*Flow of a Compressible Fluid in a Hyperbolic Channel**

HASAN ÖZOKLAV

Communicated by R. BERKER

Summary. In the first two sections of this work we have repeated some of the results, given by Prof. RATIP BERKER, about plane and steady motion of gases, and then we have applied these results to the motion in a channel whose wall is in the form of a hyperbola. We have solved the resulting boundary value problem of a linear partial differential equation of hyperbolic type, we have determined various magnitudes belonging to the motion, and we have developed the results as far as numerical calculations.

In section 1, we give the *compatibility equations* of gases. It can be shown that it is possible to eliminate the pressure and the density between the equations satisfied by the motion and thus obtain a certain number of relations which contain only the velocity field $\mathbf{V}$ and the external force $\mathbf{F}$. In the case where the motion is steady and the external force nonexistent, these relations, called the *compatibility equations*, take the following form (γ denotes the acceleration):

$$(A) \quad \gamma \cdot \operatorname{curl} \gamma = 0,$$

$$(B) \quad \operatorname{curl} \left(\frac{\operatorname{curl} \gamma \wedge \mathbf{V} - \theta \gamma}{\mu} \right) = 0,$$

$$(C) \quad \operatorname{grad} \frac{\mu}{\theta} \cdot \operatorname{curl} \gamma = 0,$$

where $\theta = \operatorname{div} \mathbf{V}$ and $\mu = \mathbf{V} \cdot \gamma$. In writing these expressions we have excluded the *uniform motions*, that is, those for which $\mathbf{V} \cdot \gamma = 0$; these are very special motions.

In the second section we suppose that the motion is plane and that the streamlines are isometric. We suppose that the isometric net formed by the streamlines and their orthogonal trajectories is given by the following equations:

* Doctoral thesis in fluid mechanics, Technical University of Istanbul, 1954.

$$\alpha(x, y) = \text{const. (orthogonal trajectories),}$$

$$\beta(x, y) = \text{const. (streamlines),}$$

and that the element of arc is of the form $ds^2 = h^2(d\alpha^2 + d\beta^2)$; here we put $w = 1/h$. Writing the compatibility equations we get the following result for the scalar velocity V : We have

$$V = g(\beta) \cdot G(\alpha, \beta)$$

where $g(\beta)$ is an arbitrary function and $G(\alpha, \beta)$ is a function satisfying the equation

$$(1) \quad \frac{G''_{\alpha\beta}}{G} = \frac{w''_{\alpha\beta}}{w}.$$

We give the expressions for the density, pressure and Mach's number in section 3.

In section 4 it is verified that the solution satisfies MUNK & PRIM's substitution principle. The presence of the arbitrary function $g(\beta)$ in the expression for V corresponds to the substitution principle.

In section 5 the above method is applied to the special problem of the motion in a channel whose wall is in the form of a hyperbola. The equation (1), belonging to this problem, takes a simple form by a convenient change of variable. To this simple equation we applied RIEMANN's method of integration; the Riemann function of the equation is expressed by the hypergeometric function and this function is expressed by the complete elliptic functions. If the distribution of the velocities on the section of entrance of the channel and on the wall is given, the velocity at any interior point of the channel can be obtained; the relation giving this velocity is explicitly stated. When the velocity is determined then all other magnitudes are easily obtained.

In section 6 we have made a simple hypothesis on the distribution of velocities along the wall, and under this hypothesis we have calculated numerically the velocity on the narrow section and on an intermediate section of the channel. Further, we have given the expression of the numerical values of the pressure along the wall of the channel.

1. Compatibility equations of gases. Let us consider the motion of a gas, that is, of a compressible fluid. We suppose the motion to be steady and the gas to be perfect and to have no heat conductivity. If we denote the external force by $\mathbf{F}$, the velocity by $\mathbf{V}$, the acceleration by $\boldsymbol{\gamma}$, the density by ρ , the pressure by p and the specific entropy by s , then the motion satisfies the following equations:

$$(1.1) \quad -\text{grad } p = \rho(\boldsymbol{\gamma} - \mathbf{F}),$$

$$(1.2) \quad \text{div } (\rho \mathbf{V}) = 0,$$

$$(1.3) \quad \rho = \rho(p, s),$$

$$(1.4) \quad \mathbf{V} \cdot \text{grad } s = 0.$$

The first of these equations is EULER's dynamical equation, the second is the continuity equation and the third is the state equation. The last equation is a consequence of the principle of conservation of energy, and states that the specific entropy is constant along each streamline ([1], p. 22).

These equations constitute a system of non-linear partial differential equations composed of six scalar equations containing six unknown scalar functions. Our knowledge about the solutions of this system of equations, in the case where the motion is not without vorticity, is very limited and does not extend beyond the results given in PRIM's important work published in 1952 [2].

If we leave aside the motions without vorticity and consider those with vorticity, a completely new method rests on the following considerations. It is possible to eliminate ρ , p and s among these six equations and get relations containing only $\mathbf{V}$ and $\mathbf{F}$. More precisely, it is possible to eliminate ρ and p between equations (1.1) and (1.2). We call the relations obtained in this way the *compatibility equations*. The compatibility equations for gases were first established by A. FRIEDMANN [3], [4]. FRIEDMANN did not suppose the motions to be steady, that is, he took the most general case. Certainly his compatibility equations are more complicated than ours. The elimination, in our case, can be done as follows.

