*A Class of Quasi-Pseudo-Conformal Transformations in the Theory of Functions of Two Complex Variables**

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1. A class of quasi-pseudo-conformal transformations. When developing the theory of functions of two complex variables we are interested in following, at least to a certain degree, the pattern used in the theory of functions of one complex variable. The possibility of solving the Dirichlet problem for harmonic functions of two real variables is a powerful tool in the theory of conformal mapping. The class of B-harmonic functions (real parts of analytic functions of two complex variables) is too small to allow the solution of the boundary value problem, even for domains with a distinguished boundary surface and with values prescribed only on this surface. Therefore it is indicated (*e.g.* when generalizing the Nevanlinna approach to the theory of two complex variables) that one should introduce functions of some suitable *extended class*. One can define, in the case of domains with a distinguished boundary surface, various extended classes of functions. See BERGMAN [3, 4, 8, 10], BREMERMAN [13, 14], LOWDENSLAGER [19, 20].

It is of considerable interest to introduce extended classes of functions for other types of domains in the space of two complex variables, and to study mappings defined by *pairs of functions of a certain extended class*.

A transformation of a domain in the space of two complex variables by a pair $\{Z_k(z_1, z_2)\}$, $k = 1, 2$, of analytic functions of z_1, z_2 with a non-vanishing and finite functional determinant is denoted as a PC (*pseudo-conformal*) transformation.

As is well known, domains of the space of two complex variables cannot in general be transformed into each other by PC transformations. For many purposes it is therefore useful to introduce more general transformations which then permit us to map into each other larger classes of domains. Obviously, mappings of this type can be defined in various ways. In particular, it is of

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interest to consider at first QPC (*quasi-pseudo-conformal*) transformations of a simple structure and for special classes of domains.

Remark. While PC transformations change an infinitesimal hypersphere

$$(1) \quad |z_1|^2 + |z_2|^2 < \epsilon^2$$

into an infinitesimal hyperellipsoid of very special type, namely into

$$(2) \quad \sum_{\nu=1}^2 [|\alpha_{11}^{(\nu)}|^2 |Z_1|^2 + |\alpha_{21}^{(\nu)}|^2 |Z_2|^2 + 2 \operatorname{Re} (\alpha_{11}^{(\nu)} \bar{\alpha}_{21}^{(\nu)}) Z_1 \bar{Z}_2] < \epsilon^2,$$

QPC transformations change (1) into

$$(3) \quad \sum_{\nu=1}^2 \sum_{\kappa=1}^2 |a_{1\kappa}^{(\nu)}|^2 |Z_1|^2 + |a_{2\kappa}^{(\nu)}|^2 |Z_2|^2 + 2 \operatorname{Re} [a_{1\kappa}^{(\nu)} a_{2\kappa}^{(\nu)} Z_1 \bar{Z}_2 + a_{\kappa 1}^{(\nu)} a_{\kappa 2}^{(\nu)} Z_{\kappa}^2 + a_{1,3-\kappa}^{(\nu)} a_{2\kappa}^{(\nu)} Z_1 Z_2] < \epsilon^2.$$

Here $\alpha_{\kappa\mu}^{(\nu)}$ are complex constants. (3) represents a general hyperellipsoid.

One of the problems of the theory of PC mappings is the distortion theory, *i.e.*, the investigation of changes of measures when applying PC transformations possessing some specified properties.

Since one of the aims of introducing QPC mappings is to complement the theory of PC transformations, it is natural to require that the results obtained should be independent of the particular representative chosen from a class of PC-equivalent domains. Therefore, instead of considering the distortion of euclidean measures, we investigate the distortion of measures based on a metric connected with the kernel function, see [1, 3, 4, 5, 6, 11], which is invariant with respect to PC transformations. In the present paper we consider QPC transformations which change the hypersphere into a Reinhardt circular domain. Let x_{κ} , $\kappa = 1, 2, 3, 4$, denote the cartesian coordinates of the four dimensional space. A Reinhardt circular domain can be obtained by rotating a plane domain satisfying certain conditions and located in the plane $x_2 = x_4 = 0$ around the x_1 - and x_3 -axes, respectively. Therefore, if we define a conformal mapping of this plane domain into a circle with the center at the origin, this describes a transformation of a Reinhardt circular domain $\mathfrak{R}$ into a hypersphere.* A mapping of this type is considered in the present paper and it is shown that if a vector field $\{u\}$ defined in the hypersphere and satisfying certain conditions is transformed into a vector field $\{U\}$ defined in $\mathfrak{R}$, we obtain (under some additional conditions) bounds for the distortion of the line element determined by the non-euclidean metric (2.1).**

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2. Auxiliary considerations. In the present section we consider domains which

*This is no longer a PC transformation.

** (2.1) = formula (1) of §2.

are PC equivalent to Reinhardt circular domains.* As has been shown in [1, 5, 6], the non-euclidean line element

$$(1) \quad dS^2(Z_1, Z_2) = \sum_{m=1}^2 \sum_{n=1}^2 T_{m\bar{n}} dZ_m d\bar{Z}_n,$$

$$T_{m\bar{n}} = \frac{\partial^2 \ln K}{\partial Z_m \partial \bar{Z}_n}, \quad m = 1, 2, n = 1, 2,$$

is invariant with respect to PC transformations. Here K is the kernel function of the domain.

As indicated in §1 we shall consider QPC mappings of the hypersphere r which is located in the z_1, z_2 -space into Reinhardt circular domains which lie in Z_1, Z_2 -space.

Instead of cartesian coordinates we shall often use doubly polar coordinates $(R_m, \Phi_m)^{**}$ and (r_m, φ_m) , $m = 1, 2$. They are connected with the cartesian coordinates by the relations

$$X_{2m-1} + iX_{2m} = Z_m = R_m \exp \{i\Phi_m\},$$

$$x_{2m-1} + ix_{2m} = z_m = r_m \exp \{i\varphi_m\}, \quad m = 1, 2.$$

In doubly polar coordinates the equation of the unit hypersphere r in the z_1, z_2 -space assumes the form

$$(2) \quad r_1^2 + r_2^2 < 1$$

while the equation of the Reinhardt circular domain $\mathfrak{R}$, to be considered here in the Z_1, Z_2 -space, becomes

$$(3) \quad R_1^2 + R_2^2 < P^2(\Theta), \quad \Theta = \operatorname{arccot} \frac{R_2}{R_1}$$

(Θ is the angle which the ray OD forms with the positive R_1 -axis). P has to satisfy certain conditions which will be specified later.

The QPC transformation which maps the unit hypersphere r into the domain $\mathfrak{R}$, described above, will be defined as follows: Let $g(\zeta)$, $g(0) = 0$, $g'(0) > 0$, be the analytic function of the complex variable ζ which transforms

$$(4) \quad n = [|\zeta| < 1, 0 < \arg \zeta < \tfrac{1}{2}\pi]$$

into the domain $\mathfrak{R}$ of the Z -plane bounded by the curve $R = P(\Theta)$ and the rays $\arg Z = 0$, $\arg Z = \tfrac{1}{2}\pi$. (See Figs. 1 and 2.) We require that $g(1) = A$, $g(i) = B$, where A and B are the points $P(0)$ and $P(\tfrac{1}{2}\pi)$ of the boundary of $\mathfrak{R}$. In a conformal mapping the images of three boundary points $\zeta = a, 0, b$ can be prescribed and

These are domains which admit the group of transformations $Z_k^ = Z_k \exp \{i\theta_k\}$, $k = 1, 2$, into themselves.

