*Interpolation Problems for Functions of Several Complex Variables**

GEORGE SPRINGER

Communicated by S. BERGMAN

1. For analytic functions of one complex variable, the following interpolation problem has a well known solution in terms of the Bergman kernel function. Let $\mathcal{L}^2$ denote the class of holomorphic functions f in a domain D which satisfy $\iint_D |f|^2 dx dy < \infty$. Then among all functions satisfying the normalizing condition $f(z_0) = 1$ for $z_0 \in D$ we seek the function for which $\iint_D |f|^2 dx dy$ is a minimum. BERGMAN [2] extended this problem to functions f of two complex variables which are holomorphic in a domain D and for which $\|f\|^2 = \iiint_D |f|^2 dV$ is finite, dV representing the measure of the four dimensional volume element. He again specified the value of f at one point and sought the minimum of $\|f\|$. The problem was solved by the Bergman kernel for the domain D .

The analytic functions f of n complex variables assume the value a not at isolated points but on analytic manifolds of complex dimension $n - 1$. Thus it would seem natural to pose an interpolation problem in which the functions are normalized to assume prescribed values on some given $n - l$ dimensional manifolds.

Let D be a bounded domain in the space C_n of n complex variables. Let $\mathcal{L}^2(D)$ denote the class of functions g which are holomorphic in D and for which

$$(1) \qquad \|g\|^2 = \int_D |g|^2 dV < \infty$$

where dV represents the measure of the $2n$ real dimensional volume element in C_n .

Let E be an arbitrary holomorphy domain containing $\bar{D}$ in its interior. To each point P of E we associate a neighborhood U_P and a function f_P which is holomorphic in U_P and let M_P be the $(n-1)$ -manifold in which

*This work was partially supported by the National Science Foundation grants NSF-G3661 and NSF-G1976.

$f_P(Z_1, \dots, Z_n) = 0$. If Q is another point in E , then f_Q is defined in U_Q and if $U_P \cap U_Q \neq 0$, then we require that f_P/f_Q be holomorphic and different from zero in $U_P \cap U_Q$.

We define as the set M the union of all M_P , i.e. $M = \bigcup_{P \in E} M_P$. Then the second Cousin problem [9] tells us that if we can find a continuous function h having the property that h/f_P is continuous and different from zero, then there exists a holomorphic function f having this property in each U_P . Thus f is continuous in $\bar{D}$ and vanishes precisely on M .

For every function $\phi \in \mathcal{L}^2(D)$ (i.e. ϕ is holomorphic in D and $\|\phi\| < \infty$), we have that $h = g + f\phi$ assumes the same values as g on M and that $\|h\| < \infty$. Let $H_M^2(D)$ denote the class of functions h which are holomorphic in D and for which $\|h\| < \infty$ and for which $(h - g)/f$ is holomorphic and in $\mathcal{L}^2(D)$. We seek the function $h \in H_M^2(D)$ for which $\|h\|$ is a minimum.

We begin by orthonormalizing a countable dense set of functions ϕ_n for $\mathcal{L}^2(D)$, using the weight function $|f|^2$:

$$(\phi_n, \phi_k)_f = \int_D |f|^2 \phi_n \bar{\phi}_k dV = \delta_{nk}.$$

We may define the weighted Bergman kernel function [8],

$$K_f(Z, \bar{T}) = \sum_{n=1}^{\infty} \phi_n(Z) \overline{\phi_n(T)},$$

where $Z = (z_1, \dots, z_n)$ and $T = (t_1, t_2, \dots, t_n)$ are points of D . Every function $h \in H_M^2(D)$ has the representation

$$h = g + f \cdot \left(\sum_{n=1}^{\infty} a_n \phi_n \right),$$

the convergence being in the norm $\|h\|$ and hence uniform in any compact subset of D , and the a_n are given by

$$\left(\frac{h - g}{f}, \phi_n \right)_f = a_n.$$

Then to determine the minimum of $\|h\|$ we differentiate the expression

$$\left\| g + f \cdot \left(\sum_{n=1}^{\infty} a_n \phi_n \right) \right\|^2$$

with respect to a_n and get

$$a_n = - \int_D g \bar{f} \bar{\phi}_n dV.$$

Thus we have the following representation for the minimizing function h_0 :

$$(2) \quad h_0(Z) = g(Z) - f(Z) \int_D g(T) \overline{f(T)} K_f(Z, \bar{T}) dV_T.$$

