The Kernel-Function and Conformal Invariants

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1. Introduction. In this paper we present some applications to the theory of conformal mapping of certain invariants which are closely related to the Bergman kernel-function. In particular, it will be shown that such invariants, which can be expressed in terms of the solutions to certain extremal problems of a rather simple type, may be used to determine the conformal modulus of a doubly-connected domain. It will be readily apparent that the methods employed here may be used to define an unlimited number of invariants which could also be used in similar investigations. It appears likely that Theorems 1 and 2, which deal with doubly-connected domains, can be generalized in such a manner as to relate some of these invariants to the conformal moduli of domains of higher connectivity. We hope to return to such problems in a later paper.

2. The Bergman Kernel-function of a Plane Domain. In this section we present briefly, and without any proofs, the most essential facts concerning the kernel-function of a plane domain. For a detailed exposition the book [1] of BERGMAN and his original papers may be consulted.

Let a domain D be given in the plane of the complex variable $z = x + iy$, and let H_D be the family of all functions which are analytic, single-valued, and quadratically integrable in D , *i.e.*,

$$(1) \quad \iint_D |f(z)|^2 dx dy < \infty.$$

If the inner product is defined in the usual manner,

$$(2) \quad (f, g) = \iint_D f(z) \overline{g(z)} dx dy,$$

H_D becomes a (separable) Hilbert space. (We assume that the boundary of D

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does not consist entirely of isolated points; in such a case H_D contains only the function $f \equiv 0$.) If the functions $\varphi_1, \varphi_2, \dots$ constitute an orthonormal basis in H_D , then the series²

$$(3) \quad K(z, t) = \sum_{k=1}^{\infty} \varphi_k(z) \overline{\varphi_k(t)}$$

converges for every pair of points $z, t \in D$, and the function $K(z, t)$ thus defined is, despite its seeming dependence on the basis $\{\varphi_k\}$, completely determined by the domain D . This is shown by establishing the following fact: For all functions of H_D which satisfy the condition $f(t) = 1$, the inequality $\iint_D |f|^2 dx dy \geq K(t, t)^{-1}$ holds, and the indicated lower bound is attained by the function $K(z, t)/K(t, t)$ and no other function. Thus $K(z, t)$ is characterized independently of any basis; it is the Bergman kernel-function of D .

With each function $f \in H_D$ and any orthonormal basis $\{\varphi_k\}$ we associate the Fourier expansion

$$(4) \quad f(z) \sim \sum_{k=1}^{\infty} f_k \varphi_k(z), \quad f_k = (f, \varphi_k);$$

in contrast to the usual situation in real analysis, the expansion (4) converges to $f(z)$ not only in norm, but pointwise, and uniformly in every compact subset of D .

Let $X_0, X_1, \dots, X_n$ be any finite set of complex numbers, and let t_0 be any point of the domain D . Then of all functions in H_D which satisfy the conditions

$$(5) \quad f(t_0) = X_0, \quad \left. \frac{df}{dz} \right|_{z=t_0} = X_1, \quad \dots, \quad \left. \frac{d^n f}{dz^n} \right|_{z=t_0} = X_n,$$

we denote the unique one which minimizes $\iint_D |f|^2 dx dy$ as $f^{X_0, X_1, \dots, X_n}$, and the minimum value of this integral is denoted as $\lambda^{X_0, X_1, \dots, X_n}$. (The dependence of the function $f^{X_0, X_1, \dots, X_n}$ and the *minimum integral* $\lambda^{X_0, X_1, \dots, X_n}$ on the point t_0 is not usually expressly indicated, but must be kept in mind.) Of particular importance are the functions $f^1, f^{01}, f^{001}, \dots$ associated with any fixed point t_0 of D and the corresponding quantities $\lambda^1, \lambda^{01}, \lambda^{001}, \dots$. It is easily shown that the functions $f^1/\sqrt{\lambda^1}, f^{01}/\sqrt{\lambda^{01}}, f^{001}/\sqrt{\lambda^{001}}, \dots$ constitute an orthonormal basis in H_D ; the expansion resulting when this particular basis is employed in (3),

$$(6) \quad K(z, t) = \frac{f^1(z) \overline{f^1(t)}}{\lambda^1} + \frac{f^{01}(z) \overline{f^{01}(t)}}{\lambda^{01}} + \frac{f^{001}(z) \overline{f^{001}(t)}}{\lambda^{001}} + \dots,$$

is particularly useful in solving certain types of extremal problems in which the conditions are imposed at the point t_0 . Of great importance in many investigations is the obvious fact that the λ 's are monotone functions of the domain; thus, if $t_0 \in D_1 \subset D_2$, then