**These are coordinates which are polar coordinates in each of the planes $Z_k = 0$, $k = 1, 2$.

FIGURE 1. The domain π in the z -plane. FIGURE 2. The domain $\mathfrak{L}$ in the Z -plane.

therefore a function $g(\zeta)$ satisfying the above requirements does exist. We assume

(A) $R = P(\theta)$ is a convex curve, $P'(\theta)$ is continuous,

$P'(0) = P'(\frac{1}{2}\pi) = 0$, and P' satisfies the Hölder condition

$$|P'(\theta_1) - P'(\theta_2)| \leq M_1 |\theta_1 - \theta_2|^\beta, \quad \beta > 0, \infty > M_1 > 0.$$

By reflecting π along the real and the imaginary ζ -axes we define the function $g(\zeta)$ in the unit circle of the ζ -plane.

The QPC mapping of the hypersphere $\mathfrak{r}$ (in the $r_1, r_2, \varphi_1, \varphi_2$ -space) into the Reinhardt circular domain $\mathfrak{R}$ (in the R_1, R_2, Φ_1, Φ_2 -space) is given by

$$(5) \quad \begin{aligned} R \cos \Theta &= \operatorname{Re} [g(re^{i\theta})], & R^2 &= R_1^2 + R_2^2, \\ R \sin \Theta &= \operatorname{Im} [g(re^{i\theta})], & r^2 &= r_1^2 + r_2^2, \end{aligned}$$

$$\Phi_k = \varphi_k, \quad k = 1, 2.$$

From (A) it follows that there exists a constant a such that

$$(6) \quad \frac{1}{a} \leq |g'(\zeta)| \leq a < \infty, \quad g'(\zeta) = \frac{dg(\zeta)}{d\zeta}, \quad |\zeta| \leq 1,$$

and g' satisfies the Hölder condition

$$(7) \quad |g'(\zeta_1) - g'(\zeta_2)| \leq M |\zeta_1 - \zeta_2|^\alpha, \quad |\zeta| \leq 1,$$

where M and $\alpha > 0$ are independent of ζ_1, ζ_2 . See *e.g.* [17], [18], [22] p. 330.

To obtain the desired inequalities for the distortion of non-euclidean measures we shall use the method of the minimum integral.

The main idea of this method consists in the following. Let $\mathfrak{Q}_{\mathfrak{B}}$ be the class of analytic functions of two complex variables which are regular in a domain $\mathfrak{B}$. We assume that $\mathfrak{B}$ has the property that there exist infinitely many analytic functions $f(\bar{Z}_1, \bar{Z}_2)$ such that

$$(8) \quad M(f) = \int_{\mathfrak{B}} |f(\bar{Z}_1, \bar{Z}_2)|^2 d\omega < \infty, \\ d\omega = d\bar{X}_1 d\bar{X}_2 d\bar{X}_3 d\bar{X}_4, \quad \int_{\mathfrak{B}} = \iiint\limits_{\mathfrak{B}}, \quad f(\bar{Z}_1, \bar{Z}_2) \in \mathfrak{Q}_{\mathfrak{B}}.$$

If we determine the minimum $\Lambda_{\mathfrak{B}}^1$ of (8) under the condition

$$(9) \quad f(Z_1, Z_2) = 1, \quad (Z_1, Z_2) \in \mathfrak{B},$$

we obtain

$$(10) \quad \Lambda_{\mathfrak{B}}^1(Z_1, Z_2) = [K_{\mathfrak{B}}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2)]^{-1}$$

where $K_{\mathfrak{B}}$ is the kernel function of $\mathfrak{B}$.

The minimum $\Lambda_{\mathfrak{B}}^{[2]}$ of $M(f)$, under the conditions

$$(11) \quad f(Z_1, Z_2) = 0, \quad U_1 \frac{\partial f}{\partial Z_1} + U_2 \frac{\partial f}{\partial Z_2} = 1,$$

is

$$(12) \quad \Lambda_{\mathfrak{B}}^{[2k]}(Z_1, Z_2; U_1, U_2) = \frac{1}{K(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) \sum_{m, \bar{n}=1}^2 T_{m\bar{n}} U_m \bar{U}_{\bar{n}}}, \\ T_{m\bar{n}} = \frac{\partial^2 \ln K(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2)}{\partial Z_m \partial \bar{Z}_{\bar{n}}}, \quad K = K_{\mathfrak{B}}.$$

See [1], [6] pp. 46 ff.

The quantities $\Lambda_{\mathfrak{B}}^1$ and $\Lambda_{\mathfrak{B}}^{[2]}$ are monotone functionals of the domain, and therefore if $(Z_1, Z_2) \in \mathfrak{U} \subset \mathfrak{B}$, then

$$(13) \quad \Lambda_{\mathfrak{G}}^1(Z_1, Z_2) \leq \Lambda_{\mathfrak{G}}^1(Z_1, Z_2),$$

$$(14) \quad \Lambda_{\mathfrak{G}}^{[2]}(Z_1, Z_2; U_1, U_2) \leq \Lambda_{\mathfrak{G}}^{[2]}(Z_1, Z_2; U_1, U_2).$$

See [1], [5], [7] p. 3 ff.

Let $dS_r(z_1, z_2; dz_1, dz_2)$ define the line-element of the non-euclidean metric (1) when using for K the kernel function of r , while $dS_{\mathfrak{R}}(Z_1, Z_2)$ is the corresponding metric (1) where K is the kernel function of $\mathfrak{R}$.

As mentioned before, when estimating $dS_{\mathfrak{R}}(Z_1, Z_2)/ds_{\mathfrak{R}}(z_1, z_2)$ it is convenient to use two different procedures, depending upon whether the point (z_1, z_2) is located very near or far from the boundary of r .

Functions

$$(15) \quad \Phi_{mn}(Z_1, Z_2) = \frac{Z_1^{m-1} Z_2^{n-1}}{a_{mn}^{\frac{1}{2}}},$$

$$a_{mn} = \frac{\pi^2}{2m + 2n - 2} \int_0^{\frac{1}{2}\pi} P^{2m+2n-2}(\Theta) \cos^{2m-2}\Theta \sin^{2n-2}\Theta d\Theta,$$

$$m, n = 1, 2, \dots,$$

represent a complete set of orthonormal functions for a Reinhardt circular domain (3).

Forming the sum

$$(16) \quad K^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) = \sum_{m=1}^M \sum_{n=1}^N \frac{|Z_1|^{2m-2} |Z_2|^{2n-2}}{a_{mn}}$$

we obtain an expression which approximates the kernel function quite well inside the domain $\mathfrak{R}$.

