Some Compact Solvmanifolds and Locally Affine Spaces

LOUIS AUSLANDER

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INTRODUCTION

Much of the early algebraic work in the study of Lie groups was centered about the classification of the simple Lie groups and related topics. In contradistinction to this, MALCEV in [5] initiated a study of nilpotent Lie groups. This work deals mainly with the existence of closed subgroups of nilpotent Lie groups such that the homogeneous space is compact. (We will often refer to the homogeneous space of a nilpotent Lie group as a nilmanifold.) More explicitly, MALCEV in [5] provides solutions to the following three problems:

Problem 1. Given a connected, simply connected, nilpotent Lie group N , when does there exist a closed subgroup Γ such that N/Γ is compact?

Problem 2. Give an abstract group theoretic characterization of the fundamental groups of compact nilmanifolds.

Problem 3. Given two compact nilmanifolds with isomorphic fundamental groups, are the manifolds homeomorphic?

There has been some work in recent years [4], [6] trying to extend MALCEV's results to compact homogeneous spaces of solvable Lie groups or, more briefly, solvmanifolds. In [6], MOSROW showed that the answer to Problem 3 is in the affirmative for compact solvmanifolds. In [4], WANG gave a necessary and sufficient condition for a group Γ to be the fundamental group of a solvmanifold (not necessarily compact). It is the purpose of this paper to show that a certain class of groups Γ consists of fundamental groups of *compact* solvmanifolds. In other words, we give a partial solution of Problem 2 for the case of compact solvmanifolds.

In order to simplify the presentation of the material in this paper, we will adopt the following conventions: Z^s will denote the direct sum of the additive group of integers taken s times, R will denote the real field and C will denote the complex field. The symbols $Z^t \rightarrow Z^t \otimes R$ or $Z^t \rightarrow Z^t \otimes C$ will denote the

isomorphism which maps $z \in Z'$ onto $z \otimes 1 \in Z' \otimes R$ or $Z' \otimes C$. We will also always use the symbol Γ to denote a group satisfying the following diagram:

$$1 \rightarrow Z^s \rightarrow \Gamma \rightarrow Z' \rightarrow 1$$

where this is an exact sequence.

We will now state our main theorems.

Theorem A. *Given a group Γ there exists a solvable Lie group S satisfying the following commutative diagram:*

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^s & \rightarrow & \Gamma & \rightarrow & Z' & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Z^s \otimes C & \rightarrow & S & \rightarrow & Z' \otimes R & \rightarrow & 1 \end{array}$$

where the horizontal arrows denote an exact sequence and Γ is mapped isomorphically into S .

Theorem B. *Given a group Γ , then Γ is the fundamental group of a compact solvmanifold which we will call $S(\Gamma)$.*

In §5 we give a sufficient condition for a group Γ to be imbedded as a discrete subgroup of a connected, simply connected, solvable Lie group S such that S/Γ is compact. In §6 we study the groups Γ themselves and give a representation of such groups in terms of generators and relations.

The last part of this paper, §§7, 8, and 9, is devoted to the study of compact, complete, locally affine spaces with fundamental groups Γ . We will denote such spaces by A^n/Γ . By a locally affine space, we mean a manifold with a complete affine connection with curvature and torsion zero. Such a space can always be identified with the quotient space of the n -dimensional affine space A by a discrete group Γ of affine transformations.

Theorem C. *A^n/Γ is homeomorphic to $S(\Gamma)$.*

An immediate corollary of Theorem C is that two compact, complete, locally affine manifolds with isomorphic fundamental groups Γ are homeomorphic. To see that this class of locally affine spaces is not vacuous, one merely notes [1], where it is shown that every three-dimensional compact solvmanifold has such a representation. It should be pointed out that §§7, 8, and 9 can essentially be read independently of the rest of the paper. In fact, the methods in these paragraphs could be used to give an alternative proof of Theorem A.

1. Preliminaries to the proof of Theorem A. Given Γ satisfying the exact diagram

$$1 \rightarrow Z^s \rightarrow \Gamma \rightarrow Z' \rightarrow 1$$

we can extend the action of Z' on Z^s by linearity to give an action of Z' on $Z^s \otimes C$. By the standard theory of group extensions [2], there exists a homomorphism of

