b$. It then follows by comparison of p_0 defined by (14) and p defined by (19) that $p_0 = p$ (provided we define $p = 0$ for $x > b$). Thus u and p so defined solve Problem II.

It follows from relation (11) that the maximum for Problem I cannot exceed the maximum for Problem II. However, u and p defined by (18) and (19) clearly satisfy the differential equation (1) and the conditions (2), (3), (4), (5), and (6) of Problem I. By (11), $E = H$, so the two maxima are equal. This shows that u and p are a solution of Problem I, and the proof of Theorem 1 is complete.

Theorem 2. *Problem I has a unique solution.*

Proof. Now let u and p denote any solution of Problem II and let $E(u, p)$ be defined by (13). Since E is a maximum, the following proportionality holds:

$$\lambda p^{\frac{1}{2}} = p^{-\frac{1}{2}} \int_x^\infty q u \, dx$$

for some constant λ . Otherwise the Schwarz inequality employed above would be a strict inequality and E could actually be increased. This proportionality is the same as (14), so we are again led to (15), (16), and (17a). Because $\delta F \leq 0$, it is clear that the expression in square brackets must vanish in any interval where $u > 0$. Thus in such an interval $u = 1 - \lambda x$. Again, because $\delta F \leq 0$ it is seen that the expression in square brackets must be non-positive in any interval where $u = 0$. It follows from these two conditions that there is a constant b such that $u = 0$ for $x > b$ and $u = 1 - x/b$ for $x < b$. It is then seen that p is given by (19) and that b is given by (21). Thus the solution of Problem II is unique.

The maxima of Problem I and Problem II are equal, so a solution of I is also a solution of II. This shows that the solution of I is unique, and the proof is completed. A similar uniqueness proof can be given for the problem treated next.

Theorem 1a. *Suppose that in Problem I there is an additional restriction that for a given constant B*

$$(24) \quad b \leq B.$$

Then the statement of Theorem I is still valid provided b defined by (21) satisfies (24). Otherwise a solution of Problem I is given by $b = B$ and

$$(18a) \quad u(x) = 1 - x/b'' \quad x \leq B,$$

$$(19a) \quad p(x) = \int_x^B (b' - x)q(x) dx,$$

$$(20a) \quad H = \int_0^B (1 - x/b')q(x) dx$$

where b' is given by

$$(21a) \quad b' = \left(K + \int_0^B x^2 q(x) dx \right) / \int_0^B x q(x) dx.$$

Proof. If b defined by (21) satisfies the restriction $b \leq B$, then the proof of Theorem 1 holds in every detail. Otherwise define b' by (21a). This means that

$$K = \int_0^B x q(x)(b' - x) dx.$$

Hence if $u(x) = 1 - x/b'$ for $x \leq B$ and $u(x) = 0$ for $x \geq B$, we have

$$(25) \quad 1/b' = K^{-1} \int_0^\infty x q u dx.$$

Substituting u , so defined, in (17) shows that the bracketed expression vanishes in the range $0 \leq x \leq B$ and $v = 0$ for $x > B$. Thus $\delta F \leq 0$. It is then easily shown that (18a), (19a), and (20a) furnish a solution to Problem II and to Problem I.

(The maximization question treated here may be regarded as a "programming problem." In a previous paper [2] it was shown how to formulate a dual minimization problem for a certain class of such problems. It would be of interest to know if a dual problem can be set up in the present case.)

3. The rectangular fin. The variational problem just considered is now related to the question of optimum tapering of cooling fins. First we consider a fin in the shape of a rectangular plate whose sides are of lengths a and b . Let x denote the distance to one of the sides of length a . The thickness of the plate is taken to be given by a function $p(x)$. Heat is supplied at constant temperature to the side at $x = 0$. By a suitable choice of units, the temperature at this edge may be taken to be unity, and the ambient temperature may be taken to be zero. Let c_1 be the coefficient of thermal conductivity of the material of the plate, and let c_2 be the combined radiation and convection coefficient of the surface of the plate. It is then customary to assume that the temperature $u(x)$ satisfies

$$(26) \quad \frac{d}{dx} \left[c_1 p(x) \frac{du}{dx} \right] = 2c_2 u.$$

The expression on the right gives the heat dissipation for a unit area of the plate; the factor 2 takes into account both sides of the plate. Clearly, equation (26) can be written in the form (1) with $q = 2c_2/c_1 = c$, where c denotes another constant. Moreover, u satisfies both boundary conditions (2) and (3).

A question in the design of such a fin is the choice of $p(x)$ and b to maximize the dissipation for a given weight fin. The dissipation is equal to $ac_1 \int_0^b cu \, dx = ac_1 H$, and the volume of the fin is $a \int_0^b p \, dx = aK$. In other words, the problem is to maximize H for a given K .

