Representation of a Markovian Semigroup and Its Infinitesimal Generator

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Consider an order preserving semigroup of linear transformations P^t on a space of functions. Such semigroups are of interest in the theory of Markoff processes. It is well known that the Markoff process associated with such a semigroup possesses many desirable regularity features in case the space of functions is a $\mathcal{C}(X)$ (all continuous functions vanishing at infinity on a locally compact Hausdorff space X) and the semigroup P^t is strongly continuous. In Section 1 such a semigroup acting on an algebra of functions is represented as acting on a $\mathcal{C}(X)$. In general, the semigroup is not strongly continuous. There is a maximal linear subspace $\mathcal{U}$ of $\mathcal{C}(X)$ on which P^t is strongly continuous. The set $\mathcal{A}$ of all f such that f and f^2 are both in $\mathcal{U}$ is shown to be an algebra, but an example shows that $\mathcal{A}$ need not be invariant under the action of P^t . Thus the desired situation that P^t be strongly continuous on an algebra of functions corresponds to a regularity assumption of an algebraic nature.

Section 2 considers the problem of representing the infinitesimal generator in terms of a family of linear functionals defined at each point. Such a representation (somewhat similar to that of HUNT [3] in the spatially homogeneous case) is established, but it holds only for functions in the multiplier algebra $\mathfrak{M}$ of the domain $\mathfrak{D}$ of the infinitesimal generator. An example is given to show that $\mathfrak{M}$ may reduce to the constant functions alone, and that such a representation on $\mathfrak{D}$ itself is not always possible. Again, the representation is only of interest when a regularity condition of an algebraic nature is satisfied.

1. Representations and strong continuity. Let $\mathcal{C}$ be a real commutative C^* algebra. Examples are all bounded measurable functions on a measurable space, $\mathcal{L}^\infty$ on a measure space, or the set $\mathcal{C}(X)$ of all continuous functions vanishing at infinity on a locally compact Hausdorff space X . The norm in these examples is the supremum norm (or essential supremum norm in the case of $\mathcal{L}^\infty$). Any

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such algebra is isomorphic to $\mathcal{C}(X)$ for a unique locally compact Hausdorff space X , called the *spectrum* of $\mathcal{C}$ (see [4]). Therefore we may restrict ourselves to the case $\mathcal{C} = \mathcal{C}(X)$ without loss of generality. However, one is sometimes interested in a different representation of $\mathcal{C}$, such as $\mathcal{L}^\infty$ (in which case the spectrum X is a rather abstract object), so that the assumptions and conclusions of this section will be formulated algebraically (without referring to points in X) whenever possible.

Let P^t , $0 \leq t < \infty$, be a family of linear transformations on $\mathcal{C}$ such that

$$(1.1) \quad \begin{aligned} P^t f &\geq 0 \text{ if } f \geq 0, \\ P^{t+s} f &= P^t P^s f, \\ \|P^t f\| &\leq \|f\|. \end{aligned}$$

This will be referred to briefly as a *sub-Markovian semigroup*.

If P^t is a sub-Markovian semigroup on $\mathcal{C} = \mathcal{C}(X)$, then for each x in X the linear functional $f \rightarrow P^t f(x)$ is continuous and positive. By the Riesz-Markoff theorem there is a regular Borel measure $p^t(x, \cdot)$ such that

$$(1.2) \quad P^t f(x) = \int_X f(y) p^t(x, dy).$$

By (1.1) $p^t(x, X) \leq 1$. If $p^t(x, X) = 1$ for all x in X , the semigroup P^t is called *Markovian*. If $\mathcal{C}$ has a unit 1 (i.e. if X is compact) then P^t is Markovian if and only if $P^t 1 = 1$.

For many purposes it is sufficient to consider only Markovian semigroups. If P^t is a sub-Markovian semigroup on $\mathcal{C}$, let $\tilde{\mathcal{C}}$ be the direct sum of $\mathcal{C}$ and the real numbers, with multiplication defined by

$$(1.3) \quad (f \oplus \alpha)(g \oplus \beta) = (fg + \alpha g + \beta f) \oplus \alpha\beta$$

Define $\tilde{P}^t$ on $\tilde{\mathcal{C}}$ by

$$(1.4) \quad \tilde{P}^t(f \oplus \alpha) = P^t f \oplus \alpha.$$

Theorem 1.1. *If P^t is a sub-Markovian semigroup on $\mathcal{C}$ then $\tilde{P}^t$ is a Markovian semigroup on $\tilde{\mathcal{C}}$. $\|P^t f - f\| \rightarrow 0$ as $t \rightarrow 0$ if and only if $\|P^t(f \oplus \alpha) - f \oplus \alpha\| \rightarrow 0$ as $t \rightarrow 0$.*