$H^2(Z^t, Z^s)$ into $H^2(Z^t, Z^s \otimes C)$. Hence the image in $H^2(Z^t, Z^s \otimes C)$ of the cohomology class which determines Γ determines a unique group G satisfying the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^s & \rightarrow & \Gamma & \rightarrow & Z^t \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & Z^s \otimes C & \rightarrow & G & \rightarrow & Z^t \rightarrow 1 \end{array}$$

where $\Gamma \rightarrow G$ is an isomorphism. Let the automorphism of $Z^s \otimes C$ induced by $z \in Z^t$ be denoted by $\text{ad}(z)$ and let $z_1, \dots, z_i$ be a basis for Z^t . Also, let u^* be an automorphism of $Z^s \otimes C$ such that $(u^*)^2 = \text{ad}(z_1)$ and let the group generated by u^* and $\{\text{ad}(z) \mid z \in Z^t\}$ be abelian. Further, let u be a 1-1 mapping of G onto itself which extends the mapping of $Z^s \otimes C$ onto itself given by u^* . Define $[u, g] = u(g)g^{-1}$ for $g \in G$ and $[g_1, g_2] = g_1g_2g_1^{-1}g_2^{-1}$ for g_1 and g_2 in G . Then it is easy to compute that a necessary and sufficient condition for u to be an automorphism of G is that

$$(1) \quad [u, g_1g_2] = [u, g_1]g_1[u, g_2]g_1^{-1}$$

Now define a mapping u of G into G which extends u^* by the equation

$$(2) \quad [\text{ad}(z_1), g] = u^*([u, g]) \cdot [u, g]$$

for all $g \in G$. Let us assume that $u^* + I$, where I is the identity automorphism of $Z^s \otimes C$, is non-singular. Then $u : g \rightarrow [u, g]g^{-1}$, where $[u, g]$ is defined by equation (2), will be a 1-1 mapping and extend the mapping u^* . We will show that equation (1) is also satisfied. *We will henceforth use additive notation whenever all the terms considered are in $Z^s \otimes C$.* We know that for $\text{ad}(z_1)$

$$(3) \quad [\text{ad}(z_1), g_1g_2] = [\text{ad}(z_1), g_1] + \text{ad}(g_1)[\text{ad}(z_1), g_2].$$

Substituting from equation (2) gives

$$(4) \quad (u^* + I)[u, g_1g_2] = (u^* + I)[u, g_1] + \text{ad}(g_1)(u^* + I)[\text{ad}(z_1), g_2].$$

Now since u^* and $\text{ad}(g_1)$ commute, we have

$$(5) \quad (u^* + I) \text{ad}(g_1) = \text{ad}(g_1)(u^* + I).$$

Combining equations (4) and (5) with the fact that $u^* + I$ has an inverse easily gives equation (3). We may summarize these results in the following lemma.

Lemma 1. *Let*

$$1 \rightarrow Z^s \otimes C \rightarrow G \rightarrow Z^t \rightarrow 1$$

be given and let $z_1, \dots, z_i$ be a basis for Z^t . Let u^ be an automorphism of $Z^s \otimes C$ such that $(u^*)^2 = \text{ad}(z_1)$, $u^* + I$ is an automorphism of $Z^s \otimes C$ and the group of automorphisms of $Z^s \otimes C$ generated by u^* and $\{\text{ad}(z) \mid z \in Z^t\}$ is abelian. If we define $u : G \rightarrow G$ by $u(g) = [u, g]g^{-1}$ for $g \in G$, where $[u, g]$ is defined by*

$$[\text{ad}(z_1), g] = u^*([u, g]) + [u, g],$$

then u is an automorphism of G extending u^* on $Z^s \otimes C$.

Lemma 2. *Let us assume that the hypothesis of Lemma 1 is satisfied; then there exists a group $G(1; \frac{1}{2})$ satisfying the following diagram:*

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 \rightarrow & Z^s \otimes C & \rightarrow & G & \rightarrow & Z^t & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \rightarrow & Z^s \otimes C & \rightarrow & G(1; \frac{1}{2}) & \rightarrow & Z^t & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 1 \rightarrow & 1 & \rightarrow & Z/(2) & \rightarrow & Z/(2) & \rightarrow 1 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 1 & & 1 & & 1 &
 \end{array}$$

where vertical and horizontal arrows denote exact sequences and $Z/(2)$ denotes the integers modulo 2.

Proof: Let Z^{t-1} denote the subgroup of Z^t generated by $z_2, \dots, z_t$. Let G' be the normal subgroup of G which is the preimage on G of Z^{t-1} . Then u , defined in Lemma 1, clearly is an automorphism of G' , since, for all $g \in G$, $u(g) = ag$, where $a \in Z^s \otimes C$ and $Z^s \otimes C \subset G'$. Now let Z act on G' by powers of the automorphism u . Then, by the general theory of group extensions over non-abelian kernels [3], there is one and only one group $G(1; \frac{1}{2})$ which can satisfy

$$1 \rightarrow G' \rightarrow G(1; \frac{1}{2}) \rightarrow Z \rightarrow 1$$

for this action of Z on G' . Clearly if Z acts on G' by even powers of u , we have

$$1 \rightarrow G' \rightarrow G \rightarrow Z \rightarrow 1.$$

The above statements easily imply the lemma.