² BERGMAN and some other authors employ the notation $K(z, \bar{t})$ instead of $K(z, t)$.

$$(7) \quad \lambda_{D_1}^{X_0 X_1 \cdots X_n} \leq \lambda_{D_2}^{X_0 X_1 \cdots X_n}.$$

The possibility of expressing quantities of geometrical significance (*e.g.*, the invariants C and δ defined below) in terms of the λ 's and using the monotonicity property (7) to analyze the behavior of such quantities is the basis of BERGMAN's *method of the minimum integral*. (Cf. [1, p. 39].)

If the domain D of the z -plane is mapped 1-1 and conformally onto the domain $\tilde{D}$ of the ζ -plane by the mapping-function $\zeta = w(z)$, the kernel-functions of the two domains are related by the identity

$$(8) \quad K_D(z, t) = K_{\tilde{D}}(\zeta, \tau) w'(z) \overline{w'(t)}, \quad \tau = w(t).$$

Identifying the points z and t , we conclude from (8) that the positive definite quadratic form

$$(9) \quad ds^2 = K(z, z) |dz|^2 = K(z, z) \{(dx)^2 + (dy)^2\}$$

is conformally invariant, and thus associates with each equivalence-class of domains an intrinsic Riemannian metric. (Cf. [1, Chapter III].) By an explicit computation (which, as shown in the appendix, becomes particularly simple if the expansion (6) is employed) one obtains for the Gaussian curvature C of BERGMAN's *invariant metric* (9) the remarkable formula³

$$(10) \quad C = -2(\log K)_{,z\bar{z}}/K = -2(\lambda^1)^2/\lambda^{01}.$$

By exploiting the monotonicity property (7) of the λ 's, one can show (*cf.* [1, pp. 36-39]) that as the point t approaches the boundary of the domain, $C(t)$ approaches the value -4π (provided that certain mild conditions are imposed on the boundary).

3. The Modulus of a Doubly-connected Domain. According to the Riemann mapping theorem, every simply-connected domain (aside from trivial exceptions) can be mapped conformally onto the unit disc with any prescribed point mapping into the center. Accordingly, the curvature C of the invariant metric (9) has a constant value, namely -4π , in any such domain. On the other hand, ZARANKIEWICZ [3] showed, by expressing the kernel-function of an annulus in closed form and then explicitly computing the curvature, that the latter is not

³ Here and elsewhere we use the notation

$$K = K(z, z), \quad u_z = \frac{\partial u}{\partial z} = \frac{1}{2} \left(\frac{\partial u}{\partial x} - i \frac{\partial u}{\partial y} \right), \quad u_{\bar{z}} = \frac{\partial u}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial u}{\partial x} + i \frac{\partial u}{\partial y} \right).$$

Thus

$$u_{z\bar{z}} = \frac{1}{4} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

constant;⁴ since every doubly-connected domain is conformally equivalent to some annulus, it follows that every such domain has non-constant curvature. He then showed that from a knowledge of the curvature at every point of a doubly-connected domain the modulus of the domain (*i.e.*, the unique ratio of inner to outer radius of any annulus onto which the domain can be mapped) can be computed, and the mapping-function onto the annulus can be effectively constructed.

In this section we develop two new methods for computing the modulus of a doubly-connected domain. In contrast to the method of [3], which requires the determination of a single conformal invariant throughout the domain, we employ two invariants, but require their values only at an arbitrary sequence of points converging to the boundary in the first method (Theorem 1), while the second method (Theorem 2) enables us to compute the modulus from the values of the same pair of invariants at a single, completely arbitrary, point of the domain.

We begin by noting that if J is any conformally invariant function, then so is $J_{z\bar{z}}/K$; for, under a conformal mapping $\zeta = w(z)$ of the domain D onto the domain $\tilde{D}$, we have, taking account of (8) and the Cauchy-Riemann equations,

$$(11) \quad \frac{J_{z\bar{z}}}{K_D} = \frac{J_{\zeta\bar{\zeta}}}{K_{\tilde{D}}} \frac{\left| \frac{dw}{dz} \right|^2}{\left| \frac{dw}{dz} \right|^2} = \frac{J_{\zeta\bar{\zeta}}}{K_{\tilde{D}}}.$$