Proof. Let X be the spectrum of $\mathcal{C}$ and let $\tilde{X}$ be the one point compactification of X by a point ∞ (so that if X is already compact then ∞ is an isolated point in $\tilde{X}$). Then $\tilde{X}$ is the spectrum of the algebra $\tilde{\mathcal{C}}$. (In fact, this is one definition of the one point compactification.) Let

$$(1.6) \quad \begin{aligned} \tilde{p}^t(x, \cdot) &= p^t(x, \cdot) + (1 - p^t(x, X))\delta_\infty; & x \neq \infty, \\ \tilde{p}^t(\infty, \cdot) &= \delta_\infty \end{aligned}$$

where δ_∞ is the unit mass concentrated at ∞ . Then, by (1.4),

$$(1.7) \quad \tilde{P}^t \tilde{f}(\tilde{x}) = \int_{\tilde{X}} \tilde{f}(\tilde{y}) \tilde{p}^t(\tilde{x}, d\tilde{y})$$

for all $\tilde{f}$ in $\tilde{\mathcal{C}} = C(\tilde{X})$ and $\tilde{x}$ in $\tilde{X}$. Consequently, $\tilde{P}^t \tilde{f} \geq 0$ if $\tilde{f} \geq 0$.

By (1.4),

$$\tilde{P}^{t+s}(f \oplus \alpha) = P^{t+s}f \oplus \alpha = P^t P^s f \oplus \alpha = \tilde{P}^t \tilde{P}^s(f \oplus \alpha).$$

Also, $\|\tilde{P}^t \tilde{f}\| \leq \|\tilde{f}\|$ since $\tilde{p}^t(\tilde{x}, \tilde{X}) = 1$. Thus $\tilde{P}^t$ satisfies (1.1) and is sub-Markovian. Since $\tilde{p}^t(\tilde{x}, \tilde{X}) = 1$ (in fact, the unit of $\tilde{\mathcal{C}}$ is $0 \oplus 1$ and $\tilde{P}^t(0 \oplus 1) = 0 \oplus 1$), $\tilde{P}^t$ is Markovian. Finally,

$$\|\tilde{P}^t(f \oplus \alpha) - f \oplus \alpha\| = \|(P^t f - f) \oplus 0\| = \|P^t f - f\|$$

concluding the proof.

The semigroup P^t is called *strongly continuous* in case $\|P^t f - f\| \rightarrow 0$ as $t \rightarrow 0$ for each f in $\mathcal{C}$. If P^t is not strongly continuous on $\mathcal{C}$ we may seek a sub-algebra on which it is strongly continuous. Let $\mathcal{V}$ be the set of all f in $\mathcal{C}$ such that $\|P^t f - f\| \rightarrow 0$ as $t \rightarrow 0$. Then $\mathcal{V}$ is a linear space. If $f_n \in \mathcal{V}$, $f_n \rightarrow f$, then

$$\|P^t f - f\| \leq \|P^t f - P^t f_n\| + \|P^t f_n - f_n\| + \|f_n - f\|.$$

For any $\epsilon > 0$, choose n so large that $\|f_n - f\| \leq \epsilon$. By (1.1), $\|P^t f - P^t f_n\| \leq \epsilon$ for all t . Now choose t so small that $\|P^t f_n - f_n\| \leq \epsilon$. Then $\|P^t f - f\| \leq 3\epsilon$, so that f is also in $\mathcal{V}$, and $\mathcal{V}$ is closed. If f is in $\mathcal{V}$ then $P^t f$ is in $\mathcal{V}$, since

$$\|P^s P^t f - P^t f\| = \|P^t(P^s f - f)\| \leq \|P^s f - f\|.$$

Thus $\mathcal{V}$ is a closed linear subspace of $\mathcal{C}$, invariant under P^t , and P^t is strongly continuous on $\mathcal{V}$. $\mathcal{V}$ is the maximal subspace of $\mathcal{C}$ with these properties.

However, $\mathcal{V}$ need not be a sub-algebra of $\mathcal{C}$. For example, let

$$P^t = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad f = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

for all $t \geq 0$. Then P^t is a Markovian semigroup on $\mathcal{C}(X)$, where X has three points. The function f is in $\mathcal{V}$ but f^2 is not in $\mathcal{V}$.

Let $\mathcal{A}$ be the set of all f in $\mathcal{C}$ such that f and f^2 are in $\mathcal{V}$.

Theorem 1.2. $\mathcal{A}$ is a closed sub-algebra of $\mathcal{C}$ and is the maximal sub-algebra of $\mathcal{C}$ continued in $\mathcal{V}$.

Proof. Imbed $\mathcal{C}$ in $\tilde{\mathcal{C}}$, as in Theorem 1.1, and let $\tilde{\mathcal{V}}$, $\tilde{\mathcal{A}}$ be defined as above for $\tilde{P}^t$ on $\tilde{\mathcal{C}}$. Then f is in $\mathcal{V}$ if and only if each $f \oplus \alpha$ is in $\tilde{\mathcal{V}}$, and f is in $\mathcal{A}$ if and only if each $f \oplus \alpha$ is in $\tilde{\mathcal{A}}$, since $(f \oplus \alpha)^2 = (f^2 + 2\alpha f) \oplus \alpha^2$ and each $f^2 + 2\alpha f$ is in $\mathcal{V}$ if and only if f is in $\mathcal{A}$. Thus it is sufficient to prove the theorem for the case that $\mathcal{C}$ has a unit 1 and $P^t 1 = 1$.