2. Extending automorphisms.

Lemma 3. *Let A be a finitely generated abelian subgroup of $GL(n, C)$, where A has generators $a_1, \dots, a_k$ and $GL(n, C)$ denotes the complex general linear group. There then exist one-parameter subgroups $L_i(\tau)$ through a_i , $i = 1, \dots, k$, such that the group generated by the $L_i(Z)$ for all i and Z is abelian, $L_i(1) = a_i$ and $L_i(\xi_1) \cdot L_i(\xi_2) = L_i(\xi_1 + \xi_2)$. Further, $L_i(1/2^n) \neq I$, where I is the identity element of $GL(n, C)$, is non-degenerate for all i and n .*

Proof: Since A is abelian, we can simultaneously reduce its elements to the form

$$\begin{bmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_t \end{bmatrix}$$

where each matrix N_α , $\alpha = 1, \dots, t$, is triangular with only one value on the main diagonal. Hence if we show how to construct a one-parameter group for matrices of the form

$$N_\alpha = \begin{bmatrix} \zeta & & * \\ & \ddots & \\ 0 & & \zeta \end{bmatrix},$$

we can extend them in the obvious way to give a definition of $L_i(\tau)$. Now

$$\begin{bmatrix} \zeta & & * \\ & \ddots & \\ 0 & & \zeta \end{bmatrix} = \begin{bmatrix} |\zeta| & & 0 \\ & \ddots & \\ 0 & & |\zeta| \end{bmatrix} \begin{bmatrix} e^\theta & & 0 \\ & \ddots & \\ 0 & & e^\theta \end{bmatrix} \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

If we consider the multiplicative group of reals there is a unique one-parameter group from 1 to $|\rho|$. Similarly, choosing an orientation for the circle group, there is a unique one-parameter group from the origin to e^θ which has the same orientation as that assigned to the circle. This gives a one-parameter group from 1 to e^θ . Since the last matrix is inside a nilpotent Lie group there is a unique one-parameter subgroup to

$$\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix},$$

and this one-parameter subgroup will have entries which are polynomials in the group parameter off the diagonal. We may extend this construction to each N_α in a_i . This will construct one-parameter groups $L_i(\tau)$, and it is easy to make $L_i(1) = a_i$. Further, $L_i(1/2^n) + I$ will be non-singular provided e^θ does not equal -1 in some N_α . But under our assignment e^θ can never equal -1 at the square root of any number.

Hence all that remains is to show that for any two one-parameter groups $L_i(\tau)$ and $L_j(\tau)$ all the elements commute. This easily reduces to verifying that

$$\begin{bmatrix} \zeta(s) & & 0 \\ & \ddots & \\ 0 & & \zeta(s) \end{bmatrix} \begin{bmatrix} 1 & & g_{ij}(s) \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \mu(t) & & 0 \\ & \ddots & \\ 0 & & \mu(t) \end{bmatrix} \begin{bmatrix} 1 & & k_{ij}(t) \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

commute if they commute for all integral values of the group parameters s and t , where $g_{ij}(s)$ and $k_{ij}(t)$ are polynomials in the group parameters. Now the commutator for any values of s and t gives

$$\begin{bmatrix} 1 & & h_{ij}(s, t) \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

where the $h_{ij}(s, t)$ are polynomials in s and t . Since $h_{ij}(s, t)$ vanish for all integral values of s and t , they must identically vanish. Hence $L_i(s)$ and $L_i(t)$ commute element-wise. This proves the lemma.

3. Proof of Theorem A. Given a group Γ satisfying the diagram

$$1 \rightarrow Z^s \rightarrow \Gamma \rightarrow Z^t \rightarrow 1$$

we construct the group G satisfying the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^s & \rightarrow & \Gamma & \rightarrow & Z^t \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & Z^s \otimes C & \rightarrow & G & \rightarrow & Z^t \rightarrow 1 \end{array}$$

as in §1. We may now apply Lemma 3 where A is the image of Z^t in $GL(n, C)$ induced by the action of Z^t on $Z^s \otimes C$. Lemma 3 then gives us an extension of the action of Z^t on $Z^s \otimes C$ to an action of $Z^t \otimes R$ on $Z^s \otimes C$ satisfying the condition on $z_1 \otimes \frac{1}{2} \in Z^t \otimes R$ that is in the hypothesis to Lemma 2. We can therefore apply Lemma 2 to construct a group

$$1 \rightarrow Z^s \otimes C \rightarrow G(1; \frac{1}{2}) \rightarrow Z^t \rightarrow 1.$$