In the case of special domains, namely, in the case of domains

$$(17) \quad \mathfrak{A}_p = [a |Z_1|^{2/p} + |Z_2|^2 < 1], \quad p \text{ positive, } 0 < a < \infty,$$

the kernel function $K(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2)$ can be written in closed form, namely,

$$(18) \quad K_{\mathfrak{A}_p}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) = \frac{a^p(1 - |Z_2|^2)^{p-2}[(p+1)(1 - |Z_2|^2)^p + (p-1)a^p |Z_1|^2]}{\pi^2[(1 - |Z_2|^2)^p - a^p |Z_1|^2]^3}.$$

See [2] p. 82.*

Let us denote

$$(19) \quad K^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) = \sum_{m=1}^M \sum_{n=1}^N |\Phi_{mn}(Z_1, Z_2)|^2.$$

*In this paper we assume that p is integral and $a < 1$. Both restrictions are superfluous. The formula is valid for every positive number p and $0 < a < \infty$.

For the remainder, we can write

$$\begin{aligned} K(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) - K^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) &= R^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) \\ &= S^{(0,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) + S^{(M,0)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) \\ &\quad - S^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2), \end{aligned}$$

where*

$$(20) \quad S^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) = \sum_{m=M+1}^{\infty} \sum_{n=N+1}^{\infty} |\Phi_{mn}(Z_1, Z_2)|^2.$$

We can determine bounds for $S^{(M,N)}$ using the method of the minimum integral as follows: $S^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2)$ is the minimum of the integral (8) under conditions (9) and

$$(21) \quad \int_{\mathfrak{R}} f(\bar{Z}_1, \bar{Z}_2) \Phi_{mn}(\bar{Z}_1, \bar{Z}_2) d\omega = 0, \quad m = 1, 2, \dots, M, \quad n = 1, 2, \dots, N.$$

Therefore if $\mathfrak{A}_p \subset \mathfrak{R}$, and we determine the minimum of (8) under the conditions (9) and (21), we obtain

$$(22) \quad R_{\mathfrak{A}_p}^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2) \leq R_{\mathfrak{R}}^{(M,N)}(Z_1, Z_2; \bar{Z}_1, \bar{Z}_2).$$

A formal computation yields

$$(23) \quad \begin{aligned} S^{(M,N)} &= \frac{1}{\pi^2 p} \sum_{m=M+1}^{\infty} r_1^{2m} a^{p(m+1)} [(m+1)p][(m+1)p+1] \\ &\quad \cdot \sum_{n=N+1}^{\infty} r_2^{2n} \frac{[(m+1)p+2] \cdots [(m+1)p+n+1]}{n!}. \end{aligned}$$

Now

$$\begin{aligned} &\frac{[(m+1)p+2] \cdots [(m+1)p+N+2]}{(N+1)!} r_2^{2N+2} \left\{ 1 + \frac{(m+1)p+N+3}{(N+2)} r_2^2 \right. \\ &\quad \left. + \frac{[(m+1)p+N+3][(m+1)p+N+4]}{(N+2)(N+3)} r_2^4 + \cdots \right\} \\ (24) \quad &= \frac{[(m+1)p+2] \cdots [(m+1)p+N+2]}{(N+1)!} \\ &\quad \cdot r_2^{2N+2} F[(m+1)p+N+3, 1, N+2, r_2^2] \end{aligned}$$

where F is the hypergeometric function.

According to [21] p. 12

Since $(Z_1 - a_1)^{m-1}(Z_2 - a_2)^{n-1}$ is equal to a linear combination of expressions $Z_1^{m-1}Z_2^{n-1}$, $m = 1, \dots, M$, $n = 1, 2, \dots, N$, the conditions (20) are preserved when the coordinate system is shifted. We have a slightly modified situation when we apply an analytic rotation $Z_k^ = (\alpha_{k1}Z_1 + \alpha_{k2}Z_2)$, $k = 1, 2$.

$$F[(m+1)p+N+3, 1, N+2, r_2^2] = (N+1) \int_0^1 \frac{(1-t)^N dt}{(1-tr_2^2)^{(m+1)p+N+3}}.$$

Thus*

$$\begin{aligned} S_{\mathfrak{A}_p}^{(M,N)} &= \frac{1}{N!\pi^2 p} \int_0^1 \frac{(1-t)^N r_2^{2N+2}}{(1-tr_2^2)^{N+3}} \\ &\times \sum_{m=M+1}^{\infty} \frac{[(m+1)p+2] \cdots [(m+1)p+N+2] r_1^{2m} a^{(m+1)p} [(m+1)p][(m+1)p+1]}{(1-tr_2^2)^{(m+1)p}} dt \\ (25) \quad &= \frac{1}{N!\pi^2 p} \int_0^1 \frac{(1-t)^N r_2^{2N+2}}{(1-tr_2^2)^{N+3} r_1^2} \left\{ T \frac{d^{N+3}}{dT^{N+3}} \left(\frac{T^{p(M+2)+N+2}}{1-T} \right) \right\} dt \end{aligned}$$

where

$$T = \frac{ar_1^{2/p}}{1-tr_2^2}.$$

When we consider the kernel function inside the domain sufficiently far from the boundary, $S^{(M,N)}$, and therefore also $R^{(M,N)}$, is a comparatively small quantity. Introducing conveniently chosen interior and exterior domains of comparison we obtain bounds for $R_{\mathfrak{A}}^{(M,N)}$, namely $R_{\mathfrak{A}}^{(M,N)} \leq R_{\mathfrak{B}}^{(M,N)} \leq R_{\mathfrak{C}}^{(M,N)}$ where the difference $R_{\mathfrak{A}}^{(M,N)} - R_{\mathfrak{B}}^{(M,N)}$ is small. If, however, we approach the boundary and, in particular, if we wish to obtain limit relations, this procedure cannot be used, since the kernel function goes to infinity and Λ^1 and $\Lambda^{[2]} \rightarrow 0$. Consequently we apply a different procedure for evaluation of Λ^1 and $\Lambda^{[2]}$. This procedure will be described in §3.

3. Bounds for the distortion in the neighborhood of the boundary. As indicated in §2, in the present section we determine bounds for the distortion $|dS_{\mathfrak{A}}/dS_{\mathfrak{C}}|$ at a point which lies in the neighborhood of a boundary point $P = \{P_1, P_2\}$. The method used here consists in conveniently forming interior and exterior domains of comparison $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \supset \mathfrak{B} \supset \mathfrak{C}$. (The boundaries of $\mathfrak{A}$ and of $\mathfrak{B}$ pass through the boundary point P .) Using these domains and determining values for Λ^1 and $\Lambda^{[2]}$ for them, see (2.10) and (2.11), we obtain the desired limit relations. In order to use this procedure we have to prove first a few auxiliary lemmas.