In this case, let $\mathcal{C} = \mathcal{C}(X)$, let x be in X , and f in $\mathcal{C}$. Then $(f - f(x))^2 = f^2 - 2f(x)f + f(x)^2$ is in $\mathcal{V}$. (Note that $f(x)$ denotes a constant.) In particular,

$P^t(f - f(x))^2 \cdot (x) \rightarrow 0$ as $t \rightarrow 0$. We shall need to show that this occurs uniformly in x .

Let $\epsilon > 0$. Then there is a $\delta > 0$ (depending on x , ϵ , and f) such that, for all $t \leq \delta$,

$$(1.8) \quad \|P^t(f - f(x))^2 - (f - f(x))^2\| \leq \epsilon,$$

$$(1.9) \quad 4 \|f\| \|P^t f - f\| \leq \epsilon,$$

since $(f - f(x))^2$ and f are in $\mathfrak{U}$. Let

$$d_x = P^t(f - f(x))^2 - (f - f(x))^2 = (P^t f^2 - f^2) - 2f(x)(P^t f - f).$$

By (1.8), $-\epsilon \leq d_x \leq \epsilon$. If y is also in X , then $\|d_x - d_y\| = \|2(f(x) - f(y))(P^t f - f)\| \leq \epsilon$ by (1.9). Therefore $-2\epsilon \leq d_y \leq 2\epsilon$ for all y in X . In particular, $|d_y(y)| \leq 2\epsilon$ for all y in X , so that $P^t(f - f(y))^2 \cdot (y) \rightarrow 0$ uniformly in y as $t \rightarrow 0$. Since $(f - f(y))^2 \geq 0$, this means that for each $\epsilon > 0$

$$(1.10) \quad p^t(y, D(\epsilon, y)) \rightarrow 0$$

uniformly in y , as $t \rightarrow 0$, where

$$(1.11) \quad D(\epsilon, y) = \{z : |f(z) - f(y)| > \epsilon\}.$$

In fact,

$$P^t(f - f(y))^2 \cdot (y) = \int (f(z) - f(y))^2 p^t(y, dz) \geq \epsilon^2 p^t(y, D(\epsilon, y)).$$

Now let f be in $\mathfrak{A}$ and g in $\mathfrak{U}$. Then

$$\begin{aligned} (P^t(fg) - fg) \cdot (y) &= \int (f(z)g(z) - f(y)g(y))p^t(y, dz) \\ &= \int (f(z) - f(y))g(z)p^t(y, dz) + \int f(y)(g(z) - g(y))p^t(y, dz). \end{aligned}$$

The second term tends to 0 uniformly in y , since g is in $\mathfrak{U}$ and f is bounded. As to the first term, its absolute value is bounded by $\|g\| \int |f(z) - f(y)| p^t(y, dz)$, which tends to 0 uniformly in y by (1.10). Consequently, fg is in $\mathfrak{U}$. Therefore $\mathfrak{A}$ is the set of all f in $\mathfrak{U}$ such that fg is in $\mathfrak{U}$ whenever g is in $\mathfrak{U}$. Since $\mathfrak{U}$ is a linear space, this means that $\mathfrak{A}$ is an algebra. (In fact, if f_1 and f_2 are in $\mathfrak{A}$, we must have $(f_1 + f_2)g = f_1g + f_2g$ in $\mathfrak{U}$, and $(f_1 f_2)g = f_1(f_2g)$ is in $\mathfrak{U}$ since f_2g is in $\mathfrak{U}$.) If f_n is in $\mathfrak{A}$ and $f_n \rightarrow f$, then, for each g in $\mathfrak{U}$, $f_n g \rightarrow fg$, so that fg is also in $\mathfrak{U}$, since $\mathfrak{U}$ is closed. Thus $\mathfrak{A}$ is a closed subalgebra of $\mathfrak{C}$. By definition of $\mathfrak{A}$, it is then the maximal subalgebra of $\mathfrak{C}$ contained in $\mathfrak{U}$, concluding the proof.

The algebra $\mathfrak{A}$ is not necessarily invariant under P^t , as the following example shows. The description is in intuitive terms, but there is no difficulty in stating the formulas involved. Let X consist of two copies of $(-\infty, 0] \cup [1, \infty)$. A particle in the process moves to the right at uniform speed until it hits 0. It then transfers to the other copy with probability $\frac{1}{2}$, and in either case continues

moving to the right from 1 with uniform speed. This gives a Markovian semigroup P^t on $\mathcal{C}(X)$. In this example the algebra $\mathcal{A}$ consists of those functions in $\mathcal{C}(X)$ which take the same value on the four points 0, 1. However, $\mathcal{A}$ is not invariant under P^t . The spectrum X_0 of the algebra $\mathcal{A}$ is X with the four points 0, 1 identified. The process considered on X_0 is not Markovian, for if the particle at a given time is at the crucial point, more knowledge about the future is obtained by knowing which copy of $(-\infty, 0]$ the particle was in during the past.