We can reapply the whole of the above argument successively to $z_2, \dots, z_t$ and we will have a group $G(1, \dots, t; \frac{1}{2}) = G(\frac{1}{2})$. By induction we can construct groups $G(1/2), \dots, G(1/2^n), \dots$. This system of groups will be a direct system with isomorphisms into as assigned by Lemma 2. Let S^* be the limit group of this direct system of groups. Then if $R'(2^w)$ denotes the diadic rationals taken t times, it is easy to see that S^* satisfies the diagram

$$1 \rightarrow Z^s \otimes C \rightarrow S^* \rightarrow R'(2^w) \rightarrow 1.$$

Now let $Z^s \otimes C$ and $R'(2^w)$ be considered as topological groups with the usual topologies. Then the homomorphism of $R'(2^w)$ in $GL(n, C)$ induced by the action of $R'(2^w)$ on $Z^s \otimes C$ is continuous. Hence S^* becomes a topological group. It is easy to see that this topology is metrizable and S^* is an incomplete metric space. Let S denote the completion of S^* . Then S is a topological group and satisfies the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^s & \rightarrow & \Gamma & \rightarrow & Z^t \rightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \rightarrow & Z^s \otimes C & \rightarrow & S & \rightarrow & Z^t \otimes R \rightarrow 1 \end{array}$$

This completes the proof of Theorem A.

4. Proof of Theorem B. In S we may consider the subgroup $\mathfrak{g}$ of $Z^s \otimes C$ consisting of pure imaginary vectors. Then $\Gamma\mathfrak{g}$ is a closed subgroup of S . Now there is a general theorem that if S is homeomorphic to a euclidean space then the fundamental group of $S/\Gamma\mathfrak{g}$ is isomorphic to $\Gamma\mathfrak{g}/\mathfrak{g}$ or Γ . To see that $S/\Gamma\mathfrak{g}$ is compact, we merely have to note that it is a fibre bundle with a t -dimensional

torus as base space and an s -dimensional torus as fibre. This completes the proof of Theorem B.

5. Refinement of Theorem A.

Lemma 4. *Let*

$$1 \rightarrow Z^s \otimes R \rightarrow G \rightarrow Z^t \rightarrow 1$$

be given where Z^t acts linearly on $Z^s \otimes R$. We will denote this action of Z^t on $Z^s \otimes R$ by $\text{ad}(z)$, $z \in Z^t$. Assume $\{\text{ad}(z) \mid z \in Z^t\}$ can be extended to $\{\text{ad}_1(z \otimes r) \mid z \otimes r \in Z^t \otimes R\}$ acting on $Z^s \otimes R$. Let $z_1, \dots, z_i$ be a basis for Z^t . Then $\text{ad}(z)$ can be extended to $\text{ad}_2(z \otimes r)$ acting on $Z^s \otimes R$ in such a way that $\text{ad}_2(z_i \otimes 1/2^n) + I$, where I is the identity automorphism, is non-degenerate for all i and n .

Proof: Over the real field $\text{ad}_1(z) = x \cdot y$ for $z \in \text{ad}_1(Z^t \otimes R)$, where x is semi-simple, y is nilpotent and $xy = yx$. This decomposition is unique. Then if $\text{ad}_1(z_1) = x_1 y_1$, since z and z_1 commute, we must have that all the x, y, x_1, y_1 commute pair-wise. Further, if these matrices commute then $\text{ad}_1(z_1)$ and $\text{ad}_1(z)$ will commute. Let $z_1, \dots, z_i$ be a basis for Z^t . Let $\text{ad}_1(z_i \otimes R) = x_i(s) \cdot y_i(s)$. We will define $\text{ad}_2(z_i \otimes R) = x_i^*(s) \cdot y_i(s)$, where $x_i^*(s)$ is defined as follows: Over the complex field

$$x_i(s) = \begin{bmatrix} \zeta_1(s) & 0 \\ & \ddots \\ 0 & \zeta_i(s) \end{bmatrix}.$$

Define

$$x_i^*(s) = \begin{bmatrix} \mu_1(s) & 0 \\ & \ddots \\ 0 & \mu_i(s) \end{bmatrix}$$

where $\mu_i(s) = \rho_i(s)$ if $\rho_i(1) \neq 1$ and $\mu_i(s) = 1$ if $\rho_i(1) = 1$. It is then easy to verify that if $x_i(s)$ commutes with a matrix M then $x_i^*(s)$ will also commute with M and that $x_i^*(s)$ is a real matrix with respect to any basis for which $x_i(s)$ is a real matrix. The conclusion that $\text{ad}_2(z_i \otimes 1/2^n) + I$ is non-degenerate is also easily verified.