In particular, let J denote the invariant $\log(-C)$; in view of (10), C is negative throughout any domain, so J is well defined. By a computation, which we

⁴ It is perhaps worthwhile to demonstrate the non-constancy of the curvature without using the explicit form of the kernel-function. According to the final statement in §2, the curvature, if constant, would be $\equiv -4\pi$. (Actually, the proof can be carried out without this observation.) From (10) and the obvious rotational symmetry of K we would get

$$(\log K)_{z\bar{z}} = \frac{1}{4r} \frac{d}{dr} \left(r \frac{d}{dr} \log K \right) = 2\pi K, \quad r = |z|.$$

It is an elementary consequence of the monotonicity property (7) and of the relation $K = 1/\lambda^4$ that K becomes infinite at both boundary curves (*cf.* [1, p. 38]); this condition determines a unique solution of the above differential equation, namely

$$K = \frac{\pi}{16r^2 \log^2 R} \sec^2 \left(\frac{\pi \log r}{2 \log R} \right),$$

if we normalize the radii of the annulus to be R and $1/R$. By "polarization" (setting $r^2 = z\bar{z}$ and then replacing $\bar{z}$ by $\bar{t}$) we get

$$K(z, t) = \frac{\pi}{16z\bar{t} \log^2 R} \sec^2 \left\{ \frac{\pi}{4 \log R} (\log z + \log \bar{t}) \right\}.$$

However, if either $\log z$ or $\log \bar{t}$ is increased by $2\pi i$, the right side of the last equation changes value, while $K(z, t)$ must be single-valued. Thus we have obtained a contradiction.

relegate to the appendix, we obtain the striking fact that the new invariant, which we term γ , can, like C , be expressed in terms of BERGMAN's *minimum integrals*, namely

$$(12) \quad \gamma = \frac{(\log(-C))_{,z}}{K} = \frac{\lambda^1 \lambda^{01}}{\lambda^{001}} - \frac{3(\lambda^1)^2}{\lambda^{01}}.$$

Now, as shown in [3], the kernel-function of the annulus $R < |z| < 1$ is given by⁵

$$(13) \quad K(z, t) = \frac{1}{\pi z \bar{t}} \left\{ \wp(\log z + \log \bar{t}; 2 \log \frac{1}{R}, 2\pi i) + \frac{\zeta(\pi i; 2 \log \frac{1}{R}, 2\pi i)}{\pi i} \right\},$$

where $\wp$ and ζ are the functions of WEIERSTRASS appearing in the theory of elliptic functions (cf. [2, pp. 161, 175]). From (10), (12), and (13) we readily obtain for C and γ the formulas

$$(14) \quad C = \frac{-2\pi A}{(\wp + \alpha)^3}, \quad A = 2\wp^3 + 6\alpha\wp^2 + \frac{1}{2}g_2\wp + (g_3 - \frac{1}{2}\alpha g_2),$$

$$\gamma = \frac{-3\pi A}{(\wp + \alpha)^3} + \frac{\pi N}{A^2(\wp + \alpha)},$$

$$(15) \quad N = 24\wp^7 + 168\alpha\wp^6 + (30g_2 + 144\alpha^2)\wp^5 + (96g_3 + 6\alpha g_2)\wp^4 \\ + (168\alpha g_3 - 24\alpha^2 g_2 - \frac{3}{2}g_2^2)\wp^3 + \left(\frac{15}{2}\alpha g_2^2 - 12g_2 g_3 + 72\alpha^2 g_3\right)\wp^2 \\ + \left(\frac{1}{8}g_2^3 - 12g_2^2 + 9\alpha^2 g_2^2 - 6\alpha g_2 g_3\right)\wp + \left(\frac{1}{8}\alpha g_2^3 - 12\alpha g_2^2 + 6\alpha^2 g_2 g_3\right).$$

Here

$$\wp = \wp(2 \log |z|; 2 \log \frac{1}{R}, 2\pi i), \quad \alpha = \frac{\zeta(\pi i; 2 \log \frac{1}{R}, 2\pi i)}{\pi i},$$

and g_2, g_3 denote the "invariants" of $\wp$ determined by the periods $2 \log (1/R)$, $2\pi i$. (Cf. [2, p. 167].) The derivatives of $\wp$ which appear in the process of computing the formulas for C and γ are eliminated by taking account of the differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ satisfied by this function. Letting $2 \log |z| = Z$ and employing the Laurent expansion

$$(16) \quad \wp\left(Z; 2 \log \frac{1}{R}, 2\pi i\right) = \frac{1}{Z^2} + \frac{1}{20} g_2 Z^2 + \frac{1}{28} g_3 Z^4 + \dots$$