A satisfactory situation is obtained if a regularity condition of an algebraic nature is satisfied, namely

$$(1.12) \quad \text{if } f \text{ is in } \mathcal{V} \text{ then } f^2 \text{ is in } \mathcal{V}.$$

If (1.12) is satisfied, then $\mathcal{A} = \mathcal{V}$ and $\mathcal{A}$ is invariant under P^t . Then we may let X be the spectrum of $\mathcal{A}$ and P^t is represented as a strongly continuous semigroup on $\mathcal{C}(X)$.

2. The infinitesimal generator. It will be assumed from now on that P^t is a Markovian semigroup on $\mathcal{C}(X)$, where X is a compact Hausdorff space. The general case may be treated by means of Theorem 1.1. Let A be the infinitesimal generator of P^t in the sense of HILLE [2] and let $\mathcal{D}$ be its domain. That is, $\mathcal{D}$ consists of all f in $\mathcal{C}(X)$ such that

$$(2.1) \quad \lim_{t \rightarrow 0} t^{-1}(P^t f - f)$$

converges in $\mathcal{C}(X)$ to a function f_0 as $t \rightarrow 0$, and Af is defined to be f_0 . The space $\mathcal{D}$ is dense in the space $\mathcal{V}$ of all f such that $\|P^t f - f\| \rightarrow 0$ as $t \rightarrow 0$, and is itself a Banach space in the norm

$$(2.2) \quad \|f\|_{\mathcal{D}} = \|f\| + \|Af\|,$$

since A is a closed operator (see [2]).

We wish to establish a representation of the infinitesimal generator A similar to that of HUNT [3] for the spatially homogeneous case. An example will be given later to show that such a representation by means of a family of linear functionals attached to the points of X does not always hold on the domain $\mathcal{D}$ itself.

Let $\mathfrak{N}$ be the set of all f such that fg is in $\mathcal{D}$ whenever g is in $\mathcal{D}$. Since $P^t 1 = 1$, 1 is in $\mathcal{D}$ (with $A1 = 0$), and so $\mathfrak{N}$ is contained in $\mathcal{D}$. The set $\mathfrak{N}$ is clearly an algebra. Let

$$(2.3) \quad \|f\|_{\mathfrak{N}} = \sup_{g \in \mathcal{D}, g \neq 0} \|fg\|_{\mathcal{D}} / \|g\|_{\mathcal{D}},$$

so that $\|f\|_{\mathfrak{N}}$ is the norm of the operator on $\mathcal{D}$ given by $g \rightarrow fg$.

Theorem 2.1. $\mathfrak{N}$ is a Banach algebra in the norm (2.3).

Proof. The linear operator $g \rightarrow fg$ is everywhere defined on the Banach space $\mathcal{D}$. If $g_n \rightarrow g$ in $\mathcal{D}$ and $fg_n \rightarrow h$ in $\mathcal{D}$ then *a fortiori* $g_n \rightarrow g$ and $fg_n \rightarrow h$ in $\mathcal{C}(X)$,

so that $h = fg$. Therefore multiplication by f is a closed operator. By the closed graph theorem, it is a bounded operator and $\|f\|_{\mathfrak{M}} < \infty$. If f_n is a Cauchy sequence in $\mathfrak{M}$ it is also a Cauchy sequence in $\mathcal{C}(X)$. (In fact, by (2.3) with $g = 1$ and (2.2),

$$(2.4) \quad \|f\|_{\mathfrak{M}} \geq \|f\|_{\mathfrak{D}} \geq \|f\|$$

so that f_n is even a Cauchy sequence in $\mathfrak{D}$.) Let f be the limit in $\mathcal{C}(X)$ of f_n . Then, for g in $\mathfrak{D}$, $f_n g$ is a Cauchy sequence in $\mathfrak{D}$ with limit fg , so that fg is in $\mathfrak{D}$ and f is in $\mathfrak{M}$. Therefore $\mathfrak{M}$ is complete. Since $\|\cdot\|_{\mathfrak{M}}$ is the operator norm and $\|1\|_{\mathfrak{M}} = 1$ by (2.3), $\mathfrak{M}$ is a Banach algebra, concluding the proof. In case $\mathfrak{D}$ itself is an algebra, then $\|\cdot\|_{\mathfrak{M}}$ and $\|\cdot\|_{\mathfrak{D}}$ are equivalent norms, by the closed graph theorem and (2.4).

Most of the literature on Banach algebras is concerned with complex Banach algebras. If we allow all functions above to be complex-valued then M becomes a self-adjoint complex Banach algebra with complex conjugation as involution. There is no important difference involved, and we shall deal with real-valued functions only.