Theorem 1. *Given a group Γ , where the action of Z^t on $Z^s \otimes R$ can be extended to an action (linear) of $Z^t \otimes R$ on $Z^s \otimes R$, then there exists a group S satisfying the diagram*

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^s & \rightarrow & \Gamma & \rightarrow & Z^t & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Z^s \otimes R & \rightarrow & S & \rightarrow & Z^t \otimes R & \rightarrow & 1 \end{array}$$

However, the action of $Z^t \otimes R$ on $Z^s \otimes R$ in S may be different from that originally assigned.

The proof of this theorem is essentially the same as that given for Theorem A and will be omitted.

6. On the groups Γ . In this section we will always be dealing with groups of the form

$$1 \rightarrow A \rightarrow G \rightarrow B \rightarrow 1$$

with A and B abelian groups. We will use additive notation whenever all the terms we are considering are in A . If this is not the case, we will use multiplicative notation.

Lemma 5. *Let G satisfy the above definition. Then if x, y and z are arbitrary elements of G and $xyx^{-1}y^{-1} = [x, y]$, we have*

$$[x[z, y]] + [z[y, x]] + [y[x, z]] = 0.$$

Proof: From the definition of $[x, y]$, we have

$$xy = [x, y]yx.$$

Taking the inner automorphism of G induced by z and making use of the definition of the bracket, we get

$$[z, x]x[z, y]y = z[x, y]z^{-1}[z, y]y[z, x]x.$$

Then

$$[z, x]x[z, y]x^{-1}xy = z[x, y]z^{-1}[z, y]y[z, x]y^{-1}yx$$

or, transposing yx and using additive notation,

$$[z, x] + x[z, y]x^{-1} + [x, y] = z[x, y]z^{-1} + [z, y] + y[z, x]y^{-1}.$$

Now $[y, x] = -[x, y]$ and $x[z, y]x^{-1} - [z, y] = [x[z, y]]$. Hence

$$[x[z, y]] + [z[y, x]] + [y[x, z]] = 0.$$

Lemma 6. *Let Z^t acting on Z^s be given and let this action be denoted by $\text{ad}(z)$, $z \in Z^t$. Let $f: Z^t \times Z^t \rightarrow Z^s$ be given satisfying the following relations:*

1. $f(z, r) = -f(r, z)$, all $r, z \in Z^t$.
2. $f(rs, z) = \text{ad}(r)f(s, z) + f(r, z)$, all $r, s, z \in Z^t$.
3. If $[z, a] = \text{ad}(z)a - a$, $a \in Z^s$, $z \in Z^t$, we have $[z, f(r, s)] + [r, f(z, s)] + [s, f(z, r)] = 0$.

Then f is a 2 cocycle in $C^2(Z^t, Z^s)$ and hence determines a unique group extension.

Proof: $\delta f(z, r, s) = \text{ad}(z)f(r, s) - f(zr, s) + f(z, rs) - f(z, r)$. Now $f(zr, s) = \text{ad}(z)f(r, s) + f(zs)$ and

$$f(z, rs) = -f(rs, z) = \text{ad}(r)f(z, s) + f(z, r).$$

Combining these formulas with the condition that $f(z, r) = -f(r, z)$ gives the left-hand side of condition 3. Since, by hypothesis, this is zero, we must have that $\delta f = 0$ or f is a cocycle.

Theorem 2. Let Z^t acting on Z^s be given. Let $z_1, \dots, z_t$ be a basis for Z^t . Assume we have a mapping $f^*: \{z_i^\alpha \times z_j^\beta\} \rightarrow Z^s$, where $i, j = 1, \dots, t$ and α, β equal ± 1 , satisfying conditions 1 and 3 of Lemma 6. Assume, further, that if $f^*(ab, c) \in Z^s$ is defined by

$$(2') \quad f^*(ab, c) = \text{ad}(a)f^*(b, c) + f^*(a, c)$$

then $f^*(ab, c)$ is well defined when a, b, c equal z_i^α , where $i = 1, \dots, t$ and $\alpha = \pm 1$. Then f^* determines a unique mapping $f: Z^t \times Z^t \rightarrow Z^s$ which extends f^* and f satisfies the conditions of Lemma 6.

Proof: Assume that a function f_1 has been defined on some proper subset S_1 of Z^t containing $z_1^\alpha, \dots, z_t^\alpha$, $\alpha = \pm 1$, and satisfying conditions 1, 3 and 2' where now $\alpha, c \in S_1$ and b equals z_i^α , $i = 1, \dots, t$, $\alpha = \pm 1$. We will show that we can enlarge the set S_1 to a set S_2 consisting of S_1 and az_i^α , for some $a \in S_1$, $\alpha = \pm 1$, and z_i a basis element, and construct a function $f_2: S_2 \times S_2 \rightarrow Z^s$ satisfying the conditions 1, 3 and 2' with now $a, c \in S_2$ and $b = z_i^\alpha$. Since any element of Z^t can be obtained by multiplying elements of S_2 by basis elements or their inverses one at a time, this will complete the proof of the theorem.