The space $H_M^2(D)$ is not a linear space. Let us enlarge $H_M^2(D)$ by defining the space $\mathfrak{L}_M^2(D)$ to consist of all functions h of the form $h = cg + f\phi$ where c is any complex constant and $g + f\phi \in H_M^2(D)$. The new space $\mathfrak{L}_M^2(D)$ is obviously a linear space and we may proceed to define its Bergman kernel function K_M as follows: Let $\{\psi_n\}$ be a countable dense set of functions in $\mathfrak{L}_M^2(D)$ so that each $h \in \mathfrak{L}_M^2(D)$ can be written as $h = \sum_{n=1}^{\infty} a_n \psi_n$, the convergence being in the norm of $\mathfrak{L}^2$ and hence uniform in any compact subset of D . Then we orthonormalize the ψ_n so that

$$(\psi_n, \psi_k) = \int_D \psi_n \bar{\psi}_k dV = \delta_{nk}$$

and define the Bergman kernel

$$(3) \quad K_M(Z, \bar{T}) = \sum_{n=1}^{\infty} \psi_n(Z) \overline{\psi_n(T)}.$$

Then, for each n , we have

$$\psi_n(Z) \equiv c_n g(Z) + f(Z) \chi_n(Z), \quad \chi_n \in \mathfrak{L}^2(D),$$

and

$$\begin{aligned} \psi_n(Z) \overline{\psi_n(T)} &= |c_n|^2 g(Z) \overline{g(T)} + c_n \overline{g(T)} f(Z) \overline{\chi_n(T)} \\ &\quad + \bar{c}_n \overline{g(T)} f(Z) \chi_n(Z) + f(Z) \overline{f(T)} \chi_n(Z) \overline{\chi_n(T)}. \end{aligned}$$

If $Z = T \in M$, then $f(Z) = 0$ and we have

$$\psi_n(T) \overline{\psi_n(T)} = |c_n|^2 |g(T)|^2,$$

and hence, for $T \in M$,

$$K_M(T, \bar{T}) = \gamma^2 |g(T)|^2$$

where $\gamma^2 = \sum_{n=1}^{\infty} |c_n|^2$ is a positive constant.

The Bergman kernel K_M has its characteristic reproducing property in the class $\mathfrak{L}_M^2(D)$ and, in particular, if $h \in H_M^2(D)$, then

$$(h(Z), K_M(Z, \bar{T})) = \int_D h(Z) \overline{K(Z, \bar{T})} dV_Z = h(T).$$

If $T \in M$, we have $h(T) = g(T)$, and, applying the Schwarz inequality, we obtain

$$(4) \quad |g(T)| = |(h(Z), K_M(Z, \bar{T}))| \leq \|h\| \cdot \|K_M(Z, \bar{T})\|.$$

However,

$$\|K_M(Z, \bar{T})\|^2 = (K_M(Z, \bar{T}), K_M(Z, \bar{T})) = K_M(T, \bar{T}) = \gamma^2 |g(T)|^2.$$