If x is in X then $f \rightarrow f(x)$ is a multiplicative linear functional on $\mathfrak{M}$, and if $f(x) = 0$ for all x then $f = 0$. Consequently, $\mathfrak{M}$ is a semi-simple Banach algebra. It will be shown (see Corollary 2.1) that every multiplicative linear functional on $\mathfrak{M}$ arises in this way.

Theorem 2.2. *If ϕ is a twice continuously differentiable real function of a real variable and f is in $\mathfrak{M}$ then $\phi(f)$ is in $\mathfrak{M}$.*

Proof. Let I be a bounded interval. By the Weierstrass approximation theorem there is a sequence of polynomials ψ_n converging uniformly on I to ϕ'' . Consequently, there is a sequence of polynomials ϕ_n (with $\phi_n'' = \psi_n$) such that ϕ_n converges to ϕ in the C^2 norm on I . Let ρ_n be defined by

$$(2.4) \quad \phi_n(s) = \phi_n(r) + \phi_n'(r)(s - r) + \frac{1}{2}\phi_n''(r)(s - r)^2 + \rho_n(r, s).$$

If $\epsilon > 0$ then

$$(2.5) \quad |\rho_n(r, s) - \rho_m(r, s)| \leq \epsilon(s - r)^2$$

for r and s in I , if n and m are large enough to ensure that $|\phi_n'' - \phi_m''| \leq \frac{1}{2}\epsilon$ on I . In fact,

$$(s - r)^{-2}(\rho_n(r, s) - \rho_m(r, s)) = (s - r)^{-2} \int_r^s \int_r^u (\phi_n''(v) - \phi_m''(v)) dv du - \frac{1}{2}(\phi_n''(r) - \phi_m''(s)),$$

which is in absolute value less than ϵ .

If f is in $\mathfrak{M}$ and g is in $\mathfrak{D}$ then $\phi_n(f)g$ is in $\mathfrak{D}$ since ϕ_n is a polynomial, and $\phi_n(f)g \rightarrow \phi(f)g$ in $\mathcal{C}(X)$. Since $\mathfrak{D}$ is the domain of the closed operator A , we need only show that $A(\phi_n(f)g)$ converges in $\mathcal{C}(X)$ to know that $\phi(f)g$ is in $\mathfrak{D}$

for all g in $\mathfrak{D}$, and hence that $\phi(f)$ is in $\mathfrak{M}$. To prove that $A(\phi_n(f)g)$ converges in $\mathfrak{C}(X)$, choose the interval I large enough to contain the range of f . Letting $s = f$, $r = f(x)$ in (2.4) we have

$$(2.6) \quad \begin{aligned} \phi_n(f)g &= \phi_n(f(x))g + \phi'_n(f(x))(f - f(x))g \\ &\quad + \frac{1}{2}\phi''_n(f(x))(f - f(x))^2g + \rho_n(f(x), f)g' \end{aligned}$$

so that

$$(2.7) \quad A(\phi_n(f)g - \phi_m(f)g) = d_{nm} + Ar_{nm}$$

where

$$(2.8) \quad r_{nm} = (\rho_n(f(x), f)g - \rho_m(f(x), f)g)$$

and d_{nm} tends uniformly to 0 as n and m tend to ∞ , since $|\phi_n - \phi_m| + |\phi'_n - \phi'_m| + |\phi''_n - \phi''_m|$ tends uniformly to 0 on I . Let $\epsilon > 0$ and suppose n and m large enough to ensure (2.5). Then $|P^t r_{nm}| \leq \epsilon \|g\| P^t(f - f(x))^2$. Consequently,

$$|Ar_{nm}(x)| \leq \epsilon \|g\| A(f - f(x))^2(x) \leq \epsilon \|g\| \sup_y A(f - f(y))^2(y).$$

Since ϵ is arbitrarily small, this proves the uniform convergence of $A(\phi_n(f)g)$, and concludes the proof of the theorem.

Corollary 2.1. *$\mathfrak{M}$ is a regular Banach algebra. Every multiplicative linear functional on $\mathfrak{M}$ is of the form $f \rightarrow f(x)$ for some x in X .*

Proof. Let W be the set of all multiplicative linear functionals on $\mathfrak{M}$; i.e. the Gelfand maximal ideal space or spectrum of $\mathfrak{M}$. Let $\tilde{f}(w) = w(f)$ for all f in $\mathfrak{M}$ and w in W . The proof of Theorem 2.2 shows that if ϕ is of class C^2 then $(\phi(f))^\sim = \phi(\tilde{f})$, since $\phi(f)$ may be approximated in $\mathfrak{M}$ by polynomials in f . That is, the algebra $\tilde{\mathfrak{M}}$ of all $\tilde{f}$ is also closed under the application of C^2 functions. Let W_0 be a compact subset of W and w a point in $W - W_0$. Since $\tilde{\mathfrak{M}}$ is a real algebra which separates points, there is an $\tilde{f}$ such that $\tilde{f} \leq \frac{1}{2}$ on W_0 and $\tilde{f}(w) \geq 1$, by the Stone-Weierstrass theorem. Let ϕ be a function of class C^2 which is 0 on $(-\infty, \frac{1}{2}]$ and 1 on $[1, \infty)$. Then $\phi(\tilde{f})$ is in $\tilde{\mathfrak{M}}$ and $\phi(\tilde{f})$ is 0 on W_0 and 1 at w . The existence of such a function in $\tilde{\mathfrak{M}}$ means, by definition, that $\mathfrak{M}$ is a regular Banach algebra.