Let $a \in S_1$ and z_i be such that az_i is not in S_1 , but is in S_2 . Define $f_2: S_2 \times S_2 \rightarrow Z^t$ by $f_2|_{S_1 \times S_1} = f_1$ and

$$\begin{aligned} f_2(az_i, s) &= \text{ad}(a)f_2(z_i, s) + f_2(a, s), \\ f_2(s, az_i) &= -f_2(az_i, s), \\ f_2(az_i, az_i) &= 0. \end{aligned} \quad \text{for all } s \in S_1,$$

We now have to show that this definition of f_2 on S_2 satisfies conditions 1, 2' and 3. Clearly 1 is satisfied. We have to verify 2'.

Part I. To show that $f_2(z_i az_i, s) = f_2(az_i z_i, s)$. First

$$\begin{aligned} f_2(z_i az_i, s) &= \text{ad}(z_i)f_2(az_i, s) + f_2(z_i, s), \\ f_2(az_i z_i, s) &= \text{ad}(az_i)f_2(z_i, s) + f_2(az_i, s). \end{aligned}$$

Now $f_2(az_i, s) = \text{ad}(a)f_2(z_i, s) + f_2(a, s)$. Hence

$$\begin{aligned} f_2(z_i az_i, s) &= \text{ad}(z_i) \text{ad}(a)f_2(z_i, s) + \text{ad}(z_i)f_2(a, s) + f_2(z_i, s) \\ &= \text{ad}(az_i)f_2(z_i, s) + f_2(z_i a, s) = f_2(az_i z_i, s) \end{aligned}$$

But

$$\begin{aligned} f_2(az_i z_i, s) &= \text{ad}(a)[\text{ad}(z_i)f_2(z_i, a) + f_2(z_i, s)] + f_2(a, s) \\ &= \text{ad}(a)f_2(z_i z_i, s) + f_2(a, s) \end{aligned}$$

and by induction $f_2(z_i, z_i, s) = f_2(z_i z_i, s)$. Hence

$$f_2(az_i z_i, s) = f_2(az_i z_i, s).$$

Part II. Assume $bz_k = az_i z_i$. We have to show $f_2(bz_k, s) = f_2(az_i z_i, s)$. Then either $k = i$ or j and we are done by the calculations in Part I or $a = a'z_k$ and $a'z_i z_i = b$. Now

$$\begin{aligned} f_2(a'z_k z_i z_i, s) &= \text{ad}(z_i) f_2(a'z_k z_i, s) + f_2(z_i, s) \\ &= \text{ad}(z_i)(f_2(z_i, s) + \text{ad}(z_i) f_2(a'z_k, s)) + f_2(z_i, s) \\ &= \text{ad}(z_i) f_2(z_i a'z_k, s) + f_2(z_i, s) \\ &= f_2(z_i z_i a'z_k, s) = f_2(bz_k, s) \end{aligned}$$

There now only remains to show that the Jacobi relation or relation 3 holds. But it is straightforward to verify that if (x, y, a) and (x, y, b) satisfy the Jacobi identity and $f_2(ab, s)$ satisfies condition 2', then (x, y, ab) satisfies the Jacobi identity. This completes the proof of Theorem 2.

7. Preliminaries to Theorem C. Let $\Gamma \subset A(n)$, $A(n)$ the group of affine transformations of euclidean n -space, be the fundamental group of a compact, complete, n -dimensional, locally affine manifold, i.e., an affinely connected manifold with curvature and torsion zero. Assume further that Γ satisfies the diagram

$$1 \rightarrow Z^* \rightarrow \Gamma \rightarrow Z' \rightarrow 1.$$

Then $s + t = n$ for, by the theorem of EILENBERG and MACLANE concerning spaces whose universal covering spaces are acyclic, if Γ is the fundamental group of an n -dimensional manifold of such type, it cannot be the fundamental group of a p -dimensional manifold, $p \neq n$, of such type. But, by Theorem A, Γ is the fundamental group of an n -dimensional compact solvmanifold. This proves that $s + t = n$.