Putting $\mathbf{j} = \mathbf{F} - \boldsymbol{\gamma}$ and taking the curl of both sides of (1.1) we get

$$(1.5) \quad \text{curl } (\rho \mathbf{j}) = 0.$$

Writing equations (1.5) and (1.2) more explicitly we get

$$(1.6) \quad \rho \text{ curl } \mathbf{j} + \text{grad } \rho \wedge \mathbf{j} = 0,$$

$$(1.7) \quad \rho \text{ div } \mathbf{V} + \text{grad } \rho \cdot \mathbf{V} = 0.$$

If we form the scalar product of both sides of (1.6) with $\mathbf{j}$ we get

$$(A_1) \quad \mathbf{j} \cdot \text{curl } \mathbf{j} = 0.$$

Putting

$$(1.8) \quad \mathbf{u} = \frac{1}{\rho} \text{grad } \rho = \text{grad } \ln \rho$$

in equations (1.6) and (1.7), we get

$$(1.9) \quad \text{curl } \mathbf{j} + \mathbf{u} \wedge \mathbf{j} = 0,$$

$$(1.10) \quad \text{div } \mathbf{V} + \mathbf{u} \cdot \mathbf{V} = 0.$$

Forming the vector product of both sides of (1.9) with $\mathbf{V}$ we get

$$\mathbf{V} \wedge \text{curl } \mathbf{j} + \mathbf{V} \wedge (\mathbf{u} \wedge \mathbf{j}) = 0.$$

Writing the double vector product on the first side of this equation, according to the known formula, we get

$$(1.11) \quad \mathbf{V} \wedge \operatorname{curl} \mathbf{j} + \mathbf{u}(\mathbf{V} \cdot \mathbf{j}) - \mathbf{j}(\mathbf{u} \cdot \mathbf{V}) = 0.$$

Putting $\mathbf{u} \cdot \mathbf{V} = -\operatorname{div} \mathbf{V}$, taken from equation (1.10), in equation (1.11) we get

$$(1.12) \quad \mathbf{V} \wedge \operatorname{curl} \mathbf{j} + \mathbf{u}(\mathbf{V} \cdot \mathbf{j}) + \mathbf{j} \operatorname{div} \mathbf{V} = 0.$$

Firstly we suppose $\mathbf{V} \cdot \mathbf{j} = 0$. In this case, when no external force exists, the velocity magnitude is constant along each streamline and every particle of the fluid is in uniform motion. When this condition is satisfied, we will call the motion of the gas uniform motion.

We leave aside the case for which $\mathbf{V} \cdot \mathbf{j} = 0$ and the uniform motion when there is no external force. We can solve equation (1.12) with respect to $\mathbf{u}$, so that we get

$$(1.13) \quad \mathbf{u} = \operatorname{grad} \ln \rho = \frac{\operatorname{curl} \mathbf{j} \wedge \mathbf{V} - \mathbf{j} \operatorname{div} \mathbf{V}}{\mathbf{V} \cdot \mathbf{j}}.$$

Taking the curl of both sides of this equation we get

$$(B_1) \quad \operatorname{curl} \left(\frac{\operatorname{curl} \mathbf{j} \wedge \mathbf{V} - \mathbf{j} \operatorname{div} \mathbf{V}}{\mathbf{V} \cdot \mathbf{j}} \right) = 0.$$

Equations (A₁) and (B₁) contain only $\mathbf{V}$ and $\mathbf{F}$, they are the compatibility equations for the motions we considered.

These equations, which are necessary conditions for $\mathbf{V}$, are at the same time sufficient; that is to say, when a solution $\mathbf{V}$ is obtained from (A₁) and (B₁), it is easy to see that the $\mathbf{u}$ given by (1.13) satisfies (1.9) and (1.10); indeed, we get

$$\mathbf{u} \cdot \mathbf{V} = -\operatorname{div} \mathbf{V}$$

and

$$\mathbf{u} \wedge \mathbf{j} = \frac{(\operatorname{curl} \mathbf{j} \wedge \mathbf{V}) \wedge \mathbf{j}}{\mathbf{V} \cdot \mathbf{j}} = \frac{\mathbf{V}(\mathbf{j} \cdot \operatorname{curl} \mathbf{j}) - \operatorname{curl} \mathbf{j}(\mathbf{V} \cdot \mathbf{j})}{\mathbf{V} \cdot \mathbf{j}} = -\operatorname{curl} \mathbf{j}$$

from (1.13). We now see that when a solution of (A₁) and (B₁) is found, then ρ and p can be obtained from (1.13) and (1.1). When we consider equations (1.3) and (1.4) the compatibility equation becomes (see [14])

$$(C_1) \quad \operatorname{grad} \frac{\mu}{\theta} \cdot \operatorname{curl} \mathbf{j} = 0,$$

where $\theta = \operatorname{div} \mathbf{V}$ and $\mu = \mathbf{V} \cdot \mathbf{j}$.

When there is no external force ($\mathbf{F} = 0$) equations (A₁), (B₁) and (C₁) take the form

$$(A) \quad \boldsymbol{\gamma} \cdot \operatorname{curl} \boldsymbol{\gamma} = 0,$$

$$(B) \quad \operatorname{curl} \left(\frac{\operatorname{curl} \boldsymbol{\gamma} \wedge \mathbf{V} - \theta \boldsymbol{\gamma}}{\mu} \right) = 0,$$

$$(C) \quad \operatorname{grad} \frac{\mu}{\theta} \cdot \operatorname{curl} \boldsymbol{\gamma} = 0,$$

where $\mu = \mathbf{V} \cdot \boldsymbol{\gamma}$.