For each x in X let $\pi(x)$ in W be the multiplicative linear functional $f \rightarrow f(x)$. By definition of the topology of W , π is a continuous mapping of X into W , so that $\pi(X)$ is a compact set in W . Suppose there is a point w in $W - \pi(X)$. Then by regularity there is an f in $\mathfrak{M}$ such that $\tilde{f} = 0$ on $\pi(X)$ and $\tilde{f}(w) = 1$. But $f = 0$ on $\pi(X)$ means $f(x) = 0$ for all x in X , so that $f = 0$, $\tilde{f}(w) = w(f) = 0$, and we have a contradiction. This concludes the proof.

An example will be given later to show that $\mathfrak{M}$ need not separate the points of X even if P^t is strongly continuous on $C(X)$.

For x in X , let $\mathcal{I}_x$ be the set of all f in $\mathfrak{M}$ such that $f(x) = 0$. Thus $\mathcal{I}_x$ is a maximal ideal in $\mathfrak{M}$. By $\mathcal{I}_x^k$ is meant the ideal generated by functions of the

form $f_1 \cdots f_k$ with $f_1, \dots, f_k$ in $\mathcal{G}_x$. A function in $\mathcal{G}_x^k$ may be said to have a zero of order k at x . The ideal $\mathcal{G}_x^k$ is not necessarily closed.

Theorem 2.3. *For each x in X there is a unique regular Borel measure $a(x, \cdot)$ on the spectrum $W_x = \pi(X) - \{\pi(x)\}$ of $\mathcal{G}_x$ such that*

$$(2.9) \quad Ah.(x) = \int \tilde{h}(w)a(x, dw)$$

for all h in $\mathcal{G}_x^3$.

If $\mathfrak{N}$ separates the points of X , i.e. $\pi(X) = X$, then $a(x, \cdot)$ is a measure on $X - \{x\}$ itself, and $Ah.(x) = \int h(y) a(x, dy)$. In this case, we may forget the distinction between h and $\tilde{h}$.

Proof. Let $\mathcal{G}_x^+$ be the subset of $\mathcal{G}_x$ consisting of positive functions. If f is in $\mathcal{G}_x^+$ then $Af.(x) \geq 0$ since $Af.(x) = \lim_{t \rightarrow 0} t^{-1}P^t f(x)$. If f is in $\mathcal{G}_x^+$ then the linear functional $g \rightarrow A(fg).(x)$ is positive on $\mathcal{G}_x$ and

$$(2.10) \quad |A(fg).(x)| \leq Af.(x) \|g\|$$

since $|P^t(fg).(x)| \leq P^t f.(x) \|g\|$ for each g . By the Riesz-Markoff theorem there is a finite positive regular Borel measure $a_f(x, \cdot)$ on W_x such that

$$(2.11) \quad A(fg).(x) = \int \tilde{g}(w)a_f(x, dw).$$

By the two theorems in §26I and §26J of [4], there is a regular Borel measure $a(x, \cdot)$ (positive but not necessarily finite) on W_x such that

$$(2.12) \quad A(fg).(x) = \int \tilde{f}(w)\tilde{g}(w)a(x, dw)$$

whenever f is in $\mathcal{G}_x^+$ and g is in $\mathcal{G}_x$.

If f is an arbitrary element of $\mathcal{G}_x$ then f^2 is in $\mathcal{G}_x^+$, so that (2.9) holds for all h of the form $f^2 g$, where f and g are in $\mathcal{G}_x$. If e is also in $\mathcal{G}_x$, then (2.9) holds also for h of the form $efg = \frac{1}{2}(e+f)^2 g - \frac{1}{2}e^2 g - \frac{1}{2}f^2 g$. Since (2.9) also holds for linear combinations of functions of this form, it holds for all h in $\mathcal{G}_x^3$, concluding the proof of the theorem.

Theorem 2.3 establishes the representation of $Af(x)$ for functions f which are sufficiently smooth ($f \in \mathfrak{N}$) and which have a zero at x of the third order ($f \in \mathcal{G}_x^3$). This is the global or integral part of A . It remains to find the second and first order parts of A . There is no zero order part since $A1 = 0$.