Lemma 1: *Let the function $g(\xi)$, introduced on p. 939, transform $\mathfrak{b} = \{|\xi| < 1\}$ into the domain $\mathfrak{B}$. (See p. 941.) Further let c be a point in $|\xi| < 1$, C its image in the Z -plane; finally let $\delta_{\mathfrak{b}}$ (and $\delta_{\mathfrak{B}}$) denote the distances of ξ (of $g(\xi)$) from the respective boundaries. Then there exists a positive constant a such that*

$$(1) \quad \frac{1}{a} \leq \frac{\delta_{\mathfrak{B}}}{\delta_{\mathfrak{b}}} \leq a < \infty.$$

Here a is independent of the choice of c . (The constant a is the same as that appearing in (2.6).)

*Since the expression in the bracket $\{\dots\}$ has a factor r_1^2 , (25) is regular also for $r_1 = 0$.

Proof. Let $\widehat{OCD}$ be the image of ocd and let $CHG = \delta_{\mathfrak{g}}$ be the distance of the point C from the boundary. See Fig. 1 and 2. Then

$$(2) \quad \delta_{\mathfrak{c}} = cd \leq cg \leq \widehat{cg} \leq a \cdot \widehat{CG} = a\delta_{\mathfrak{g}}.$$

Analogously,

$$(3) \quad \delta_{\mathfrak{g}} = CG \leq CD \leq \widehat{CD} \leq a \cdot cd = a\delta_{\mathfrak{c}}.$$

Here $cd, cg, CG, CD, \dots$ denote the distances between the points c and d , etc., while $\widehat{cg}, \widehat{CD}, \dots$ denote the length of the segments of curves connecting c and g , etc.

Lemma 2: For the mapping function $g(\zeta)$ the inequalities

$$(4) \quad a^{-1} |\zeta - \zeta_0| \leq |g(\zeta) - g(\zeta_0)| \leq a |\zeta - \zeta_0|$$

hold.

Proof. According to (1)

$$(5) \quad |\zeta - \zeta_0| \leq \widehat{\zeta n \zeta_0} \leq a |Z - Z_0|, \quad Z_0 = g(\zeta_0), \quad Z = g(\zeta),$$

$$(6) \quad |Z - Z_0| \leq \widehat{ZmZ_0} \leq a |\zeta - \zeta_0|.$$

Here $\widehat{\zeta n \zeta_0}$ denotes the image in the ζ -plane of the straight line ZZ_0 while $\widehat{ZmZ_0}$ is the image in the Z -plane of the straight line $\zeta_0\zeta$. Then (4) follows from (5) and (6).

Lemma 3: Let (2.5) transform a line element l in the z_1z_2 -space of euclidean length de into a line element L in the Z_1, Z_2 -space of euclidean length dE . Then the inequality

$$(7) \quad \frac{1}{a} \leq \left| \frac{dE}{de} \right| \leq a < \infty$$

holds.

Proof. Since rotation leaves the length of a line element unchanged we can assume that one of the endpoints of l lies in the plane $\varphi_1 = \varphi_2 = 0$. The corresponding endpoint of L will lie in the plane $\Phi_1 = \Phi_2 = 0$. Then

$$(8) \quad de^2 = |d\zeta|^2 + |\zeta|^2 \cos^2 \theta d\varphi_1^2 + |\zeta|^2 \sin^2 \theta d\varphi_2^2,$$

$$(9) \quad dE^2 = |dZ|^2 + |Z|^2 \cos^2 \Theta d\Phi_1^2 + |Z|^2 \sin^2 \Theta d\Phi_2^2, \quad \varphi_k = \Phi_k, \quad k = 1, 2.$$

Bounds for $|dZ/d\zeta|$ have been given in (2.6). (We note that $Z(\zeta) = g(\zeta)$.) $|\zeta \sin \theta|$ is the distance of the point ζ from the axis $\text{Im } \zeta = 0$, while $|Z \sin \Theta|$ is the distance of Z from $\text{Im } Z = 0$. Repeating the proof of Lemma 2, we obtain

that

$$(10a) \quad a^{-1} |\zeta \sin \theta| \leq |Z \sin \Theta| \leq a |\zeta \sin \theta|,$$

and, analogously,

$$(10b) \quad a^{-1} |\zeta \cos \theta| \leq |Z \cos \Theta| \leq a |\zeta \cos \theta|$$

since $|\zeta \cos \theta|$ is the distance of ζ from $\operatorname{Re} \zeta = 0$. Combining (10a), (10b), and (2.6) we obtain (7).

When deriving the inequalities in the neighborhood of a boundary point P of $\Re$, which is the image of p , we introduce coordinates (N_1, N_2) possessing the property that the plane $N_1 = 0$ passes through the point P , see p. 944, and coincides with the analytic plane tangent to the boundary of $\Re$ at P . We assume further that the normal to the boundary of $\Re$ at P lies in the plane $N_2 = 0$. Let p be the image of P on the boundary of the hypersphere r .

For simplicity we can assume that the point P lies in $\Phi_1 = \Phi_2 = 0$, and that p lies in $\varphi_1 = \varphi_2 = 0$.

The coordinates (n_1, n_2) are denoted as normal for r at the point p , while (N_1, N_2) are normal for $\Re$ at P . See [1] and [7] p. 9, in particular, see Footnote 1.

Notation. Since we sometimes use coordinates with the origin at the center of the hypersphere r and sometimes coordinates which are normal with respect to the boundary point p , we shall list in the following the corresponding notations. In the case of the hypersphere we denote by $z_\kappa = x_{2\kappa-1} + ix_{2\kappa}$ coordinates with the origin at the center. In this case the components of the vectors will be denoted by $u_\kappa = v_{2\kappa-1} + iv_{2\kappa}$. Coordinates normal with respect to the point p of the boundary will be denoted by $n_\kappa = t_{2\kappa-1} + it_{2\kappa}$. When using these coordinates the components of the vector will be denoted by $m_\kappa = q_{2\kappa-1} + iq_{2\kappa}$.

In the case of Reinhardt circular domains we use a similar notation, but with the corresponding capital letters.

Lemma 4. Let $[n_1 = 0]$ be the analytic plane tangential to the boundary of r at the point p . Further, let $[N_1 = 0]$ be the analytic plane tangential to the boundary of $\Re$ at the point P . p is assumed to lie in the plane $\mathfrak{p} = [\varphi_1 = \varphi_2 = 0]$ while P lies in $\tilde{\mathfrak{p}} = [\Phi_1 = \Phi_2 = 0]$. P is the image of p , when applying the transformation (2.5). Then

$$(11) \quad \lim_{t_1 \rightarrow 0} \frac{T_1}{t_1} \equiv \lim_{n_1 \rightarrow 0} \left(\frac{N_1 + \bar{N}_1}{n_1 + \bar{n}_1} \right) = |g_\theta(e^{i\theta})|, \quad g_\theta(e^{i\theta}) = \frac{dg(e^{i\theta})}{d\theta}.$$

Proof. Let $\zeta = x_1 + ix_3$, $Z = X_1 + iX_3$ be the cartesian coordinates in the planes $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$, respectively. The unit function $g(\zeta)$ maps the unit circle $[|\zeta| < 1]$ into the domain $\mathfrak{B} = \Re \cap [\Phi_1 = \Phi_2 = 0]$. In this transformation the line element $d\theta$, at the point $\zeta = e^{i\theta}$, goes into the line element $dg(e^{i\theta})$. We introduce the complex coordinates $h_1 = t_1 + it_3$ and $H_1 = T_1 + iT_3$, where $t_1 [T_1]$ coincides with the interior normal at the point $p [P]$ and $t_3 [T_3]$ coincides with the tangent to the corresponding boundary curves.