Theorem 1 gives the following solution of the problem. Substituting $q = c$ in relation (21) gives

$$(27) \quad b = (6K/c)^{\frac{1}{3}}.$$

Then (19) and (20) give

$$(28) \quad p(x) = \frac{1}{2}c(b-x)^2$$

and

$$(29) \quad H = (\frac{3}{4}c^2K)^{\frac{1}{3}}.$$

As a practical consideration in the design, suppose that the maximum fin length permitted is B . If $B < (6K/c)^{\frac{1}{3}}$, then Theorem 1a gives the following solution:

$$(27a) \quad b' = \frac{6K + 2cB^3}{3cB^2},$$

$$(28a) \quad p(x) = (B-x) \left(\frac{2K}{B^3} + \frac{cB}{6} - \frac{cx}{2} \right),$$

$$(29a) \quad H = \frac{cB(12K + cB^3)}{12K + 4cB^3}.$$

4. The circular fin. We now consider a fin in the shape of a circular plate. Let $t(r)$ be the plate thickness where r denotes the radial distance. Then the basic differential equation is

$$(30) \quad \frac{d}{dr} \left[c_1 r t(r) \frac{du}{dr} \right] = 2c_2 ru.$$

Heat is supplied to the fin at an inner radius, say $r = a$. It is then convenient to introduce the variable $x = r - a$, which gives the distance to the inner radius. The differential equation becomes

$$(31) \quad \frac{d}{dx} \left[(x+a)t \frac{du}{dx} \right] = (x+a)cu.$$

This equation is in the form (1) with $p = (x+a)t$ and $q = (x+a)c$. The boundary conditions (1) and (2) apply as before if we take the outer radius to be $a+b$.

The dissipation of the fin is given by $2\pi c_1 \int_0^b qu \, dx = 2\pi c_1 H$. The volume of the fin is $2\pi \int_0^b p \, dx = 2\pi K$. Again the problem is to maximize H for a given K .

The solution of the problem is again given by Theorem 1. Equation (21) for

b becomes

$$(32) \quad b^4 + 2ab^3 = 12Kc^{-1}.$$

Since this equation is of the fourth degree, it is best solved numerically. Equations (19) and (20) become

$$(33) \quad p(x) = cba(b - x) + \frac{1}{2}c(b - a)(b^2 - x^2) - \frac{1}{3}c(b^3 - x^3),$$

$$(34) \quad H = \frac{1}{2}cba + \frac{1}{6}cb^2.$$

If there is a maximum permitted radius, say $a + B$, then Theorem 1a can be applied as before.

5. Approximation methods. It is desirable to have methods which yield approximations to the solution u of the differential equation (1) under the given boundary conditions. For example, while the optimal tapering of fins has been explicitly determined, nevertheless it is of value to know if a departure from the optimum taper would result in a significant loss of thermal output.

In some cases the exact solution is known. If in equation (30) the thickness t is constant, then the solution can be expressed in terms of Bessel functions. Such problems are treated in reference [3].

A lower bound to the dissipation can be obtained from the maximum principle (10). For a given choice of p the expression E is maximized relative to some finite set of trial functions u .

An upper bound to the dissipation can be obtained from the following minimum principle. The expression

$$(35) \quad E^*(u) = \int_0^b \left[p \left(\frac{du}{dx} \right)^2 + qu^2 \right] dx$$

is minimized for the class of functions which satisfy the boundary conditions (2) and (3). As is well known, the Euler equation resulting is the Sturm-Liouville equation (1). Moreover, integrating (35) by parts and using (1), it is seen that the dissipation H is the minimum of E^* . Thus if u is a function which satisfies the boundary condition (2),

$$(36) \quad E(u) \leq H \leq E^*(u).$$

It is not necessary to assume (3) because it is a natural boundary condition.

Another approximation method is based on the fact that the kernel of the integral equation (9) is positive. Given a function u_1 , let a function u_2 be defined by

$$(37) \quad u_2 = - \int_0^x p^{-1} \left[\int_x^b qu_1 dx \right] dx + 1.$$

Suppose that $u_1 \geq u$ where u is the solution of equation (9). (In particular this could be $u_1 = 1$.) Then it is seen by comparing (37) and (9) that $u_2 \leq u$. Thus

$\int_0^b q u_2 dx$ gives a lower bound for the dissipation. Likewise if $u_1 \leq u$, then $u_2 \geq u$ and $\int_0^b q u_2 dx$ is an upper bound for the dissipation.

It is clear that if p is increased (decreased), the dissipation is increased (decreased). The same statement holds for q . This can be proved on the basis of the following variational expression

$$(38) \quad E = \int_0^\infty \left[2u - u^2 - (pq)^{-1} \left(\int_v^\infty u \, dy \right)^2 \right] dy.$$

This is relation (13) in terms of a new independent variable $y = \int_0^x q \, dx$. It gives another method for getting upper and lower bounds by comparison with certain exact solutions. In particular, exact solutions can be obtained when p and q are step functions. Then the solution of (1) is reducible to a finite system of linear equations in a finite number of unknowns.

In a previous paper [1], W. D. SERBYN and the writer developed certain variational methods for obtaining lower bounds for solutions of equation (1). These methods, suitably modified, could be applied to cooling fin problems

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