Equation (A) states that the vector field $\boldsymbol{\gamma}(xyz)$ is a complex-laminar field.¹

It is well known that, in the mechanics of continuous media, the inverse method² is a very useful one [5]. It has given important results for the motions of incompressible perfect and viscous fluids [6] and gases, in elasticity, and in plasticity.

The compatibility equations we obtained above make the applications of the inverse method to gases more convenient. In this way it becomes possible to solve many problems.

In this work we will apply this method to a special problem so that we can show how this method may be developed as far as numerical calculations.

2. Plane motion, the case where the trajectories are isometric. In this section we will consider the plane steady motion of a perfect fluid. Further, we suppose that there is no external force and that the streamlines and their orthogonal trajectories form an isometric net.³

Let

$$(2.1) \quad x = x(\alpha, \beta), \quad y = y(\alpha, \beta)$$

be the equations of these two families of curves; and let $\beta = \text{const.}$ represent the streamlines and $\alpha = \text{const.}$ their orthogonal trajectories. It will be convenient to use these curves as coordinate curves. The element of arc length in these coordinates is of the form

$$(2.2) \quad ds^2 = h_1^2 d\alpha^2 + h_2^2 d\beta^2$$

where $h_1 = h_1(\alpha, \beta)$, $h_2 = h_2(\alpha, \beta)$; and since we have supposed the net to be isometric we must have

$$h_1 = h_2 = h.$$

Let $\mathbf{e}_1$ be a unit vector on the curves $\beta = \text{const.}$ in the direction of increasing α and $\mathbf{e}_2$ be a unit vector on the curves $\alpha = \text{const.}$ in the direction of increasing β .

¹ This means that $\boldsymbol{\gamma} = \lambda \operatorname{grad} v$.

² In studying a partial differential equation (or a system of partial differential equations) of mathematical physics, instead of the boundary conditions, some analytic, geometric, kinematic or physical conditions are prescribed which are valid not only on the boundary but in the whole region.

³ G. HAMEL has stated this hypothesis for an incompressible viscous fluid and has found an important result [7]. For related works see [5] and [6].

β (Fig. 1). Accordingly we can write

$$(2.3) \quad \mathbf{V} = V(\alpha, \beta) \mathbf{e}_1$$

for the velocity vector. From the compatibility conditions (A), (B) and (C) found in section 1 the equations (A) and (C) are satisfied by themselves in the case of plane flow.

Let us express the conditions (B) in terms of the curvilinear coordinates we introduced above. For this purpose it will be convenient to put

Fig. 1.

$$(2.4) \quad \frac{1}{h} = w(\alpha, \beta).$$

Accordingly we get

$$(2.5) \quad \gamma_1 = wV \frac{\partial V}{\partial \alpha}, \quad \gamma_2 = V^2 \frac{\partial w}{\partial \beta}$$

as the coordinates of acceleration. Furthermore, we get the following results:

$$(2.6) \quad \operatorname{div} \mathbf{V} = w^2 \frac{\partial}{\partial \alpha} \left(\frac{V}{w} \right),$$

$$(2.7) \quad \operatorname{curl} \boldsymbol{\gamma} = w^2 \left[\frac{\partial}{\partial \alpha} \left(\frac{V^2}{w} \frac{\partial w}{\partial \beta} \right) - \frac{\partial}{\partial \beta} \left(V \frac{\partial V}{\partial \alpha} \right) \right] \mathbf{e}_3$$

for $\operatorname{div} \mathbf{V}$ and $\operatorname{curl} \boldsymbol{\gamma}$ contained in condition (B). Here $\mathbf{e}_3$ represents a unit vector perpendicular to the plane such that $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ form a right system. Finally we get

$$\begin{aligned}
(2.8) \quad \mathbf{u} = \text{grad } \ln \rho &= \frac{\text{curl } \boldsymbol{\gamma} \wedge \mathbf{V} - \boldsymbol{\gamma} \text{div } \mathbf{V}}{\mathbf{V} \cdot \boldsymbol{\gamma}} \\
&= -w \frac{\partial}{\partial \alpha} \left(\ln \frac{V}{w} \right) \mathbf{e}_1 + \frac{-w \frac{\partial}{\partial \beta} \left(V \frac{\partial V}{\partial \alpha} \right) + V \frac{\partial}{\partial \alpha} \left(V \frac{\partial w}{\partial \beta} \right)}{V \frac{\partial V}{\partial \alpha}} \mathbf{e}_2
\end{aligned}$$

for the relation (1.13). Now we recall that we have left aside, in section 1, the case for which $\mathbf{V} \cdot \boldsymbol{\gamma} = 0$; the velocity is not constant in absolute value on every streamline, which means that $V'_\alpha \neq 0$. Taking the curl of both sides of equation (2.8) we get the condition (B):

$$(2.9) \quad \frac{\partial}{\partial \alpha} \left[\frac{w \frac{\partial}{\partial \beta} \left(V \frac{\partial V}{\partial \alpha} \right) - V \frac{\partial}{\partial \alpha} \left(V \frac{\partial w}{\partial \beta} \right)}{w V \frac{\partial V}{\partial \alpha}} \right] - \frac{\partial^2}{\partial \beta \partial \alpha} \left(\ln \frac{V}{w} \right) = 0.$$