Thus $\|h\| \geq 1/\gamma$ gives us a lower bound for $\|h\|$. However, for $T \in M$ and arbitrary $Z \in D$, we have

$$\psi_n(Z) \overline{\psi_n(T)} = |c_n|^2 g(Z) \overline{g(T)} + \bar{c}_n \overline{g(T)} f(Z) \chi_n(Z),$$

so that, for $T \in M$, $Z \in D$,

$$K_M(Z, \bar{T})\gamma^2\overline{g(T)}g(Z) + \overline{g(T)}f(Z) \sum_{n=1}^{\infty} \bar{c}_n\chi_n(Z),$$

and if $g(T) \neq 0$,

$$\frac{K_M(Z, \bar{T})}{\gamma^2\overline{g(T)}} = g(Z) + \frac{f(Z)}{\gamma^2} \sum_{n=1}^{\infty} \bar{c}_n\chi_n(Z).$$

Thus if $g(T) \neq 0$, we see that

$$\frac{K_M(Z, \bar{T})}{\gamma^2\overline{g(T)}} \in H_M^2(D).$$

(If $g(T) = 0$ in M , then the minimizing function $h_0 \equiv 0$ in D , so we may assume that $g(T) \neq 0$ for some $T \in M$.) Furthermore,

$$\left\| \frac{K_M(Z, \bar{T})}{\gamma^2\overline{g(T)}} \right\|^2 = \frac{K_M(T, \bar{T})}{\gamma^4 |g(T)|^2} = \frac{1}{\gamma^2}$$

and we see that this is the unique function in $H_M^2(D)$ for which $\|h\|$ is a minimum and that the minimum value is γ^{-1} . The uniqueness follows from (4) since equality can hold only when h is a multiple of K_M . From (5), we see that, for $T \in M$, $K(Z, \bar{T})/\gamma^2\overline{g(T)}$ is a function of Z alone.

The equation (2) also gave us a representation for the minimizing function in terms of the weighted kernel K_f of the class $\mathcal{L}^2(D)$ using the weighting function $|f|^2$. Thus, for fixed $T \in M$, we obtain the identity

$$(6) \quad \frac{K_M(Z, \bar{T})}{K_M(T, \bar{T})} g(T) = g(Z) - f(Z) \int_D g(W) \overline{f(W)} K_f(Z, \bar{W}) dV_W,$$

relating the kernels K_M and K_f .

2. In the preceding section, we discussed an interpolation problem for a class of holomorphic functions in a domain D . In this section we shall briefly sketch how these same considerations may be carried over to a similar problem for a class of meromorphic functions in a more restrictive type of domain.

Let R be a bounded domain in C_2 which has a distinguished boundary surface.* Locally, the distinguished boundary surface is given by $Z_M = \varphi_M(\lambda_1, \lambda_2)$, $M = 1, 2$, where λ_1, λ_2 are real variables in $0 \leq \lambda_j \leq 2\pi$, $j = 1, 2$, and the φ_M , $M = 1, 2$, are continuously differentiable functions. BERGMAN [5] has introduced the area measure

$$dA = \left| \frac{\partial(\varphi_1, \varphi_2)}{\partial(\lambda_1, \lambda_2)} \right| d\lambda_1 d\lambda_2$$

on S and has shown how to construct a complete system of orthonormal functions

*The Bergman distinguished boundary surface is a two dimensional subset of the three dimensional boundary of D on which the values of a holomorphic function in D are completely determined. This is a special case of a Šilov boundary when the Šilov boundary is two dimensional [3, 7].

each regular in $\bar{D}$. Those functions are complete for the class of functions regular in $\bar{D}$.

We now assume that $D = D_0$ may be deformed into a domain D_1 by the continuous deformation $\Theta: D \times I \rightarrow C_2$, $I = \{t \mid 0 \leq t \leq 1\}$, such that Θ is given by

$$Z_1^* = h_1(Z_1, Z_2, t), \quad Z_2^* = h_2(Z_1, Z_2, t),$$

where $\Theta(D \times t) = D_t$ for $0 \leq t \leq 1$ and h_1, h_2 give us a pseudoconformal mapping of D onto D_t . We assume that $\bar{D}_0 \subset D_t$ for $0 < t \leq 1$ and that D_t is also a domain with distinguished boundary S_t which converges to S as $t \rightarrow 0$. Then the method of O. FARREL [6] can be used to obtain a complete system of orthonormal functions φ_ν , each regular on S , which are complete for the class $\mathcal{L}_S^2(D)$ of all functions f holomorphic in D with boundary values almost everywhere on S such that $\iint_S |f|^2 dA < \infty$. In particular, this can be carried out for convex domains having a distinguished boundary surface.