The imaginary h_1 -axis goes into the imaginary H_1 -axis. Since by definition

$$dh_1 = |de^{i\theta}| = d\theta, \quad dH_1 = |dg(e^{i\theta})|,$$

it follows that

$$(12) \quad \left(\frac{dT_1}{dt_1} \right)_{t_1=0} = \left(\frac{dT_3}{dt_3} \right)_{t_3=0} = \left| \frac{dg(e^{i\theta})}{de^{i\theta}} \right| = |g'(e^{i\theta})|.$$

Lemma 5. *Let the domain $\mathfrak{B}$ have the property that it lies in the hypersphere*

$$(13) \quad \mathfrak{A} = [N_1 + \bar{N}_1 - P_2 |N_1|^2 - P_2 |N_2|^2 > 0]$$

of radius $1/P_2$ and that it includes in its interior the hypersphere

$$(14) \quad \mathfrak{S} = [N_1 + \bar{N}_1 - P_1 |N_1|^2 - P_1 |N_2|^2 > 0]$$

of radius $1/P_1$. Then at every point $(N_1, N_2) \in \mathfrak{S}$ the inequality

$$(15) \quad 3 \frac{P_2}{P_1} Q(P_2) \leq (N_1 + \bar{N}_1)^2 dS_{\mathfrak{B}}^2(N_1, N_2; dN_1, dN_2) \leq 3 \frac{P_1}{P_2} Q(P_1)$$

holds. See (3.1).

$$Q(P) = \frac{S(N_1, N_2; dN_1, dN_2; P)}{1 - P(|N_1|^2 + |N_2|^2)/(N_1 + \bar{N}_1)}.$$

Proof. Since, according to the considerations of §2, see p. 941, $\Lambda^1(N_1, N_2)$ and $\Lambda^{[2]}(N_1, N_2; dN_1, dN_2)$ are *monotone* functionals of the domain and since $(N_1, N_2) \in \mathfrak{S} \subset \mathfrak{B} \subset \mathfrak{A}$ we have

$$(16) \quad \frac{2P_2}{\pi^2} \leq \frac{(N_1 + \bar{N}_1)^3}{\Lambda_{\mathfrak{B}}^1(N_1, N_2)} = (N_1 + \bar{N}_1)^3 K_{\mathfrak{B}}(N_1, N_2; \bar{N}_1, \bar{N}_2) \leq \frac{2P_1}{\pi^2},$$

$$(17) \quad \frac{6P_2}{\pi^2} \leq \frac{(N_1 + \bar{N}_1)^5}{\Lambda_{\mathfrak{B}}^{[2]}(N_1, N_2; dN_1, dN_2)} \leq \frac{6P_1}{\pi^2}.$$

According to (2.10) and (2.12)

$$(18) \quad dS_{\mathfrak{B}}^2(N_1, N_2; dN_1, dN_2) = \frac{\Lambda_{\mathfrak{B}}^1(N_1, N_2)}{\Lambda_{\mathfrak{B}}^{[2]}(N_1, N_2; dN_1, dN_2)}.$$

The inequality (15) follows from (16), (17) and (18).

Definition. Let the coordinates (N_1, N_2) be normal with respect to the point P of the boundary of $\mathfrak{B}$. (Consequently at the point P we have $N_1 = N_2 = 0$.) Let Ω_α be the cone consisting of all rays from P which form an angle $\leq \alpha < \frac{1}{2}\pi$ with the interior normal to $\mathfrak{B}$ at P . If the point $Q = (N_1, N_2)$ approaches P so that it remains always in the cone Ω_α we shall say that Q approaches the point P of $\mathfrak{B}$ in the sense $\mathbf{A}^I$. When using the coordinates normal at the point P , the cone Ω_α is given by

$$(20) \quad \operatorname{Re} N_1 > 0, \quad \frac{\sqrt{|N_1|^2 + |N_2|^2}}{\operatorname{Re} N_1} \leq \frac{1}{\cos \alpha}$$

(see also [1] p. 7).

Definition. A vector field $U(Z) = (U_1, U_2)$, $Z = (Z_1, Z_2)$, is said to satisfy the condition C_1 at the boundary point P , if, when using coordinates N_1, N_2 normal for $\mathfrak{B}$ at P and denoting by $\{M_1, M_2\}$, $M_k = M_k(N_1, N_2)$, the (complex) components of the vector field, we have

$$(21) \quad \frac{|M_1|}{\sqrt{|M_1|^2 + |M_2|^2}} \geq C_1 > 0$$

when we approach the boundary point P in the sense $\mathbf{A}^+$.

Definition. Suppose that for the vector field $U \equiv U(Z)$ the inequality (21) holds at every point $\{N_1^{(P)}, 0\}$ for which

$$(22) \quad N_1^{(P)} \leq C_2.$$

If the constants C_1 and C_2 can be chosen independently of the boundary point P , which belongs to a subdomain $\mathfrak{M}$ of the boundary of $\mathfrak{B}$, the vector field $U(Z)$ is said to satisfy the condition C in $\mathfrak{M}$.

Theorem 3.1. Suppose that the QPC transformation (2.5) maps the hypersphere $\mathfrak{r} = [|z_1|^2 + |z_2|^2 < 1]$ onto the Reinhardt circular domain $\mathfrak{R}$ in the Z_1, Z_2 -space, see (2.3). $\mathfrak{R}$ is supposed to satisfy the condition (A). See p. 940.

Let $\mathfrak{p}_1$ be a subdomain of $\mathfrak{r}$, whose boundary has a part, say m_1 , in common with the boundary of $\mathfrak{r}$. Let $\mathfrak{p}_2$ be a simply connected part of $\mathfrak{p}_1$, such that the intersection $m_2 = \mathfrak{p}_2 \cap m_1$ is a simply connected domain which lies completely inside m_1 . By a QPC transformation (2.5), a vector field $\{u(z)\}$ in $\mathfrak{r}$ is mapped onto the field $\{U(Z)\}$ of vectors in $\mathfrak{R}$. Let $\mathfrak{M}_\kappa$ be the images of m_κ , and $\mathfrak{P}_\kappa$ those of $\mathfrak{p}_\kappa$ ($\kappa = 1, 2$). We assume that the field $\{u(z)\}$ [$\{U(Z)\}$] satisfies the condition C in $\mathfrak{p}_1[\mathfrak{P}_1]$. Let, finally, $dS_{\mathfrak{r}}(z, u)$ denote the length of the line element $u(z)$, when using the invariant metric (2.1) of $\mathfrak{r}$, and let $dS_{\mathfrak{R}}(Z, U)$ denote the length of the line element $U(Z)$, when using the invariant metric of $\mathfrak{R}$. Then there exists a positive constant c such that

$$(23) \quad 0 < \frac{1}{c} \leq \left| \frac{dS_{\mathfrak{R}}(Z, U)}{dS_{\mathfrak{r}}(z, u)} \right| \leq c < \infty, \quad z \in \mathfrak{p}_2.$$

Proof. When deriving bounds for $dS_{\mathfrak{R}}/dS_{\mathfrak{r}}$ it is sufficient to consider these quantities in the neighborhood of the boundary, say in a strip around the boundary, which, when using at every point p of the boundary normal coordinates $(n_1, n_2) \equiv (n_1^{(p)}, n_2^{(p)})$, is given by

$$(24) \quad \operatorname{Re} n_1^{(p)} \geq C_3 > 0$$

where C_3 is a fixed constant independent of the choice of the point p .