For each x in X , let V_x be the quotient vector space $\mathcal{G}_x/\mathcal{G}_x^2$. For f in $\mathfrak{N}$, let

$$(2.13) \quad df(x) = (f - f(x)) + \mathcal{G}_x^2$$

be the equivalence class of $(f - f(x))$ in V_x . We may think of V_x as the tangent space at x and of df as the gradient of f . It is easy to see that if $g = f$ in a neighborhood of x then $dg(x) = df(x)$.

Theorem 2.4. *If h is in $\mathcal{S}_x^2$ then h is $a(x, \cdot)$ -integrable. The linear functional*

$$(2.14) \quad \Delta h \cdot (x) = Ah \cdot (x) - \int \tilde{h}(w)a(x, dw)$$

defined for h in $\mathcal{S}_x^2$ depends only on the values of h in a neighborhood of x , and

$$(2.15) \quad \langle df(x), dg(x) \rangle = \frac{1}{2} \Delta((f - f(x))(g - g(x))) \cdot (x)$$

is a well defined symmetric positive semi-definite bilinear form on V_x .

Proof. By (2.10),

$$(2.16) \quad \int \tilde{f}(w)^2 a(x, dw) \leq Af^2 \cdot (x)$$

for all f in $\mathcal{S}_x$, and, consequently, each linear combination of functions of the form $\tilde{f}^2$ is $a(x, \cdot)$ -integrable, proving the first statement in the theorem. If h is zero in a neighborhood of x then by regularity there is a g in I_x which is 1 on the support of h , so that $h = hg^k$ is in $\mathcal{S}_x^k$ for all k . (The statement preceding Theorem 2.4 follows from this fact for $k = 2$.) Since h is in $\mathcal{S}_x^3$, by Theorem 2.3, $\Delta h \cdot (x) = 0$. By linearity, $\Delta h \cdot (x)$ depends only on the values of h in a neighborhood of x .

If $h = f^2$, with f in $\mathcal{S}_x$, then $\Delta h \cdot (x) \geq 0$ by (2.16) and (2.14). Therefore the right hand side of (2.15) is a symmetric positive semi-definite bilinear form. If f, g, f' , and g' are such that $df(x) = df'(x)$, $dg(x) = dg'(x)$ then $(f - f(x))(g - g(x))$ and $(f' - f'(x))(g' - g'(x))$ are congruent modulo $\mathcal{S}_x^4$, by (2.13), and *a fortiori* modulo $\mathcal{S}_x^3$. By Theorem 2.3, Δ annihilates their difference. That is, the right hand side of (2.15) depends only on $df(x)$ and $dg(x)$, and the form is well defined on V_x , concluding the proof.

The bilinear form (2.15) is somewhat analogous to the ds^2 of a Riemannian metric (hence the factor $\frac{1}{2}$), and Δ serves as the Laplace-Beltrami operator, but is defined only for functions with zero gradient at x .

It was possible to define Δ on $\mathcal{S}_x^2$ because the $a(x, \cdot)$ -integral could be extended to $\mathcal{S}_x^2$. If there were some natural way of extending the $a(x, \cdot)$ -integral and Δ itself to $\mathcal{S}_x$, then we could define the first order part of A as the difference of A and these two extended functionals. As it is, we have the following theorem.

Theorem 2.5. *If $a(x, \cdot) = 0$ and Δ is 0 on $\mathcal{S}_x^2$, then*

$$A(fg) \cdot (x) = f(x)Ag \cdot (x) + g(x)Af \cdot (x)$$

for all f and g in $\mathfrak{M}$. There is a linear functional Λ on V_x such that

$$(2.17) \quad Ah \cdot (x) = \Lambda(dh(x))$$

for all h in $\mathfrak{M}$.

Proof. By Theorems 2.3 and 2.4, $Ah \cdot (x) = 0$ for h in $\mathcal{S}_x^2$. Since $A((f - f(x))(g - g(x))) = 0$ and $A(f(x)g(x)) = 0$ ($f(x)g(x)$ being a constant), we have $A(fg) \cdot (x) = f(x)Ag \cdot (x) + g(x)Af \cdot (x)$. If $dh(x) = dh'(x)$ then $h - h'$ is in

$\mathcal{G}_x^2$, so that $Ah(x) = Ah'(x)$, and $Ah(x)$ is a well defined linear functional on V_x , concluding the proof.

Theorem 2.6. *If the $df_i(x)$ form a basis for V_x , then, for all f in $\mathfrak{M}$, if $df(x) = \sum a_i df_i(x)$, let $g = f - f(x) - (\sum a_i f_i - \sum a_i f_i(x))$. Then*

$$(2.18) \quad Af \cdot(x) = \int \tilde{g}(w) a(x, dw) + \Delta g \cdot(x) + \sum a_i Af_i(x).$$

This is an immediate consequence of Theorems 2.3 and 2.4. It should be remarked that the formula of HUNT [3] (as well as the classical formula of LÉVY and others) also has the feature that a function with the same gradient as f is subtracted from f before the integration is performed. The measure G of [3] is not our $a(0, \cdot)$, since it also involves a first order convergence-producing term.