⁵ The kernel-function employed here and in [3] differs from the one defined in §2 in that the analytic functions under consideration are required to possess single-valued integrals, so that the Hilbert space employed here is a subspace, of deficiency one, of the space H_D defined in §2. This restriction is necessitated by the fact that the curvature associated with the kernel-function of the entire space H_0 fails to possess the monotonicity property, described on p. 930, which plays an essential role both here and in [3]. (Cf. [1, p. 47].)

we obtain the expansions

$$(17) \quad C = -4\pi \left\{ 1 + \left(\frac{1}{4} g_2 - 3\alpha^2 \right) Z^4 + \left(\frac{1}{2} g_3 - \alpha g_2 + 8\alpha^3 \right) Z^6 + \dots \right\},$$

$$(18) \quad \gamma = 12\pi \left\{ \left(\frac{1}{4} g_2 - 3\alpha^2 \right) Z^4 + \left(\frac{5}{4} g_3 - \frac{11}{4} \alpha g_2 + 23\alpha^2 \right) Z^6 + \dots \right\},$$

$$(19) \quad \frac{\gamma}{12\pi} + \frac{C}{4\pi} + 1 = \left(\frac{3}{4} g_3 - \frac{7}{4} \alpha g_2 + 15\alpha^3 \right) Z^6 + \dots$$

As shown in [3], $-C - 4\pi$ is always positive; from this it follows that the coefficient $(\frac{1}{4}g_2 - 3\alpha^2)$ appearing in (17) and (18) is positive for all R , $0 < R < 1$. (This can also be proven easily from the expansions (27), (28) for α and g_2 .) Thus $(-C/4\pi - 1)^{\frac{1}{3}}$ is real, and from (17) and (19) we obtain

$$(20) \quad \lim_{|z| \rightarrow 1} \frac{\frac{\gamma}{12\pi} + \frac{C}{4\pi} + 1}{\left(-\frac{C}{4\pi} - 1 \right)^{\frac{1}{3}}} = \frac{\left(\frac{3}{4} g_3 - \frac{7}{4} \alpha g_2 + 15\alpha^3 \right)}{\left(\frac{1}{4} g_2 - 3\alpha^2 \right)^{\frac{1}{3}}}.$$

Since any doubly-connected domain D can be mapped onto the annulus $R < |z| < 1$ for some unique value of R (the modulus of D) and since C and γ are conformally invariant, the relation (20) must hold in D also, the condition $|z| \rightarrow 1$ being replaced by the condition that the variable point should approach the boundary of D . Now, as shown in the appendix, the right hand side of (20) is a monotone function of R . Taking account of (10), (12), and (20) we obtain the following result:

Theorem 1. *Let D be any doubly-connected domain. Then the value of the function*

$$\frac{\frac{\lambda^1 \lambda^{01}}{\lambda^{001}} - \frac{9(\lambda^1)^2}{\lambda^{01}} + 12\pi}{\left(\frac{(\lambda^1)^2}{\lambda^{01}} - 2\pi \right)^{\frac{1}{3}}}$$

approaches at the boundary a value which completely determines the modulus of D .

We now get a stronger result by showing that the values of C and γ (or, equivalently, λ^1 , λ^{01} , λ^{001}) at any single point of D completely determine the modulus. First we consider the case where D is an annulus. For convenience, we denote the right hand sides of (14) and (15) by $f(r, R)$, $g(r, R)$, respectively ($r = |z|$). As shown in [3], $\partial f / \partial r$ is negative in the interval $R < r < \sqrt{R}$ for each fixed value of R ; i.e., C decreases from its limiting value of -4π until it reaches some lower value (depending on R) at $r = \sqrt{R}$, then for larger r it increases steadily and attains the value -4π again at $r = 1$. Therefore, if we restrict r to the interval $R < r \leq \sqrt{R}$, we can solve the equation (14) to obtain

$r = F(C, R)$; substituting into (15), we obtain $\gamma = g(F(C, R), R) = G(C, R)$. By differentiating these equations we obtain

$$(21) \quad \frac{\partial G}{\partial R} = \frac{\partial g}{\partial r} \frac{\partial F}{\partial R} + \frac{\partial g}{\partial R},$$

$$(22) \quad 0 = \frac{\partial f}{\partial r} \frac{\partial F}{\partial R} + \frac{\partial f}{\partial R}, \quad \frac{\partial F}{\partial R} = -\frac{\partial f}{\partial R} \bigg/ \frac{\partial f}{\partial r}.$$