After changing the order of differentiation in the last term of this equation, it becomes possible to integrate it once:

$$\frac{w \frac{\partial}{\partial \beta} \left(V \frac{\partial V}{\partial \alpha} \right) - V \frac{\partial}{\partial \alpha} \left(V \frac{\partial w}{\partial \beta} \right)}{w V \frac{\partial V}{\partial \alpha}} - \frac{w}{V} \frac{\partial}{\partial \beta} \left(\frac{V}{w} \right) = K(\beta)$$

where $K(\beta)$ represents an arbitrary function. Simplifying we get

$$(2.10) \quad w \frac{\partial^2 V}{\partial \alpha \partial \beta} - V \frac{\partial^2 w}{\partial \alpha \partial \beta} = K(\beta) w \frac{\partial V}{\partial \alpha}.$$

It is possible to write this last equation in a more symmetrical form by putting

$$(2.11) \quad V = g(\beta) \cdot G(\alpha, \beta).$$

Indeed, equation (2.10) can be put in the form⁴

$$(2.12) \quad \frac{G''_{\alpha\beta}}{G} = \frac{w''_{\alpha\beta}}{w}$$

since $K(\beta)$ and, accordingly, $g(\beta)$ represent arbitrary functions. For this purpose it is enough to take

$$g(\beta) = \exp \left\{ \int K(\beta) d\beta \right\}$$

where $g(\beta)$ must be arbitrary since $K(\beta)$ is arbitrary.

In this way, we have found not only one solution according to equation (2.11) for each $G(\alpha, \beta)$ satisfying (2.12), but a family of solutions. All these solutions have the same streamlines.

⁴ A paper about the deduction and study of this equation by R. BERKER will soon appear.

3. Density, pressure, state equation and Mach's number. Putting the expression

$$V = g(\beta) \cdot G(\alpha, \beta)$$

in relation (1.13) and considering (2.12) we get

$$(3.1) \quad \rho = \rho_0 \frac{w}{g^2 G},$$

where ρ_0 represents an integration constant. Using the expression (2.11) and the relation (2.5) we get

$$(3.2) \quad \begin{aligned} \gamma_1 &= w V \frac{\partial V}{\partial \alpha} = w g^2 G \frac{\partial G}{\partial \alpha}, \\ \gamma_2 &= V^2 \frac{\partial w}{\partial \beta} = g^2 G^2 \frac{\partial w}{\partial \beta} \end{aligned}$$

for the coordinates of acceleration. Carrying (3.1) and (3.2) into equation (1.1) we find

$$(3.3) \quad \frac{\partial p}{\partial \alpha} = -\rho_0 w \frac{\partial G}{\partial \alpha}, \quad \frac{\partial p}{\partial \beta} = -\rho_0 \frac{\partial w}{\partial \beta} G.$$

It is easy to see that the compatibility condition of equation (3.3) is the equation (2.12). Thus, considering equation (2.12), we get from equations (3.3), for the pressure p ,

$$(3.4) \quad p = -\rho_0 \int_{\beta_0}^{\beta} w'_s(\alpha, \beta) G(\alpha, \beta) d\beta - \rho_0 \int_{\alpha_0}^{\alpha} w(\alpha, \beta_0) G'_\alpha(\alpha, \beta_0) d\alpha + p_0$$

where α_0 , β_0 and p_0 denote arbitrary constants.

Suppose we find

$$(3.5) \quad \rho = \rho(\alpha, \beta), \quad p = p(\alpha, \beta)$$

from the results above. On the other hand, since the specific entropy is constant on each streamline according to equation (1.4) we have

$$(3.6) \quad s = s(\beta).$$

Eliminating α and β between these three equations we find

$$(3.7) \quad \rho = \rho(p, s)$$

for the state equation.

Finally, Mach's number is defined as

$$(3.8) \quad M = \frac{V}{c}$$

where V is the velocity of gas and c is the local velocity of sound. Mach's number

can easily be calculated from the results above since

$$c^2 = \frac{\partial p}{\partial \rho} = \frac{p'_\alpha}{\rho_\alpha}.$$

As a result we get

$$(3.9) \quad M = \frac{V}{c} = \frac{gG}{c} = \sqrt{1 - \frac{w'_\alpha}{w} \frac{G}{G'_\alpha}}.$$

The equations

$$(2.11) \quad V = g(\beta) \cdot G(\alpha, \beta),$$

$$(3.1) \quad \rho = \rho_0 \frac{w}{g^2 G},$$

$$(3.4) \quad p = -\rho_0 \int_{\beta_0}^{\beta} w'_\beta \cdot G \, d\beta - \rho_0 \int_{\alpha_0}^{\alpha} w(\alpha, \beta_0) G'_\alpha(\alpha, \beta_0) \, d\alpha + p_0,$$

and (3.7) and (3.9), form the complete solution of the motion.

4. The substitution principle. When a motion for a gas is known, it is possible to get from this known motion another motion for a gas, by the substitution principle of MUNK & PRIM. In general the new flow so obtained will be the flow of a different gas. The streamlines are the same in both motions. The basis of this principle is the following theorem ([1]. p. 36):

Invariance Theorem. Let $\mathbf{V}$, p , ρ be the velocity, the pressure and the density of a gas which is in steady motion and which depends on no external force. Then if m be any nonvanishing differentiable function which is constant upon each streamline ($\mathbf{V} \cdot \text{grad } m = 0$), $\mathbf{V}/m$, p , $m^2 \rho$ are the velocity, the pressure and the density of a new flow.