Let g be a meromorphic function in D which has as its pole surfaces the set B . We shall make the assumption that B does not meet S . We also assume that g has boundary values almost everywhere on S and that $\iint_S |g|^2 dA < \infty$. Then, for any function $\varphi \in \mathcal{L}_S^2(D)$, the function $h = g + \varphi$ has the same singular part as g and we may pose the problem of finding the function for which $\|h\|^2 = \iint_S |h|^2 dA$ is a minimum. We may further normalize the problem by requiring that h assumes the same values as g on a given two dimensional set A . Again we select A so that there is a function f which vanishes on A and is continuous on $\bar{D}$. Then $h = g + f\varphi$ has the same singular part as g and has the same values as g on A .

We may orthonormalize the countable dense set of functions for $L_S^2(D)$ with the weighting function $|f|^2$ and construct the weighted Bergman kernel $K_f(Z, \bar{T})$, $Z = (z_1, z_2)$, $T = (t_1, t_2)$, for this system. Then the function h_0 having minimum norm can be represented as

$$h_0(Z) = g(Z) - f(Z) \iint_S g(\zeta) \overline{f(\zeta)} K_f(Z, \bar{\zeta}) dA_\zeta,$$

by the same argument as given in Section 1.

On the other hand, we may extend the functions we consider to include the set $M^2(D)$ and functions $cg + f\varphi$ where c is any complex constant. Then $M_S^2(D)$ is a linear space and we may define its Bergman kernel $\tilde{K}(Z, \bar{T})$. We again have the relation

$$\frac{\tilde{K}(Z, \bar{T})}{\tilde{K}(T, \bar{T})} g(T) = g(Z) - f(Z) \iint_S g(\zeta) \overline{f(\zeta)} K_f(Z, \bar{\zeta}) dA_\zeta$$

for $T \in A$, $Z \in D$.

REFERENCES

1. BEHNKE, H. & P. THULLEN, *Theorie der Funktionen Mehrerer Komplexer Veränderlichen*, Berlin, Springer (1934).

2. BERGMAN, S., Über Hermiteische unendliche Formen, die zu einem Bereich gehören, *Math. Zeitschr.*, **29** (1929), pp. 640–677.
3. BERGMAN, S., Über die ausgezeichneten Randflächen in der Theorie der Funktionen von zwei komplexen Veränderlichen, *Math. Ann.*, **104** (1931), pp. 611–636.
4. BERGMAN, S., *Sur les fonctions orthogonales de plusieurs variables complexes avec les applications à la théorie des fonctions analytiques*, Gauthier-Villars, Paris (1947).
5. BERGMAN, S., Bounds for analytic functions in domains with a distinguished boundary surface, *Math. Zeitschr.*, **63** (1955), pp. 173–194.
6. FARRELL, O. J., On approximation to an analytic function by polynomials, *Bull. A.M.S.*, **40** (1934), pp. 900–914.
7. GELFAND, I., RAIKOV, O., & ŠILOV, G., Commutative Normed Rings, *Uspeki Math. Nauk N.S.*, **2** (1946), pp. 48–146.
8. NEHARI, Z. On weighted kernels, *J. d'Anal. Math.*, **2** (1952), pp. 126–149.
9. OKA, K., Sur les fonctions des plusieurs variables III. Deuxième problème de Cousin, *Journ. of Science of Hiroshima Univ.*, **9** (1939), pp. 7–19.

University of Kansas
Lawrence, Kansas