When we derive inequalities** in the neighborhood of a boundary point p

* m_1 consists of all boundary points of $\mathfrak{p}_1$ for which $|z_1|^2 + |z_2|^2 = 1$.

**The "length" of line element obtained using the metric (2.1) is independent of the choice of the coordinate system; i.e. we have $dS_{\mathfrak{r}}(z; u) = dT_{\mathfrak{r}}(n; m)$. The same holds in the Z_1, Z_2 -space.

instead of the coordinates (z_1, z_2) for the point and (u_1, u_2) for the vector we shall use coordinates (n_1, n_2) and (m_1, m_2) normal for the point p . (We shall proceed analogously in the domain $\mathfrak{H}$.) When using the normal coordinates we obtain for the line element dT_* of the invariant metric

$$(25) \quad dT_*^2(n; m) = \frac{3}{[n_1 + \bar{n}_1 - |n_1|^2 - |n_2|^2]^2} \{ (1 - |n_2|^2) |m_1|^2 \\ - (1 - \bar{n}_1)n_2 m_1 \bar{m}_2 - (1 - n_1)\bar{n}_2 \bar{m}_1 m_2 + (n_1 + \bar{n}_1 - |n_1|^2) |m_2|^2 \}, \\ n \equiv (n_1, n_2), \quad m \equiv (m_1, m_2).$$

In accordance with the assumption C at the boundary point P (image of p in $\mathfrak{H}$), we can use the hypersphere

$$(26) \quad \sigma_1^{-1}(N_1 + \bar{N}_1) - |N_1|^2 - |N_2|^2 > 0$$

as an interior domain of comparison $\mathfrak{I}$, and similarly a hypersphere (26) (with σ_1 replaced by σ_2) as an exterior domain of comparison $\mathfrak{U}$. (The quantities σ_1 and σ_2 are independent of the choice of the point P .) See [1] p. 18 ff. and p. 98 ff.

We proceed now to the computation of the values of $\Lambda_1(N)$, $N = (N_1, N_2)$, and of $\Lambda^{[2]}(N; M)$ for the interior and exterior domains of comparison $\mathfrak{I}$ and $\mathfrak{U}$.

According to (2.10)

$$(27) \quad \Lambda^1(N) = \frac{1}{K(N)} = \frac{\pi^2 [H(\sigma)]^3}{2\sigma},$$

$$(27a) \quad H(\sigma) = N_1 + \bar{N}_1 - \sigma |N_1|^2 - \sigma |N_2|^2$$

where $\sigma = \sigma_1$ for $\mathfrak{I}$, and $\sigma = \sigma_2$ for $\mathfrak{U}$. See also [1, 6, 7].

Further, as a formal computation shows, one has for $\mathfrak{I}$ and $\mathfrak{U}$ the relations

$$(28) \quad T_{1\bar{1}} = \frac{KK_{10\bar{1}0} - K_{1000}K_{00\bar{1}0}}{K^2} = 3 \frac{1 - \sigma^2 |N_2|^2}{H^2(\sigma)},$$

$$(29) \quad T_{2\bar{2}} = \frac{KK_{01\bar{0}1} - K_{0100}K_{00\bar{0}1}}{K^2} = 3\sigma \frac{N_1 + \bar{N}_1 - \sigma |N_1|^2}{H^2(\sigma)},$$

$$(30) \quad T_{1\bar{2}} = \frac{KK_{01\bar{1}0} - K_{0100}K_{00\bar{0}1}}{K^2} = -3\sigma N_2 \frac{1 - \sigma \bar{N}_1}{H^2(\sigma)},$$

where

$$K_{pq\bar{r}s} \equiv \frac{\partial^{p+q+r+s} K(N_1, N_2; \bar{N}_1, \bar{N}_2)}{\partial N_1^p \partial N_2^q \partial \bar{N}_1^r \partial \bar{N}_2^s}.$$

According to (2.12) we have

$$(31) \quad \Lambda^{[2]}(N; M) = \frac{\pi^2 H^5(\sigma)}{6\sigma S}, \\ S \equiv S(N_1, N_2; M_1, M_2; \sigma) = \{ (1 - \sigma^2 |N_2|^2) |M_1|^2 \\ - \sigma \bar{N}_2 (1 - \sigma N_1) \bar{M}_1 M_2 - \sigma N_2 (1 - \sigma \bar{N}_1) M_1 \bar{M}_2 \\ + \sigma (N_1 + \bar{N}_1 - \sigma |N_1|^2) |M_2|^2 \}.$$

According to our assumption the vector fields $\{u\}$ in $\mathfrak{p}_1$ and $\{U\}$ in $\mathfrak{P}_1$ satisfy the condition C. Therefore there exist boundary strips of finite thickness* $\mathfrak{s} = [0 < n_1 \leq C_4]$ and $\mathfrak{S} = [0 < N_1 \leq C_4]$, $C_4 > 0$, such that

$$(32) \quad \left| \frac{m_2(n)}{m_1(n)} \right| \leq C_5, \quad n = (n_1, n_2) \in \mathfrak{s},$$

$$(33) \quad \left| \frac{M_2(N)}{M_1(N)} \right| \leq C_6, \quad N = (N_1, N_2) \in \mathfrak{S}.$$

According to (27a)

$$(34) \quad 1 - \frac{1}{2}\sigma(1 + \alpha^2)T_1 \leq \frac{H(\sigma)}{2T_1} \leq 1, \quad N \in \mathfrak{S}, \quad T_1 = \frac{1}{2}(N_1 + \bar{N}_1)$$

for $|N_2| \leq \alpha N_1$, where α is a suitably chosen positive constant.