The state equation of this new flow, got in this way, is in general different from the original one; in other words, the two flows do not belong to the same gas. If we seek the conditions under which the two flows belong to the same gas we see that we must have

$$\rho = P(p) \cdot S(s)$$

for the state equation; the gases for which the state equation is in this form are called Prim gases. The substitution principle for Prim gases takes the form ([1], p. 38; [12], [13]):

The Substitution Principle. Let $\mathbf{V}$, p , ρ denote the velocity, the pressure and the density of a gas which is in steady motion and which depends on no external force. Then if m be any nonvanishing differentiable function which is constant upon each streamline ($\mathbf{V} \cdot \text{grad } m = 0$), $\mathbf{V}/m$, p , $m^2 \rho$, are the velocity, the pressure and the density of a new flow; the necessary and sufficient condition that the two flows belong to the same gas is that the gas be a Prim gas. The Mach's number is the same for the two flows.

The substitution principle is clearly satisfied by our results above. Indeed, it is seen from our expressions (3.3), (3.9), (2.11), and (3.1) that $\mathbf{V} \cdot \text{grad } m = 0$ when $g = 1/m$, that the pressure p and the Mach's number do not change, and that the velocity and the density for the two flows are related to each other as expressed by the substitution principle.

5. The flow in a channel whose wall is in the form of a hyperbola. Let us suppose that the streamlines are composed of confocal hyperbolas. This will give us the motion in a channel whose wall is in the form of a hyperbola.

The isometric net mentioned above is composed of these confocal hyperbolas and their orthogonal trajectories which are ellipses having the same focus (Fig. 2).

Equations (2.1) are in the form

$$(5.1) \quad x = a \operatorname{ch} \alpha \cos \beta, \quad y = a \operatorname{sh} \alpha \sin \beta$$

Fig. 2

for this net. The quantity w defined by (2.4) is given by the relation

$$(5.2) \quad w = \frac{1}{a} (\text{sh}^2 \alpha + \sin^2 \beta)^{-\frac{1}{2}}$$

Accordingly we get

$$(5.3) \quad \frac{\partial^2 G}{\partial \alpha \partial \beta} - \frac{3 \text{sh } \alpha \text{ch } \alpha \sin \beta \cos \beta}{(\text{sh}^2 \alpha + \sin^2 \beta)^2} G = 0$$

for the equation (2.12). This equation is a linear partial differential equation of the second order and of hyperbolic type. It is possible to solve the boundary value problem belonging to it.

Taking the branches $\beta = \beta_0$ and $\beta = \pi - \beta_0$ of the hyperbolas as boundary, in accordance with the boundary principle, we will find the motion of a gas in such a channel. Because of symmetry it is enough to take the right side of the Oy axis.

Let M be an arbitrary point on the hyperbola $\beta = \beta_0$ (Fig. 2); by the methods of analysis we can find the velocities at every point of the region $NMBCN$ when we are given the velocities along the hyperbolic arc BM and the elliptic arc BC . Let A be the vertex of the hyperbola; the velocities in the region $OABCO$ can be calculated when the velocities on BA and BC are given.

Before beginning to solve this boundary value problem we give the following result about the sense of variation of the intensity of the velocity.

The variation of the velocity on any streamline is related to the variation of $G(\alpha, \beta)$ since the velocity is given by $V = g(\beta)G(\alpha, \beta)$ and the function $g(\beta)$ is constant on each streamline. Therefore the sense of variation of the intensity of velocity on a streamline depends on the sign of G'_α . Consider any streamline PQ where P is a point on the section of entrance BC and Q is a point on the narrow section OA (Fig. 3). We suppose that the gas moves in the sense PQ . If $G'_\alpha > 0$ on PQ the quantity $|G|$ increases when we go from P to Q ; and $|G|$ decreases if $G'_\alpha < 0$.

Consider the expression of Mach's number given by the equation (3.9). We have

$$\frac{w'_\alpha}{w} = - \frac{\text{sh } \alpha \text{ch } \alpha}{\text{sh}^2 \alpha + \sin^2 \beta};$$

hence $w'_\alpha/w < 0$ in the region $OABCO$. Therefore

$$M < 1 \quad \text{if} \quad GG'_\alpha < 0 \quad \text{and} \quad M > 1 \quad \text{if} \quad GG'_\alpha > 0.$$

Finally, when we go from P to Q , the velocity magnitude increases when the motion is subsonic ($M < 1$), and it decreases when the motion is supersonic ($M > 1$). Certainly we have corresponding results for the part following the narrow section of the channel. We have then found the same classical result about the sense of variation of the velocity magnitude in the channel when we

Fig. 3

make the hypothesis of parallel sections. $w'_\alpha = 0$ in the narrows section and so $M = 1$.⁵

Now we return again to the solution of the boundary value problem; first we change the independent variable in equations (5.3).

Fig. 4.

⁵ We suppose $G'_\alpha(0, \beta) \neq 0$.

If we write

$$(5.4) \quad \operatorname{ch} 2\alpha = u, \quad \cos 2\beta = v,$$

equation (5.3) takes the form

$$(5.5) \quad \frac{\partial^2 G}{\partial u \partial v} + \frac{3}{4(u-v)^2} G = 0.$$

In this way we have reduced equation (5.3) to a known equation. Indeed, equation (5.5) is an equation which has been studied a great deal ([8], p. 56).