Substituting from (22) into (21), we obtain

$$(23) \quad \frac{\partial G}{\partial R} = \left(\frac{\partial f}{\partial r} \frac{\partial g}{\partial R} - \frac{\partial f}{\partial R} \frac{\partial g}{\partial r} \right) \bigg/ \frac{\partial f}{\partial r}.$$

Now, as shown in the appendix, the Jacobian $\partial(f, g)/\partial(r, R)$ appearing in (23), and hence $\partial G/\partial R$, has a fixed sign for all values of r and R under consideration, so that for fixed C the invariant γ is a monotone function of R ; therefore R may be expressed as a single-valued function of C and γ . Once again taking account of the invariance of C and γ , we obtain the following result:

Theorem 2. *The values of the invariants C and γ at any single point of a doubly-connected domain completely determine its modulus, which can be computed from equations (14) and (15).*

While the proofs presented in the appendix are of a tedious computational nature, it is to be hoped that some proofs less computational in nature will be discovered. If proofs can be found which do not depend on the explicit form of the kernel-function, an analysis of the relationship between various invariants (or λ 's) and the moduli of domains of higher connectivity, for which the kernel-functions are as yet unknown, might also be accomplished.

4. Integral Invariants. In addition to forming invariant functions, one can also form invariant domain-functionals. In particular, if J is any invariant function defined on a domain D , then it follows from the transformation law (8) for the kernel-function that the integral $\iint_D JK \, dx \, dy$ is, if convergent, also an invariant.

The simplest such integral would appear to be

$$\iint_D (C + 4\pi)K \, dx \, dy = \iint_D \left(\frac{4\pi}{\lambda^1} - \frac{2\lambda^1}{\lambda^{01}} \right) dx \, dy.$$

(The constant 4π must be added to C to render the integral convergent. In the case of an annulus, and hence for any doubly-connected domain, one can show explicitly, by employing (13) and (17), that the factor $C + 4\pi$ approaches zero near the boundary sufficiently rapidly to overcome the behavior of the factor K ; for domains of arbitrarily high connectivity this may be demonstrated by using the methods of [1, pp. 36–39], which are based on the monotonicity principle (7).) If this integral turned out, in the case of a doubly-connected

domain, to depend monotonely on the modulus, we would have another characterization of the latter quantity. Somewhat surprisingly, this integral turns out to have a value independent of the modulus, namely -4π . We do not present the proof, which is a simple exercise in the theory of elliptic functions. If instead of the kernel-function (13) we employ the kernel-function for the entire space H_D (cf. footnote on p. 929), the integral again turns out to have a constant value, namely zero. Similarly, using either kernel-function, one obtains the value zero for $\iint_D \gamma K \, dx \, dy$, where γ is the invariant defined in §3. It would be of interest to determine a set of invariants $\{J\}$ related to the kernel-function such that the corresponding integral invariants would enable us to compute the moduli of a given domain, but even in the case of a doubly-connected domain we have obtained only the aforementioned negative results.

APPENDIX

Here we present three computations whose results are used in §3.

(a) *Proof of (10) and (12).*

From the definitions of C and γ we immediately get

$$(24) \quad C = \frac{-2\delta}{K^3}, \quad \delta = KK_{z\bar{z}} - K_z K_{\bar{z}}$$

and

$$(25) \quad \gamma = \frac{\delta\delta_{z\bar{z}} - \delta_z\delta_{\bar{z}}}{K\delta^2} - \frac{3\delta}{K^3}.$$

Let any point t_0 of the domain D be chosen and let the corresponding minimizing functions $f^1, f^{01}, f^{001}, \dots$ and minimum integrals $\lambda^1, \lambda^{01}, \lambda^{001}, \dots$ be determined. Employing the expansion (6) and recalling the behavior of these functions at t_0 , we readily obtain the following formulas, valid only at that point:⁶

$$(26) \quad K = \frac{1}{\lambda^1}, \quad K_z = \frac{f_1^1}{\lambda^1}, \quad K_{\bar{z}} = \frac{\bar{f}_1^1}{\lambda^1}, \quad K_{z\bar{z}} = \frac{|f_1^1|^2}{\lambda^1} + \frac{1}{\lambda^{01}},$$

$$\delta = \frac{1}{\lambda^1\lambda^{01}}, \quad \delta_z = \frac{f_2^{01}}{\lambda^1\lambda^{01}}, \quad \delta_{\bar{z}} = \frac{\bar{f}_2^{01}}{\lambda^1\lambda^{01}}, \quad \delta_{z\bar{z}} = \frac{|f_2^{01}|^2}{\lambda^1\lambda^{01}} + \frac{1}{\lambda^1\lambda^{01}}.$$

Substituting from (26) into (24) and (25) we get the expressions for C and γ given in (10) and (12).

