Stability in Incompressible Systems

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1. In the (n -component vector) system of differential equations

$$(1) \quad x' = f(x) \quad \left(' = \frac{d}{dt} \right),$$

let $f(x)$ be of class C^1 and satisfy the incompressibility condition

$$(2) \quad \operatorname{div} f = 0.$$

A solution $x = x(t)$ of (1) will be called stable, in the sense of LIAPOUNOFF, if $x(t)$ exists for $-\infty < t < \infty$ and if, for every $\epsilon > 0$, there exists a $\delta = \delta_\epsilon$ such that all solutions $y = y(t)$ of (1) exist and satisfy $|x(t) - y(t)| < \epsilon$, for $-\infty < t < \infty$, whenever $|x_0 - y_0| < \delta$. (Here $x_0 = x(0)$, $y_0 = y(0)$, and the norm is the ordinary Euclidean distance.) The following will be proved:

(I) *Let the system (1) satisfy (2) and suppose that $x = x(t)$ is a bounded, stable solution of (1). Then $x(t)$ must be (Bohr) almost periodic.*

That the assertion of (I) can be false if (2) is not assumed is easily seen from a consideration of the single equation $x' = \cos^2 x$; cf. [8], p. 352, also [7], p. 731, for a similar example. In fact, all solutions of the type $x = \operatorname{Arc} \tan(t + c)$ are bounded and stable but fail to be almost periodic. That the assertion of (I) can be false if the boundedness assumption on $x(t)$ is dropped can be seen from the single equation $x' = 1$, of which every solution $x = t + c$ is stable but none is almost periodic.

It is not certain whether (I) is known, although various related theorems do occur in the literature. For instance, a result similar to (I), but when (2) is not assumed and involving a more severe type of "stability" requirement on $x(t)$, was obtained in [5]; in this connection, see the remarks of [8]. For other results relating to recurrence, almost periodicity, and stability, see [1], [3], [4], [7]. In particular, the present theorem is to be compared with a problem formulated in [7], p. 738. For a result concerning almost periodicity and a one-sided LIAPOUNOFF stability, see [9].

Two simple proofs of (I) involving results of ergodic theory will be given.

The first will depend upon a theorem of KHINTCHINE and the second will use the VON NEUMANN mean ergodic theorem.

2. Proof of (I). Consider an open sphere $S = S_\delta$ of center x_0 and radius δ , where $x(t)$ is the given bounded, stable solution of (1). Then, if δ is sufficiently small, all solution paths $y = y(t)$ issuing from S (that is, with $y_0 = y(0)$ in S) exist for all t and satisfy $|x(t) - y(t)| < \text{const}$. Let Ω denote such a set of points, so that Ω is the union of all t -images, S_t , of a sufficiently small sphere $S (= S_0)$ under the flow defined by (1). Since $x(t)$ is bounded, so also is the set Ω , so that Ω has a finite positive volume measure; the condition (2) corresponds to a measure-preserving condition of the flow on Ω .

In order to prove (I), suppose, if possible, that $x(t)$ is not almost periodic. Then, for some $\epsilon > 0$, $x(t)$ does not possess a corresponding relatively dense set of translation numbers (cf. [2], p. 32); hence there exist arbitrarily large t -intervals, $u \leq t \leq v$ ($v - u \rightarrow \infty$), such that if $x(0) = x_0$, then $x(t)$ lies outside a sufficiently small sphere with center at x_0 whenever $u \leq t \leq v$. Consequently, in view of the stability assumption on $x(t)$, there exists a number $\eta = \eta_\epsilon$ so small that every solution $y = y(t)$ of (1) satisfying $|x_0 - y_0| < \eta$ also satisfies $|y(t) - y_0| > \eta$ whenever $u \leq t \leq v$. Thus, for $u \leq t \leq v$, the t -image of the sphere $S = S_\eta$, with center x_0 and radius η , has no point in common with S itself.

From here on, it is possible to complete the proof by using, for example, KHINTCHINE's theorem (cf. [6], p. 40), namely, when applied to the set S above:

$$\lim_{v-u} \frac{\int_u^v \mu(SS_t) dt}{v-u} \geq \frac{\mu^2(S)}{\mu(\Omega)}, \quad \text{as } v-u \rightarrow \infty,$$

where μ denotes the volume measure in Ω . A contradiction results if one notes that the right side of the inequality is positive and that, since u and v can be chosen as the end-points of the interval $u \leq t \leq v$ of the preceding paragraph, the left side is zero. This contradiction implies, of course, that the assumption that $x(t)$ is not almost periodic is untenable and the proof of (I) is now complete.

3. An alternative proof of (I). The proof is identical with that of the preceding section up to the last paragraph. At that point, let $\phi(x)$ denote the characteristic function of the set $S = S_\eta$ and note that $\phi(y(t)) = 0$ on the interval $u \leq t \leq v$ considered there, whenever $y(0) = y_0$ is in S . According to the VON NEUMANN mean ergodic theorem, since ϕ is of class L^2 on the space Ω , there is an invariant function $\psi(x)$, also of class L^2 , for which

$$(3) \quad \int_{\Omega} |(v-u)^{-1} \int_u^v \phi(y(t)) dt - \psi(y_0)|^2 d\mu \rightarrow 0 \quad \text{as } v-u \rightarrow \infty,$$

where $d\mu = dy_0$ is the element of volume of the space Ω . (Incidentally, it is noteworthy that the BIRKHOFF ergodic theorem involves averages over the t -interval $0 \leq t \leq T$ (or $-T \leq t \leq 0$) when $T \rightarrow \infty$, but does not allow, as in

the mean ergodic theorem, the unrestricted limit process $v - u \rightarrow \infty$, which is needed here; in this connection, cf. a remark in [10], p. 796.) If one chooses the interval $u \leq t \leq v$ as above, $\int_u^v \phi(y(t))dt = 0$ for y_0 in S , and hence, by (3), $\psi(y_0) = 0$ almost everywhere on S ; as a consequence, $\phi\psi = 0$ almost everywhere on Ω . But $\int_{\Omega} \phi f d\mu = \int_{\Omega} \psi f d\mu$ for every invariant (real) function f of class L^2 on Ω (cf., e.g., [6], p. 28), and so, if $f = \psi$, it follows that $\psi = 0$ almost everywhere on Ω . In view of the SCHWARZ inequality, relation (3) now implies that

$$(v - u)^{-1} \int_{\Omega} \left(\int_u^v \phi(y(t))dt \right) d\mu \rightarrow 0 \quad \text{as } v - u \rightarrow \infty \quad (u, v \text{ arbitrary}),$$

and hence, by FUBINI's theorem, that $\int_{\Omega} \phi(y(t))d\mu = \int_{\Omega} \phi(y_0)d\mu = 0$. But this means that S has zero measure, a contradiction, and, as before, the proof is complete.

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