If we consider the auxiliary plane Ωuv and take

$$\operatorname{ch} 2\alpha_0 = u_0, \quad \cos 2\beta_0 = v_0,$$

we see that the rectangle $O_1 A_1 B_1 C_1 O_1$ in the Ωuv plane, shown in the figure, corresponds to the region $OABCO$ in the Oxy plane and that the values of the function $G(u, v)$ on the sides $A_1 B_1$ and $B_1 C_1$ are as follows:

$$(5.6) \quad G(u_0, v) = \varphi(v), \quad G(u, v_0) = \psi(u).$$

These functions $\varphi(v)$ and $\psi(u)$ satisfy the condition $\varphi(v_0) = \psi(u_0)$. Now our problem consists of finding the value of the function in the rectangle $O_1 A_1 B_1 C_1 O_1$.

We will solve this boundary value problem, which belongs to the equation (5.5),⁶ by RIEMANN'S method ([9], p. 311–317). Denoting the Riemann function of this equation by $R(u, v; \xi, \eta)$, we know that the function $G(u, v)$ is given by the following relation:

$$(5.7) \quad \begin{aligned} G(u, v) = & G(u_0, v_0)R(u, v; u_0, v_0) + \int_{u_0}^u R(u, v; \xi, v_0)\psi'(\xi) d\xi \\ & + \int_{v_0}^v R(u, v; u_0, \eta)\varphi'(\eta) d\eta. \end{aligned}$$

In this integral (u, v) will be treated as constant and (ξ, η) will be taken as a variable point.

We know the Riemann function of the equation (5.5) ([8], p. 84); it has a simple expression in terms of the hypergeometric function given by the relation

$$(5.8) \quad \begin{aligned} R(u, v; \xi, \eta) = & \left[1 - \frac{(u-\xi)(v-\eta)}{(v-\xi)(u-\eta)} \right]^{-\frac{1}{2}} \\ & \cdot F\left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(u-\xi)(v-\eta)}{(v-\xi)(u-\eta)} \right], \end{aligned}$$

where $F(a, b; c; z)$ is the hypergeometric function given by the equation

$$(5.9) \quad \begin{aligned} F(a, b; c; z) = & 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 \\ & + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}z^3 + \dots \end{aligned}$$

⁶ Besides the solution of the boundary value problem, an infinite number of particular solutions of the equation (5.5) are known (see [8], p. 56–71).

Carrying the expression (5.8) into the relation (5.7) one gets

$$\begin{aligned}
 (5.10) \quad G(u, v) = & G(u_0, v_0) \left[1 - \frac{(u - u_0)(v - v_0)}{(v - u_0)(u - v_0)} \right]^{-\frac{1}{2}} \\
 & \cdot F \left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(u - u_0)(v - v_0)}{(v - u_0)(u - v_0)} \right] \\
 & - \int_u^{u_0} \left[1 - \frac{(u - \xi)(v - v_0)}{(v - \xi)(u - v_0)} \right]^{-\frac{1}{2}} \\
 & \cdot F \left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(u - \xi)(v - v_0)}{(v - \xi)(u - v_0)} \right] \psi'(\xi) d\xi \\
 & - \int_v^{v_0} \left[1 - \frac{(u - u_0)(v - \eta)}{(v - u_0)(u - \eta)} \right]^{-\frac{1}{2}} \\
 & \cdot F \left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(u - u_0)(v - \eta)}{(v - u_0)(u - \eta)} \right] \varphi'(\eta) d\eta.
 \end{aligned}$$

This relation gives us the solution of the boundary value problem; it is reduced to the solution of two simple integrals.

It is known that the hypergeometric series $F(a, b; c; z)$ is convergent for $|z| < 1$. It is convergent for $|z| = 1$ too, if $c - a - b > 0$ ⁷ ([10], p. 24). This condition is satisfied since $a = b = -\frac{1}{2}$ and $c = 1$ in the relation (5.8). Therefore our hypergeometric series $F(-\frac{1}{2}, -\frac{1}{2}; 1; z)$ is convergent for $|z| \leq 1$.

It is further possible to express the function $F(-\frac{1}{2}, -\frac{1}{2}; 1; z)$ in terms of complete elliptic integrals. Indeed, writing the series (5.9) for $a = b = -\frac{1}{2}$ and $c = 1$ one gets

$$(5.11) \quad F\left(-\frac{1}{2}, -\frac{1}{2}; 1; z\right) = 1 + \left(\frac{1}{2}\right)^2 z + \left(\frac{1 \cdot 1}{2 \cdot 4}\right)^2 z^2 + \left(\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 z^3 + \dots$$

We consider now the complete elliptic integrals of the first and second type:

$$\begin{aligned}
 (5.12) \quad K(k) &= \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \\
 &= \frac{\pi}{2} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \dots \right], \\
 (5.13) \quad E(k) &= \int_0^{\frac{1}{2}\pi} \sqrt{1 - k^2 \sin^2 \varphi} d\varphi \\
 &= \frac{\pi}{2} \left[1 - \left(\frac{1}{2}\right)^2 k^2 - \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \frac{k^4}{3} - \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \frac{k^6}{5} - \dots \right].
 \end{aligned}$$

Putting $z = k^2$ in the relation (5.11) we get

$$(5.14) \quad F\left(-\frac{1}{2}, -\frac{1}{2}; 1; k^2\right) = 1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 1}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \dots$$

⁷ We suppose a, b, c real.