(b) *Proof of the monotonicity of the right-hand side of (20).*

⁶ For convenience we denote differentiation by subscripts; e.g.,

$$f_2^{01} = \left. \frac{d^2 f^{01}}{dz^2} \right]_{z=t_0}.$$

Here and in (c) we shall employ two different expansions for each of the three quantities α , g_2 , g_3 . These expansions are the following (cf. [2, pp. 190, 210]):

$$(27) \quad \alpha = -\frac{1}{12} + 2 \sum_{k=1}^{\infty} \frac{k(R^2)^k}{1 - (R^2)^k} = -\frac{u}{2\pi} + u^2 \left\{ \frac{1}{12} - 2 \sum_{k=1}^{\infty} \frac{k}{e^{2\pi k u} - 1} \right\},$$

$$(28) \quad g_2 = \frac{1}{12} + 20 \sum_{k=1}^{\infty} \frac{k^3(R^2)^k}{1 - (R^2)^k} = u^4 \left\{ \frac{1}{12} + 20 \sum_{k=1}^{\infty} \frac{k^3}{e^{2\pi k u} - 1} \right\},$$

$$(29) \quad g_3 = -\frac{1}{216} + \frac{7}{3} \sum_{k=1}^{\infty} \frac{k^5(R^2)^k}{1 - (R^2)^k} = u^6 \left\{ \frac{1}{216} - \frac{7}{3} \sum_{k=1}^{\infty} \frac{k^5}{e^{2\pi k u} - 1} \right\}, \quad u = \frac{\pi}{\log \frac{1}{R}}.$$

Now, denoting the right-hand side of (20) for brevity by $f(R)$, we differentiate and obtain

$$(30) \quad \frac{df}{dR} = \frac{\frac{d\alpha}{dR} (24\alpha^2 g_2 + 216\alpha g_3 - 14g_2^2) + \frac{dg_2}{dR} (7\alpha g_2 - 9g_3 - 12\alpha^3) + \frac{dg_3}{dR} (6g_2 - 72\alpha^2)}{(g_2 - 12\alpha^2)^{5/2}}.$$

To prove that $f(R)$ is monotone decreasing, we shall show that the numerator is negative for all R , $0 < R < 1$. If we rearrange the first expansion of each of the above three pairs into power series, namely

$$\alpha = -\frac{1}{12} + 2R^2 + 6R^4 + \dots,$$

$$g_2 = \frac{1}{12} + 20R^2 + 180R^4 + \dots,$$

$$g_3 = -\frac{1}{216} + \frac{7}{3}R^2 + 77R^4 + \dots,$$

and substitute into (30) we obtain

$$(31) \quad (g_2 - 12\alpha^2)^{5/2} \frac{df}{dR} = -960R^3 - 23040R^5 + \dots,$$

and by coarse estimates which need not be presented here we prove that the remainder of this expansion cannot be as large as the sum of the two computed terms for $R < \frac{1}{5\pi}$. Similarly, for larger values of R we use the second expansion of each of the above pairs and obtain

$$(32) \quad (g_2 - 12\alpha^2)^{5/2} \frac{df}{dR} = \frac{du}{dR} \left(-\frac{u^7}{3\pi^2} + \frac{u^6}{2\pi^3} + \dots \right), \quad \frac{du}{dR} = \frac{\pi}{R \log^2 R} > 0$$

where the two computed terms would give the entire expansion if the infinite sums

$$\sum_{k=1}^{\infty} \frac{k^m}{e^{2\pi k u} - 1}, \quad m = 1, 3, 5,$$

which approach zero very rapidly as $R \rightarrow 1$ ($u \rightarrow +\infty$), were suppressed. The sum of these two computed terms is negative for $u > 3/2\pi$, or $R > e^{-3/2\pi^2} \approx \frac{1}{7\pi}$, and, again using coarse estimates, we can show that for $R > \frac{1}{10\pi}$ the remainder

cannot cause the sign to change. Therefore, df/dR is negative for all R , and $f(R)$ decreases steadily (from $+\infty$ to $-\infty$) as R increases from zero to one.