Now $\Gamma \subset A(n) \subset GL(n+1, R)$. By a theorem of ZASSENHAUS [8], $Z^* \subset \Gamma$ has all eigenvalues 1. Hence we may apply the theorem of WANG [4], which asserts that there is a solvable subgroup $S \supset \Gamma$, $S \subset A(n)$, with the property that S has a finite number of components and each component S_α of S is simply connected. Let S_0 denote the identity component of S . Now there is a mapping $\eta : A(n) \rightarrow R^n$, where R^n is n -dimensional affine space. η is defined by associating to each element of $A(n)$ its translational component.

Lemma 7. $\eta|S_0$ is a homeomorphism of S_0 onto R^n and $R^n/\Gamma \cap S_0$ is homeomorphic to $S_0/\Gamma \cap S_0$.

Proof: We will let $\eta^* = \eta|S_0$ and for $s \in S_0$ we have $\eta^*s = s\eta^*$. To see that this is so, note that an element of S_0 can be written as (A, t) where A is a non-singular $n \times n$ matrix and t is the translation component. In this notation, the

group multiplication is given by

$$(A_1, t_1)(A_2, t_2) = (A_1 A_2, A_1 t_2 + t_1).$$

Further, $\eta^*(A, t) = t$ and $(A, t)r, r \in R^n$, is $Ar + t$. These formulas easily give that $\eta^*s = sn^*$. Hence η^* is equal to S_0/C where C is the subgroup of S_0 consisting of matrices of the form $(A, 0)$. Hence S_0/C is homeomorphic to a euclidean space. Clearly, the dimension of C must be zero or η^* is locally a homeomorphism. Hence $\eta^*(S_0)$ is open in R^n . It is clearly also closed since $\eta^*s = s\eta^*$. Hence η^* is onto R^n and is a local homeomorphism. Since R^n is simply connected, η^* is a homomorphism. This proves the lemma.

Corollary. S is a split extension $S = S_0 \cdot F$, where F is a finite abelian group. (This is a special case of a general theorem of Mostow, Self adjoint groups, *Annals of Math.*)

Corollary. The components of S project 1 - 1 onto R^n .

There is therefore a well defined representation of the elements of S by $s_0 f$, where $s_0 \in S_0$ and $f \in F$. Hence we have a well defined mapping $\theta : S \rightarrow S_0$ defined by $\theta(s_0 f) = s_0$. Define Γ acting on S_0 by $\theta(\gamma s_0)$ for $\gamma \in \Gamma$ and $s_0 \in S_0$.

Lemma 8. Let Γ act on S_0 as above, then S_0/Γ is homeomorphic to R^n/Γ .

The proof of this is now straightforward.

8. Construction I. Let $S \supset \Gamma$ be given by the Wang construction in (4). Then define S^* by the following diagram, where the cocycle determining S^* is the image of the cocycle determining S :

$$\begin{array}{ccccccc} 1 \rightarrow Z^* \otimes R & \rightarrow & S & \rightarrow & Z^t \otimes R + F & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \\ 1 \rightarrow Z^* \otimes C & \rightarrow & S^* & \rightarrow & Z^t \otimes R + F & \rightarrow & 1 \end{array}$$

Now, since S is a split extension, $S^* = S_0^* \cdot F$ is also, where S_0^* is the identity component of S^* . We can now extend the finite abelian group of automorphisms F to a torus group T of automorphisms of S_0^* . Let $S^* = S^* \cdot T$, where the dot denotes the semi-direct product. Now there is a subgroup Z^t of Z^* of finite index, which has a basis $z_1, \dots, z_i$ with the property that there are preimages $s_1, \dots, s_i$ in S_0^* of $z_1, \dots, z_i$ which lie on one-parameter subgroups $p_i(t)$ in S_0^* in such a way that $p_i(1) = s_i$.

Lemma 9. There exist in S^* one-parameter subgroups $P_i(t)$ which are pointwise invariant under the group T and equal to $p_i(t)$ modulo $Z^* \otimes C$.

Proof: Choose an inner product in $Z^* \otimes C$ relative to which T is hermitian. We may ask for each point of $p_i(t)$ to find a $z(t) \in Z^* \otimes C$ such that

$$\tau p_i(t) z(t) \tau^{-1} = p_i(t) z(t)$$

for all $\tau \in T$. Assume

$$\tau p_i(t) \tau^{-1} = p_i(t) z(t).$$

Then $\tau p_i(t) x \tau^{-1} = p_i(t) z(\tau) \cdot \tau x \tau^{-1}$. Hence in $Z^s \otimes C$ we want a solution of the equation

$$x = z(\tau) + \tau x \tau^{-1}$$

for all $\tau \in T$. Now the group of transformations of R^n onto itself given by $x \rightarrow z(\tau) + \tau x \tau^{-1}$ is a compact group, since it is a quotient group of the compact group T . But any compact group of linear transformations on R^n has a fixed point. Hence we can always answer our question in the affirmative.