Therefore, by (27) and (34),

$$(35) \quad \frac{\pi^2[1 - \frac{1}{2}\sigma(1 + \alpha^2)T_1]^3}{2\sigma} \leq \frac{\Lambda^1(N)}{(2T_1)^3} \leq \frac{\pi^2}{2\sigma}.$$

We proceed to the determination of bounds for

$$(36) \quad \frac{S}{|M_1|^2} = \left\{ (1 - \sigma^2 |N_2|^2) - 2 \operatorname{Re} \left(\sigma \bar{N}_2 (1 - \sigma N_1) \frac{M_2}{M_1} \right) \right. \\ \left. + \sigma(N_1 + \bar{N}_1 - \sigma |N_1|^2) \left| \frac{M_2}{M_1} \right|^2 \right\}.$$

According to the assumptions of the theorem, $|n_2| \leq \alpha_1 n_1$, $\alpha_1 > 0$, and $|N_2| \leq \alpha_2 N_1$, $\alpha_2 > 0$.

$$(37) \quad [B(\sigma)]^{-1} \equiv (1 - \sigma^2 \alpha_2^2 \eta^2) - 2(\sigma \alpha_2 \eta (1 + \sigma \eta)) C_6 + \sigma \eta (2 - \sigma \eta) C_6^2 \\ \leq \left\{ (1 - \sigma^2 |N_2|^2) - 2 \operatorname{Re} \left(\sigma \bar{N}_2 (1 - \sigma N_1) \frac{M_2}{M_1} \right) \right. \\ \left. + \sigma(N_1 + \bar{N}_1 - \sigma |N_1|^2) \left| \frac{M_2}{M_1} \right|^2 \right\} \\ \leq (2T_1)^{-5} [H(\sigma)]^5 [B(\sigma)]^{-1} \equiv 1 + 2\sigma \eta \alpha_2 (1 + \sigma \eta) C_6 + 2\sigma \eta C_6^2,$$

for $N_1 + \bar{N}_1 < 2\eta$. From (31) and (37) it follows that

$$(38) \quad \frac{\pi^2 (2T_1)^5 b(\sigma_1)}{6\sigma_1 |M_1|^2} \leq \Lambda_{\mathfrak{R}}^{[2]}(N; M) \leq \frac{\pi^2 (2T_1)^5 B(\sigma_2)}{6\sigma_2 |M_1|^2}.$$

Using (35) and (38) we obtain

$$(39) \quad \frac{3\sigma_2 |M_1|^2 [1 - \frac{1}{2}\sigma_1(1 + \alpha^2)T_1]^3}{(2T_1)^2 \sigma_1 B(\sigma_2)} \leq dT_{\mathfrak{R}}^2(M; N) \leq \frac{3\sigma_1 |M_1|^2}{\sigma_2 (2T_1)^2 b(\sigma_1)}.$$

*In the following we assume that $\varphi_k = 0$ and $\Phi_k = 0$ so that n_i and N_i are real quantities. This assumption involves no loss of generality due to the symmetry of the domain.

In a similar way we obtain for the unit hypersphere the relation

$$(40) \quad \frac{3 |m_1|^2 [1 - \frac{1}{2}(1 + \alpha^2)t_1]^3}{(2t_1)^2 B(1)} \leq dT_{\tau}^2(m; n) \leq \frac{3 |m_1|^2}{(2t_1)^2 b(1)}.$$

When using coordinates normal with respect to the point p and considering a point $q = (t_a, 0)$, according to Lemma 2 we obtain for the coordinates $T_1^{(q)}$, $T_2^{(q)}$ of the image Q of q , the inequality

$$(41) \quad \frac{1}{a} t_1^{(a)} \leq T_1^{(q)} \leq a t_1^{(a)}.$$

Therefore there exists a positive constant C_7 such that for every point $\{t_1, 0\} \in \mathfrak{s}$, we have

$$(42) \quad \frac{1}{C_7} \leq \left| \frac{T_1}{t_1} \right| \leq C_7 < \infty.$$

Since, further, the vector fields $\{m\}$ and $\{M\}$ satisfy the condition C, the following relation holds, as we shall show, in the strip $\mathfrak{s}$ for the m_1 and M_1 coordinates of the corresponding points:

$$(43) \quad \frac{1}{C_8} \leq \left| \frac{M_1}{m_1} \right| \leq C_8 < \infty.$$

Indeed, from Lemma 3 it follows that

$$(44) \quad \frac{1}{C_9} \leq \frac{q_1^2 + q_2^2 + q_3^2 + q_4^2}{Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2} \leq C_9,$$

see p. 948. Condition C yields

$$(45) \quad \frac{q_1^2 + q_2^2}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \geq C_{10}, \quad \frac{Q_1^2 + Q_2^2}{Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2} \geq C_{11},$$

and therefore

$$(46) \quad C_{10} \cdot \frac{1}{C_9} \cdot 1 \leq \frac{q_1^2 + q_2^2}{Q_1^2 + Q_2^2} = \frac{q_1^2 + q_2^2}{q_1^2 + q_2^2 + q_3^2 + q_4^2} \cdot \frac{q_1^2 + q_2^2 + q_3^2 + q_4^2}{Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2} \cdot \frac{Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2}{Q_1^2 + Q_2^2} \leq 1 \cdot C_9 \cdot \frac{1}{C_{11}}.$$

From (39), (40), (42), (43) it follows that

$$(47) \quad \begin{aligned} \frac{1}{C_{12}} &\leq \frac{|M_1|^2}{(T_1)^2} \frac{(t_1)^2 \sigma_2 b(1) (1 - \frac{1}{2} \sigma_1 (1 + \alpha^2) T_1)^3}{(T_1)^2 |m_1|^2 \sigma_1 B(\sigma_2)} \leq \frac{dT_{\mathfrak{R}}^2(M; N)}{dT_{\tau}^2(m; n)} \\ &\leq \frac{|M_1|^2}{(T_1)^2} \frac{|M_1|^2 (t_1)^2 \sigma_1 B(1)}{|m_1|^2 \sigma_2 b(\sigma_1) (1 - \frac{1}{2} (1 + \alpha^2) t_1)^3} \leq C_{12} < \infty \end{aligned}$$

where C_{12} is a conveniently chosen positive constant. Since by definition $dT_{\mathfrak{R}}^2(M; N) = dS_{\mathfrak{R}}^2(Z; U)$ and $dT_{\tau}^2(m; n) = dS_{\tau}^2(z; u)$, (23) holds for $(z_1, z_2) \in \mathfrak{s}$

and $\{dz_1, dz_2\}$, $\{dZ_1, dZ_2\}$ satisfying the condition C. This completes the proof of Theorem 3.1.

4. Concluding remarks. In §3 the bounds (3.23) for $dS_{\mathfrak{K}}/dS_{\mathfrak{L}}$ have been derived under the assumption that the line elements $dS_{\mathfrak{K}}(Z; U)$ and $dS_{\mathfrak{L}}(z; u)$ satisfy the condition C. (In many instances it is possible to clarify how the above ratio behaves at the boundary even when the condition C is not satisfied.) Further, it is possible to generalize our approach to domains which are PC images of general circular domains, *i.e.*, domains which admit a one-parameter group

$$(1) \quad Z_k^* = Z_k e^{i\Phi}, \quad k = 1, 2,$$

of rotations into themselves.