(c) *Proof that $\partial(f, g)/\partial(r, R) > 0$ for $R < r < \sqrt{R}$, $0 < R < 1$.*

In accordance with standard notation, let

$$2 \log \frac{1}{R} = \omega_1, \quad 2\pi i = \omega_2, \quad e_1 = p(\tfrac{1}{2}\omega_1; \omega_1, \omega_2)$$

$$e_2 = p(\tfrac{1}{2}\omega_1 + \tfrac{1}{2}\omega_2; \omega_1, \omega_2), \quad e_3 = p(\tfrac{1}{2}\omega_2; \omega_1, \omega_2).$$

As r varies from R to $\sqrt{R}$, p ($= p(2 \log r; \omega_1, \omega_2)$) varies monotonely from $+\infty$ to e_1 , and from the well known relationships $e_1 > e_2 > e_3$, $e_1 + e_2 + e_3 = 0$ it follows that $e_1 > 0$, so that $1/p$ is well defined and positive for all values of r and R under consideration. Taking account of this fact, one finds by tedious calculation that the Jacobian $\partial(f, g)/\partial(r, R)$ is, aside from factors that are obviously positive, equal to the following sum, where $P = 1/p$:

$$(33) \quad \sum_{m=0}^5 P^m \left(A_{m1} \frac{d\alpha}{dR} + A_{m2} \frac{dg_2}{dR} + A_{m3} \frac{dg_3}{dR} \right),$$

where the quantities A_{mn} , which are functions of R , are given in the following table:

$m \backslash n$	1	2	3
0	$-144\alpha^2 g_2 - 1296\alpha g_3 + 84g_2^2$	$72\alpha^3 - 42\alpha g_2 + 54g_3$	$432\alpha^2 - 36g_2$
1	$-144\alpha^3 g_2 - 3888\alpha^2 g_3 - 12\alpha g_2^2 + 360g_2 g_3$	$72\alpha^4 + 6\alpha^2 g_2 - 18\alpha g_3$	$1296\alpha^3 - 108\alpha g_2$
2	$-2592\alpha^3 g_3 - 456\alpha^2 g_2^2 + 596\alpha g_2 g_3 - 6g_2^3 + 648g_3^2$	$228\alpha^3 g_2 - 288\alpha^2 g_3 + 3\alpha g_2^2 - 9g_2 g_3$	$6g_2^2 - 216\alpha g_3 + 864\alpha^4$
3	$-360\alpha^3 g_2^2 - 288\alpha^2 g_2 g_3 - 30\alpha g_2^3 + 1944\alpha g_3^2 + 6g_2^2 g_3$	$180\alpha^4 g_2 - 72\alpha^3 g_3 + 15\alpha^2 g_2^2 - 39\alpha g_2 g_3$	$144\alpha^3 g_2 - 648\alpha^2 g_3 + 24\alpha g_2^2$
4	$-504\alpha^3 g_2 g_3 - 33\alpha^2 g_2^3 + 1296\alpha^2 g_3^2 + 21\alpha g_2^2 g_3 + 18g_2 g_2^2 - \frac{3}{4}g_2^4$	$144\alpha^4 g_3 + \frac{3}{2}\alpha^3 \alpha^3 g_2^2 - 42\alpha^2 g_2 g_3 + 18\alpha g_2^3 + \frac{3}{8}\alpha g_2^3 - \frac{3}{8}g_2^2 g_3$	$-18\alpha g_2 g_3 - 432\alpha^3 g_3 + 21\alpha^2 g_2^2 + 72\alpha^4 g_2 + \frac{3}{4}g_2^3$
5	$-9\alpha^3 g_2^3 + 15\alpha^2 g_2^2 g_3 + 18\alpha g_2 g_2^2 - \frac{3}{4}\alpha g_2^4$	$\frac{9}{2}\alpha^4 g_2^2 - 12\alpha^3 g_2 g_3 + 18\alpha^2 g_2^2 + \frac{3}{8}\alpha^2 g_2^3 - \frac{9}{8}\alpha g_2^2 g_3$	$3\alpha^3 g_2^2 - 18\alpha^2 g_2 g_3 + \frac{3}{4}\alpha g_2^3$