Let z_1 and z_2 be two such fixed points. Then

$$z_1 - z_2 = \tau(z_1 - z_2) \tau^{-1}, \quad \text{for all } \tau \in T.$$

Hence their difference lies in the linear subspace of $Z^s \otimes C$ which has eigenvalues 1 for all $\tau \in T$. Call this space $V(1)$ and its orthogonal complement $\mathcal{C}$. Hence for $p_i(t)$ we can choose a fixed $z_i(t)$ such that

$$\tau p_i(t) z_i(t) \tau^{-1} = p_i(t) z_i(t)$$

for all i and t and $z_i(t) \in \mathcal{C}$. There is only one point of $Z^s \otimes C$ with these two properties.

Definition. Let $P_i(t) = p_i(t) z_i(t)$. We will prove that $P_i(t)$ is a one-parameter group.

$$\begin{aligned} p_i(t_1) z_i(t_1) p_i(t_2) z_i(t_2) &= p_i(t_1) p_i(t_2) \operatorname{ad}(p_i^{-1}(t_2))(z_i(t_1)) z_i(t_2) \\ &= p_i(t_1 + t_2) \operatorname{ad}(p_i^{-1}(t_2)) z_i(t_1) z_i(t_2). \end{aligned}$$

Now $z_i(t_1 + t_2)$ is the unique solution of the equation $\tau p_i(t_1 + t_2) X \tau^{-1} = p_i(t_1 + t_2) X$ which lies in $\mathcal{C}$. But by a straightforward computation, we see that $\operatorname{ad}(p_i^{-1}(t_1))(z_i(t_1)) + z_i(t_2)$ satisfies these conditions. Hence $P_i(t)$ is a one-parameter group.

9. Construction II. We have in §8 constructed the group $s^* = S_0^* \cdot T$. This group is too big for our purposes. It will serve as a universal imbedding group for the groups we wish to consider. We will construct a new group $s \subset s^*$ with the property that $s \supset \Gamma$ and satisfies the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^s & \rightarrow & \Gamma & \rightarrow & Z^t & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Z^s \otimes C & \rightarrow & s & \rightarrow & Z^t \otimes R & \rightarrow & 1 \end{array}$$

The construction of s . Let $z_1, \dots, z_t$ be a basis for Z^t such that $z_i^{i^i} = \bar{z}_i$, $i = 1, \dots, t$. Choose preimages of z_i, s'_i , in $S \subset S^*$, such that $(s'_i)^{i^i} = s_i$, where s_i is the preimage of z_i defined in §8. Then $s'_i = \xi_i f_i$, where $\xi_i \in S_0$ and

$f_i \in F$. Further,

$$\xi_i f_i \xi_i f_i = \xi_i f_i \xi_i f_i^{-1} f_i^2 = \xi_i^2 z f_i^2 \quad \text{where } z \in Z^* \otimes C.$$

Hence $\xi_i = P_i(1/j_i) \cdot f_i$ modulo $Z^* \otimes C$, where $f_i \in T$. Now consider a one-parameter group $f_i(t)$ in T such that $f_i(1) = f_i$. Then by Lemma 8 $P_i(t_1)$ and $f_i(t_2)$ commute for all t_1 and t_2 . Hence $L_i(t) = P_i(t)f_i(t)$ is a one-parameter subgroup in S^* . Consider the subgroup s of S^* generated by $Z^* \otimes C$ and $L_i(t)$, $i = 1, \dots, t$. Then clearly $s \supset \Gamma$ and satisfies the diagram

$$\begin{array}{ccccccc} 1 & \rightarrow & Z^* & \rightarrow & \Gamma & \rightarrow & Z^t & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & Z^* \otimes C & \rightarrow & s & \rightarrow & Z^t \otimes R & \rightarrow & 1 \end{array}$$

Lemma 10. Let $k : S^* \rightarrow S^*/T$. Then $k|_s$ is a one to one mapping of s onto S^*_0 .

The proof of this lemma is trivial.

Proof of Theorem C: In Lemma 8, we saw that S_0/Γ as defined in §7 is homeomorphic to R^n/Γ . It is trivial to extend this to a homeomorphism of $S^*_0/\mathfrak{g}\Gamma$, where $\mathfrak{g}$ is the subgroup of $Z^* \otimes C \subset S^*_0$ consisting of pure imaginary vectors, on R^n/Γ . Now $k\gamma = \gamma k$ for $\gamma \in \mathfrak{g}\Gamma$. Hence $s/\Gamma\mathfrak{g}$ is homeomorphic to $S^*_0/\mathfrak{g}\Gamma$ and therefore to R^n/Γ . This proves Theorem C.

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Indiana University
Bloomington, Indiana