From these relations it is easily seen that

$$(5.15) \quad F\left(-\frac{1}{2}, -\frac{1}{2}; 1; k^2\right) = \frac{2}{\pi}[2E(k) - (1 - k^2)K(k)].$$

In this way we have found a relation by which we can calculate the function $F(-\frac{1}{2}, -\frac{1}{2}; 1; z)$ for positive values of z .

For the negative values of z , we put $-k^2$ instead of k^2 in (5.15) and we get

$$(5.16) \quad F\left(-\frac{1}{2}, -\frac{1}{2}; 1; -k^2\right) = \frac{2}{\pi}[2E_1(k) - (1 + k^2)K_1(k)]$$

where we have

$$K_1(k) = \int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\sqrt{1 + k^2 \sin^2 \varphi}}, \quad E_1(k) = \int_0^{\frac{1}{2}\pi} \sqrt{1 + k^2 \sin^2 \varphi} d\varphi,$$

which can be expressed in terms of $K(k)$ and $E(k)$. Indeed, it is easily seen that

FIG. 5. Variation of the velocity on the narrow section OA , for $V_0 = 0.4$ and $V_0 = 1.2$.

$$(5.17) \quad K_1(k) = \frac{1}{\sqrt{1+k^2}} K\left(\frac{k}{\sqrt{1+k^2}}\right), \quad E_1(k) = \sqrt{1+k^2} E\left(\frac{k}{\sqrt{1+k^2}}\right).$$

Carrying these expressions to (5.15) we get

$$(5.18) \quad F\left(-\frac{1}{2}, -\frac{1}{2}; 1, -k^2\right) = \frac{2}{\pi} \sqrt{1+k^2} \left[2E\left(\frac{k}{\sqrt{1+k^2}}\right) - K\left(\frac{k}{\sqrt{1+k^2}}\right) \right].$$

Finally we get the following relation

$$(5.19) \quad (1+k^2)^{-\frac{1}{2}} F\left(-\frac{1}{2}, -\frac{1}{2}; 1, -k^2\right) = \frac{2}{\pi} \left[2E\left(\frac{k}{\sqrt{1+k^2}}\right) - K\left(\frac{k}{\sqrt{1+k^2}}\right) \right]$$

which is necessary for the calculations of the integrals present in the relation (5.10); the expression on the left side is ready for the calculation of the factors of $\psi'(\xi)$ and $\varphi'(\eta)$ under the integration sign.

In this way we have reduced the problem of the motions in a channel, whose wall is in the form of a hyperbola, to a form which is convenient for numerical calculations.

FIG. 6. Variation of the velocity on the narrow section OA for $V_0 = 0.566$.

6. A numerical example. In this section we will give some numerical results of an example. We can suppose the velocity magnitude on the arc BC (Fig. 2) of the ellipse at the entrance of the channel to be equal to a constant and also this constant may be supposed to be equal to unity. This will not effect the generality of the problem. Indeed, because of the substitution principle and the function $g(\beta)$ in the relation (2.11), the velocities in the whole region can be multiplied by a scalar that is constant on each streamline but which changes arbitrarily in passing from one streamline to another. Hence, with this scalar we can get any velocity distribution that we want on the arc BC .

We suppose that the velocity on the arc AB changes linearly with u . According to these hypotheses the functions $\varphi(v)$ and $\psi(u)$ in (5.6) are given as follows:

$$\varphi(v) = -1, \quad \psi(u) = Au + B$$

where A and B are constants. From the condition $\varphi(v_0) = \psi(u_0)$ we get $B = -Au_0 - 1$. Finally, having given the expressions

$$(6.1) \quad \varphi(v) = -1, \quad \psi(u) = A(u - u_0) - 1$$

and having put the derivatives of the functions (6.1) in the relation (5.10), we get the expression

$$(6.2) \quad G(u, v) = - \left[1 - \frac{(u - u_0)(v - v_0)}{(v - u_0)(u - v_0)} \right]^{-\frac{1}{2}} \cdot F \left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(u - u_0)(v - v_0)}{(v - u_0)(u - v_0)} \right] \\ - A \int_{u_0}^u \left[1 - \frac{(u - \xi)(v - v_0)}{(v - \xi)(u - v_0)} \right]^{-\frac{1}{2}} \cdot F \left[-\frac{1}{2}, -\frac{1}{2}; 1; \frac{(u - \xi)(v - v_0)}{(v - \xi)(u - v_0)} \right] d\xi$$

for the value of the function $G(u, v)$ at any point.

To determine the form of the channel we will choose

$$\beta_0 = \frac{\pi}{3}, \quad \cos 2\beta_0 = v_0 = -\frac{1}{2}, \quad u_0 = \operatorname{ch} 2\alpha_0 = 3$$

and we will give numerical results for the velocities on the narrow section OA corresponding to $u = \operatorname{ch} 2\alpha = 1$ and for these on the arc NM of the ellipse (Fig. 2) corresponding to $u = \frac{3}{2}$. Putting the values of u_0 , v_0 and u in the formula (6.2), we get

$$(6.3) \quad G(1, v) = - \left[1 + \frac{4v + 2}{3v - 9} \right]^{-\frac{1}{2}} \cdot F \left(-\frac{1}{2}, -\frac{1}{2}; 1; -\frac{4v + 2}{3v - 9} \right) \\ - A \int_1^3 \left[1 + \frac{(\xi - 1)(2v + 1)}{3(v - \xi)} \right]^{-\frac{1}{2}} \cdot F \left[-\frac{1}{2}, -\frac{1}{2}; 1; -\frac{(\xi - 1)(2v + 1)}{3(v - \xi)} \right] d\xi$$

for the values of $G(u, v)$ on the narrowest section. The integral on the right side of this formula is calculated from the tables on complete elliptic functions [11] by using the relation (5.19). We used Simpsons's formula in these calculations.