We must show that this cumbersome expression is positive for $0 < R < 1$, $0 < P < 1/e_1$, where e_1 admits the following pair of expansions (cf. [2, p. 201]):

$$\begin{aligned}
 (34) \quad e_1 &= \frac{2 \sum_{k=1}^{\infty} (-1)^{k+1} k^2 R^{k^2}}{1 + 2 \sum_{k=1}^{\infty} (-1)^k R^{k^2}} + \frac{\sum_{k=0}^{\infty} (-1)^k (2k+1)^3 R^{k(k+1)}}{12 \sum_{k=0}^{\infty} (-1)^k (2k+1) R^{k(k+1)}} \\
 &= \frac{u^2}{4} \left\{ \frac{1 + \sum_{k=1}^{\infty} (2k+1)^2 e^{-k(k+1)\pi u}}{1 + \sum_{k=1}^{\infty} e^{-k(k+1)\pi u}} - \frac{1 + \sum_{k=1}^{\infty} (-1)^k (2k+1)^3 e^{-k(k+1)\pi u}}{3(1 + \sum_{k=1}^{\infty} (-1)^k (2k+1) e^{-k(k+1)\pi u})} \right\},
 \end{aligned}$$

where $u = \pi/\log(1/R)$, as in (27), (28), (29). Since, as R increases from 0 to 1, $1/e_1$ decreases monotonely from 12 to 0 (as is readily proven from (34)), it will more than suffice to show that the expression (33) is positive for $0 < R < 1$, $0 < P < 12$. For small values of R we substitute the first expansion of each of the pairs (27), (28), (29) into (33) and obtain

$$\begin{aligned}
 (33) &= 2R^3 \left\{ (2880 + 69120R^2 + \dots) + (528 + 65832R^2 + \dots)P \right. \\
 (35) \quad &+ (-88 + 4416R^2 + \dots)P^2 + \left(-\frac{50}{3} - 1360R^2 + \dots \right)P^3 \\
 &\left. + \left(\frac{97}{36} - \frac{10297}{288}R^2 + \dots \right)P^4 + \left(-\frac{41}{432} + \frac{5875}{288}R^2 + \dots \right)P^5 \right\}.
 \end{aligned}$$

Denoting the quantity in { } in this equation by $H(R, P)$, we readily confirm by elementary computations that

$$H(0, P) > 0, \quad \left. \frac{\partial H(R, P)}{\partial (R^2)} \right]_{R=0} > 0, \quad 0 \leq P < 12;$$

by using coarse estimates similar to the ones required in part (b), we can show that the latter inequality, and *a fortiori* the former, holds for $0 < R \leq \frac{1}{10}$, $0 \leq P < 12$. For larger values of R , we employ, as in part (b), the second expansion of each of the pairs (27), (28), (29) and obtain

$$\begin{aligned}
 (33) &= \frac{du}{dR} \left\{ (8c^2u^7 - 12c^3u^6 + \dots) \right. \\
 (36) \quad &+ \left(\frac{10}{3}c^2u^9 - 21c^3u^8 + 12c^4u^7 + \dots \right)P \\
 &+ \left(\frac{5}{9}c^2u^{11} - \frac{37}{6}c^3u^{10} + 12c^4u^9 + \dots \right)P^2 \\
 &+ \left(\frac{5}{108}c^2u^{13} - \frac{53}{72}c^3u^{12} + \frac{5}{2}c^4u^{11} + \dots \right)P^3 \\
 &+ \left(\frac{5}{2592}c^2u^{15} - \frac{347}{5184}c^3u^{14} + \frac{7}{36}c^4u^{13} + \dots \right)P^4 \\
 &\left. + \left(\frac{1}{31104}c^2u^{17} - \frac{17}{20736}c^3u^{16} + \frac{1}{192}c^4u^{15} + \dots \right)P^5 \right\},
 \end{aligned}$$

where $c = 1/2\pi$ and, as before,

$$u = \frac{\pi}{\log \frac{1}{R}}, \quad \frac{du}{dR} = \frac{\pi}{R \log^2 R} > 0.$$

As in (32), the computed terms in (36) would be exact if the infinite series appearing in (27), (28), (29) were suppressed. By rather simple computations we find that the expression obtained by disregarding these infinite series in (36) is positive for $u > \frac{1}{2}$, $0 \leq P < 12$, and that the contributions from these series cannot alter the sign of the right-hand side of (36) for $u > 1$, or $R > e^{-\pi} \approx 0.04$. Thus the Jacobian $\partial(f, g)/\partial(r, R)$ is positive for all values of r and R under consideration, and the proof of Theorem 2 is complete.

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