We conclude by discussing briefly several examples.

Example 1. Let R be the real line, x its one-point compactification by ∞ . Let a be a positive Lebesgue-integrable function on R , let

$$(2.19) \quad \alpha(x, t) = \exp \left\{ - \int_0^t a(x + u) du \right\}, \quad x \neq \infty,$$

and let

$$(2.20) \quad \begin{aligned} P^t f \cdot(x) &= \alpha(x, t) f(x + t) + (1 - \alpha(x, t)) f(\infty), \quad x \neq \infty, \\ P^t f \cdot(\infty) &= f(\infty). \end{aligned}$$

If f is in $\mathcal{C}(X)$ then $P^t f$ is in $\mathcal{C}(X)$, since, for each t , $\alpha(x, t)$ is a continuous function of x which tends to 1 as x tends to ∞ . If $f \geq 0$ then $P^t f \geq 0$, since $0 \leq \alpha(x, t) \leq 1$. Also $P^t 1 = 1$. Let

$$(2.21) \quad \epsilon(t) = \sup_{x \neq \infty} (1 - \alpha(x, t)).$$

Then $\epsilon(t) \rightarrow 0$ as $t \rightarrow 0$. If $x = \infty$ then $|P^t f(x) - f(x)| = 0$. If $x \neq \infty$, then, for f in $\mathcal{C}(X)$, the triangle inequality gives

$$\begin{aligned} |P^t f \cdot(x) - f(x)| &\leq |P^t f \cdot(x) - f(x + t)| + |f(x + t) - f(x)| \\ &\leq 2\epsilon(t) |f| + |f(x + t) - f(x)| \end{aligned}$$

which tends to 0 uniformly in x as t tends to 0, so that P^t is strongly continuous. A simple calculation shows that $P^t P^s = P^{t+s}$, so that P^t is a strongly continuous Markovian semigroup on $\mathcal{C}(X)$.

Let A be the infinitesimal generator of P^t , $\mathfrak{D}$ its domain. Let $f \geq 0$, f in $\mathfrak{D}$, $f(x) = 0$. Then $P^t f \cdot(x) - f(x) = P^t f \cdot(x) \geq (1 - \alpha(x, t)) f(\infty)$, so that if $\tilde{a}(x) = \limsup_{t \rightarrow 0} (1 - \alpha(x, t))/t$, then $Af \cdot(x) \geq \tilde{a}(x) f(\infty)$. If $\tilde{a}(x) = \infty$, then $f(\infty)$ must be 0 since f is in $\mathfrak{D}$. That is, for such a point x , any positive function in the domain of the infinitesimal generator which vanishes at x must also vanish at a certain other point in the space. If f and f^2 are both in $\mathfrak{D}$ and $\tilde{a}(x) = \infty$, then we

must have $f(\infty) = f(x)$, since $(f - f(x))^2$ is a positive function in $\mathfrak{D}$ vanishing at x . There exists an integrable function a such that $\tilde{a}(x) = \infty$ for a dense set of points x . In this case, the algebra $\mathfrak{M}$ reduces to the constant functions. *For a dense set of points x , there exists a neighborhood $N(x)$ of x such that $p^t(x, X - N(x))$ is not $O(t)$.*

Example 2. P^t is the semigroup corresponding to the heat equation on $[0, 1]$ with boundary conditions $f'(0) = 0, f'(1) = 0$. In this case, $\mathfrak{D}$ is the set of all C^2 functions satisfying the boundary conditions, and is itself an algebra, $\mathfrak{M} = \mathfrak{D}$. The bilinear form (2.15) is $f'(x)g'(x)$, and so is the 0 form on V_0 and V_1 .

Example 3. P^t is the semigroup corresponding to the heat equation on $[0, 1]$ with boundary conditions $f''(0) = 0, f''(1) = 0$ (see [1], p. 210). In this case, $\mathfrak{D}$ is the set of all C^2 functions satisfying the boundary conditions, and is not an algebra. It is easily seen that $\mathfrak{M}$ consists of all f in $\mathfrak{D}$ which, in addition, satisfy $f'(0) = 0, f'(1) = 0$. Thus $\mathfrak{M}$ separates the points of $[0, 1]$. The bilinear form (2.15) is $f'(x)g'(x)$, and is 0 on V_0 and V_1 .

REFERENCES

- [1] W. FELLER, *On differential operators and boundary conditions*, Communications on Pure and Applied Math. **8** (1955), pp. 203–216.
- [2] E. HILLE, *Functional analysis and semi-groups*, American Math. Soc. Colloquium Publications 31, New York (1948).
- [3] G. A. HUNT, *Semi-groups of measures on Lie groups*, Trans. American Math. Soc. **81** (1956), pp. 264–293.
- [4] LYNN H. LOOMIS, *An introduction to abstract harmonic analysis*, New York (1953).

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