The results for the velocity distribution on the narrow section of the channel are given in the following table. We give the values of the velocity for seven

points on the section OA . For $g = 1$ we have $V = G$. The value of the velocity at A is denoted by V_0 . Q represents a point on the narrow section OA .

β	x_Q/x_A	V_Q
60°	1.00000	V_0
65°	0.84524	$1.00793V_0 - 0.00452$
70°	0.68404	$1.01374V_0 - 0.00779$
75°	0.51764	$1.01788V_0 - 0.01015$
80°	0.34730	$1.02067V_0 - 0.01166$
85°	0.17432	$1.02225V_0 - 0.01253$
90°	0.00000	$1.02277V_0 - 0.01280$

In a similar manner we can find the results, given in the following table, for the values of $G(\frac{3}{2}, v)$ which gives the distribution of velocity on the arc NM of the ellipse (Fig. 2) corresponding to $u = \frac{3}{2}$. Here Q_1 represents a point on the arc NM of the ellipse.

β	x_{Q_1}/x_M	V_{Q_1}
60°	1.00000	V_1
65°	0.84524	$1.00423V_1 - 0.00292$
70°	0.68404	$1.00746V_1 - 0.00514$
75°	0.51764	$1.00983V_1 - 0.00677$
80°	0.34730	$1.01145V_1 - 0.00787$
85°	0.17432	$1.01239V_1 - 0.00853$
90°	0.00000	$1.01270V_1 - 0.00874$

It is possible to find the pressure on the boundary for this example, by using the relation (3.4). Indeed, carrying the relations (6.1) into (3.4) we get

$$(6.4) \quad p(\alpha, \beta_0) = p_0 + \frac{4A\rho_0}{a} (\text{sh}^2 \alpha_0 + \sin^2 \beta_0)^{\frac{1}{2}} - \frac{4A\rho_0}{a} (\text{sh}^2 \alpha + \sin^2 \beta_0)^{\frac{1}{2}}.$$

FIG. 7. Variation of the pressure on the wall BA .

By using the relation (6.4) we have the results given in the following table for 10 points on the boundary. Here K denotes a constant defined by $K = 4A\rho_0/a$.

$y_M/y_\beta = \operatorname{sh}\alpha$	$(p_M - p_\beta)/K$
1.0	0.00000
0.9	0.07388
0.8	0.14390
0.7	0.20933
0.6	0.26931
0.5	0.32288
0.4	0.36394
0.3	0.40636
0.2	0.43406
0.1	0.45110
0.0	0.45685

If we represent graphically the values pressure and velocity given in the tables above we get the curves represented by Figures 5, 6 and 7.

BIBLIOGRAPHY

- [1] C. TRUESDELL, *Vorticity and the thermodynamic state in a gas flow*. Mémorial des Sciences Mathématiques, Fasc. 119, Paris, 1952.
- [2] R. C. PRIM, Steady rotational flow of ideal gases. *Journal of Rational Mechanics and Analysis*, **1**, pp. 425–497, 1952.
- [3] A. FRIEDMANN, Sur les tourbillons dans un liquide à température variable. *Comptes Rendus de l'Acad. des Sci. de Paris*, **163**, pp. 219–222, 1916.
- [4] A. FRIEDMANN, Über Wirbelbewegung in einer kompressiblen Flüssigkeit. *Zeitschrift für angewandte Mathematik und Mechanik*, **4**, pp. 102–107, 1924.
- [5] P. F. NEMENYI, Recent developments in inverse and semi-inverse methods in the mechanics of continua. *Advances in Applied Mechanics*, edited by R. von MISES and TH. VON KARMAN, Vol. II, pp. 123–151, New York, 1951.
- [6] R. BERKER, *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible*. Paris-Lille, 1936.
- [7] G. HAMEL, Spiralförmige Bewegungen zäher Flüssigkeiten. *Jahresb. der deutschen Math. Ver.*, pp. 34–60, 1916.
- [8] G. DARBOUX, *Leçons sur la théorie générale des surfaces*, t. 2. Paris, 1915.
- [9] R. COURANT & D. HILBERT, *Methoden der mathematischen Physik*, Bd. 2, Berlin, 1937.
- [10] E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*, Cambridge, 1952.
- [11] L. M. MILNE-THOMSON, Ten-figure table of the complete elliptic integrals. *Proceedings of the London Mathematical Society, Second Series*, **33**, London, pp. 160–164, 1932.
- [12] M. MUNK & R. C. PRIM, On the multiplicity of steady gas flow having the same streamline pattern. *Proc. Nat. Acad. Sci.*, **33**, pp. 137–141, 1947.
- [13] R. C. PRIM, A note on the substitution principle for steady gas flow. *Journal of Applied Physics*, **20**, pp. 448–450, 1949.
- [14] R. BERKER, Sur les équations de compatibilité relatives au mouvement d'un gaz, *Comptes Rendus de l'Académie des Sciences de Paris*, **242**, pp. 342–344, 1956.

Technical University of Istanbul