One can associate with every circular domain a convex (three-dimensional) domain $\mathfrak{D}$ located in the upper half ($X_3 \geq 0$) of the X_1, X_2, X_3 -space. The intersection of $\mathfrak{D}$ with the plane $X_3 = 0$ is a circle of radius R , with the center at $B(0, 0, 0)$. Applying then the transformation (1) to $\mathfrak{D}$ one obtains the circular domain under consideration. See Fig. 3.

FIGURE 3. The domain $\mathfrak{D}$ (the upper half of the intersection of the circular domain $\mathfrak{K}$ with the X_3 -space).

We assume that the part δ of the boundary of $\mathfrak{D}$ which lies in $X_3 > 0$ is continuously differentiable (including the edge lying in $X_3 = 0$), and that the point A where the tangent plane to δ is perpendicular to direction of the X_3 -axis lies on this axis (so that $A = (0, 0, X_3^{(0)})$). From the point B we draw in the

X_1, X_2 -plane segments of straight lines. Each of these segments intersect the circumference of the circle in the point $C^{(\Phi)}$. If we introduce polar coordinates R, Θ, Φ so that B coincides with the origin $R = 0$, and the X_1, X_2 -plane coincides with the plane $\Theta = \frac{1}{2}\pi$, to every ray $BC^{(\Phi)}$ from B , forming the angle Φ with the positive X_1 -direction, corresponds an intersection $\mathfrak{S}(\Phi) = ABC^{(\Phi)}$ with the domain $\mathfrak{D}$. Let $\mathfrak{d}$ be the upper half of the unit sphere r in the x_1, x_2, x_3 -space with the center at $b = (x_1 = x_2 = x_3 = 0)$. The x_3 -axis intersects the boundary of the unit sphere at the point a . Let $bc^{(\varphi)}$ be the ray from the center of the unit circle, which forms (in the plane $x_3 = 0$) the angle φ with the positive x_1 -axis. See Fig. 4.

FIGURE 4. The domain $\mathfrak{d}$ (the upper half of the unit sphere).

We shall describe a mapping of the domain $\mathfrak{D}$ onto $\mathfrak{d}$ transforming the point B into b .

Let $\eta = t(\zeta, \varphi)$ be the conformal mapping of the sector $s(\varphi) = abc^{(\varphi)}$ into the sector $S(\Phi)$, $\varphi = \Phi$, so that the points $a, b, c^{(\varphi)}$ go into the points $A, B, C^{(\Phi)}$, respectively. This defines a mapping of the domain $\mathfrak{d} - ba$ into $\mathfrak{D} - BA$. However, the mapping is not defined on the segment BA since the transformation $T(\eta, \Phi)$, inverse to $t(\zeta, \varphi)$, $\varphi = \Phi$, transforms a point $E = (0, 0, X_3)$ of the segment BA into the point $e^{(\varphi)} = (0, 0, x_3(\varphi))$ of ba , which (in general) *varies when φ varies*. Therefore the mapping $T(\eta, \Phi)$ has to be modified; namely, we shall define an additional transformation of every sector $abc^{(\varphi)}$ onto itself preserving the points $a, b, c^{(\varphi)}$, which transforms every point $e^{(\varphi)}$ into the point $e^{(0)}$ (i.e., into the point of the ba -axis, when using $T(\eta, 0)$ we map the sector $ABC^{(0)}$ into the sector $abc^{(0)}$).

As mentioned before, $T(\eta, \Phi)$ transforms the point E of the AB -axis into $e^{(\varphi)}$. Let $c^{(\varphi)}e^{(\varphi)}$ be the arc of the circle connecting the point $e^{(\varphi)}$ with $c^{(\varphi)}$ and perpendicular to ab in the plane $abc^{(\varphi)}$. We denote by $\mathfrak{Q} = C^{(\Phi)}E$ its image in the sector $ABC^{(\Phi)}$. Let

$$(3) \quad \zeta = \tau(\zeta, \varphi) = \frac{(ie^{(\varphi)}e^{(0)} + i)\zeta + (e^{(\varphi)} - e^{(0)})}{(e^{(\varphi)} - e^{(0)})\zeta + (ie^{(\varphi)}e^{(0)} + i)}$$

be the linear transformation which changes the arc $\widehat{c^{(\varphi)}e^{(\varphi)}}$ into the arc $\widehat{c^{(\varphi)}e^{(0)}}$.^{*} See Fig. 5. The combination of $T(\eta, \Phi)$ with $\tau(\zeta, \varphi)$ transforms the arc $\widehat{C^{(\Phi)}E}$ into the arc $\widehat{c^{(\varphi)}e^{(0)}}$ and therefore it defines a mapping of every segment $BAC^{(\Phi)}$ onto $bac^{(\varphi)}$ which has the property that the point $E = (0, 0, X_3)$ goes into the point $e = (0, 0, x_3(0))$ and therefore is *independent of φ* .

FIGURE 5. The arcs $\widehat{c^{(\varphi)}e^{(\varphi)}}$ and $\widehat{c^{(\varphi)}e^{(0)}}$.

When defining QPC transformations in the present paper we used the fact that domains which are PC-equivalent to a circular domain have a “distinguished point” inside the domain, namely, the fixed point of the group of transformations of the domain into itself. When considering QPC transformations of domains which are PC-equivalent to a circular domain $\mathfrak{H}$ we required that the distinguished point of the circular domain $\mathfrak{H}$ be mapped into the distinguished point of the hypersphere, namely the center. On the other hand, one can also obtain distinguished points in a class of PC-equivalent domains by using different methods which can be applied to domains which do not necessarily admit PC transformations into themselves.

For instance, one can consider extremal points (maxima, minima and minimaxes) of some invariants with respect to PC transformations in a given domain $\mathfrak{B}$. We assume that there are only a finite or denumerable number of extremal points, say $P_1, P_2, \dots$. Then one can consider a second invariant, say $\mathbf{R}$ (see [6] p. 55), at the aforementioned points $P_1, P_2, \dots$. If $\mathbf{R}(P_k)$ is the smallest of these values^{**} then one can use P_k as the “distinguished point”. If, however, there are several such points, say $Q_1, Q_2, \dots$, one considers a third invariant, say J_2 (see e.g. [15]), at the points $Q_1, Q_2, \dots$. (The points $Q_1, Q_2, \dots$ represent a subsequence of points $P_1, P_2, \dots$.) If in this manner

^{*}These curves are segments of a circle with the center on the continuation of ba so that they are perpendicular to ab at the point of their intersection with ab .

^{**}Thus, if the sequence $P_1, P_2, \dots$ is infinite we require that the minimum of the invariant is actually assumed *in* the domain.

a unique point has not yet been selected, this procedure can be continued, using further invariants. It is possible to show that there are instances where this procedure permits us to determine *one* distinguished point in a finite number of steps.

Suppose we have two classes of PC-equivalent domains and in a domain of each class, say in $\mathfrak{B}_1$ and $\mathfrak{B}_2$, we can determine a distinguished point. In this case we can ask whether there exists a QPC transformation which transforms the domain $\mathfrak{B}_1$ onto $\mathfrak{B}_2$, mapping the distinguished point of the first domain into that of the second one.

This procedure also can be used in studying mappings in spaces of more than two complex variables.

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