Plastic Disturbances Whose Speed of Propagation Is Less Than the Velocity of a Shear Wave

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1. Introduction. Let Σ be a surface of discontinuity in a plastic region whose dynamical behavior is assumed to be governed by the following equations:

$$(1.1) \quad \frac{d\sigma_{ij}^*}{dt} = 2\mu(\epsilon_{ij} - \psi\sigma_{ij}^*), \quad \text{Prandtl-Reuss equations,}$$

$$(1.2) \quad \sigma_{ij,i} = \rho \frac{dv_i}{dt}, \quad \text{equations of motion,}$$

$$(1.3) \quad \frac{\partial \rho}{\partial t} + \rho_{,i}v_i + \rho v_{i,i} = 0, \quad \text{equation of continuity,}$$

$$(1.4) \quad v_{i,i} = 0, \quad \text{equation of incompressibility,}$$

$$(1.5) \quad \sigma_{ij}^*\sigma_{ij}^* = K, \quad \text{quadratic yield condition,}$$

where μ and K are (positive) material constants of the medium and

$$(1.6) \quad \epsilon_{ij} = \frac{1}{2}(v_{i,i} + v_{j,i}); \quad \psi = \frac{1}{K} \sigma_{ij}\epsilon_{ij}.$$

In these relations the quantities σ_{ij}^* are the components of the stress deviator; the remaining symbols have their usual significance. The material is referred to a system of rectangular coordinates x_1, x_2, x_3 and the comma in the above and following relations denotes partial differentiation with respect to these coordinates. A repeated index in any term of an equation indicates a summation on this index.

We assume that Σ is singular of order 1 relative to the set of functions consisting of the stress components σ_{ij} and the velocity components v_i . This implies that $[\sigma_{ij}] = 0$ and $[v_i] = 0$ where the bracket $[\]$ is used to denote the difference in the values of the enclosed quantity on the two sides of the surface Σ . Moreover, we must have relations of the form

$$(1.7) \quad [v_{i,i}] = \lambda_i v_i; \quad \left[\frac{\partial v_i}{\partial t} \right] = -G\lambda_i,$$

$$(1.8) \quad [\sigma_{ij,k}] = \xi_{ij}\nu_k; \quad \left[\frac{\partial \sigma_{ij}}{\partial t} \right] = -G\xi_{ij},$$

where $G (\geq 0)$ is the coordinate velocity of propagation of Σ in the direction of its unit normal ν . Not all of the quantities $[v_{i,i}]$ and $[\sigma_{ij,k}]$ can vanish simultaneously at any point of Σ from the above assumption regarding the order of the singularity over this surface. *Hence not all of the quantities λ_i and ξ_{ij} can vanish at points of Σ .*

Consider the well known relation

$$(1.9) \quad \rho(G - v_n) = \bar{\rho}(G - \bar{v}_n),$$

where ρ and $\bar{\rho}$ are the densities on the two sides of Σ and v_n and $\bar{v}_n$ denote the corresponding components of normal velocity of the material. But $v_n = \bar{v}_n$ since $[v_i] = 0$. Hence if we assume that the *speed* of propagation of Σ , defined by the quantity $G - v_n$, does not vanish it follows from the relation (1.9) that $[\rho] = 0$, i.e. *the density is continuous across Σ .*

In this note¹ we shall be concerned with disturbances or wave surfaces Σ whose speed of propagation $G - v_n$ is different from zero and numerically less than the velocity of an equivoluminal or *shear* wave, i.e.

$$(1.10) \quad 0 < (G - v_n)^2 < \frac{\mu}{\rho}.$$

A formula for the speed of propagation of such disturbances is derived in §5. This formula involves the stress components σ_{ij} , and hence the actual determination of the speed of propagation of the disturbance which arises in a particular dynamical situation may constitute a problem of some degree of difficulty. A case in point is the problem of the determination of the velocity of a running crack which traverses a thin plate in a direction perpendicular to the applied load.² Since such cracks have velocities which are considerably less than $\sqrt{\mu/\rho}$ it is natural to suppose that the disturbance caused by the propagation of these cracks can be represented by a surface Σ of the type under consideration. We have commented briefly on this problem in §6.

2. Condition for the speed of the disturbance to be less than the velocity of a shear wave. From (1.4) and (1.7) we have immediately

$$(2.1) \quad \lambda_i \nu_i = 0.$$

¹ This investigation was conducted at the request of the Naval Research Laboratory and has been supported by the Office of Naval Research under Contract Nonr-908(09), Indiana University, NR 041 037.

² Measurements of the stress distribution in the vicinity of running cracks and the velocities of crack propagation have recently been made at the Naval Research Laboratory. See A. A. WELLS & D. POST, *The dynamic stress distribution surrounding a running crack—A photographic analysis*, Naval Research Laboratory, Washington, D.C., Report 4935, 1957. For a comprehensive account of the theory of fracture in solids reference may be made to the article by GEORGE R. IRWIN, *Encyclopedia of Physics*, Vol. VII, Springer, Berlin, 1958.

Also from (1.2) and the compatibility conditions (1.7) and (1.8) we readily find that

$$(2.2) \quad \xi_{ij}\nu_j = -\rho(G - v_n)\lambda_i,$$

where ρ denotes the density at points of the surface Σ . Similarly, application of the compatibility conditions to the Prandtl-Reuss equations (1.1) leads to a set of relations which can be written in the form

$$(2.3) \quad (G - v_n)(\xi_{ij} - \tfrac{1}{3}\xi_{kk}\delta_{ij}) + \mu(\lambda_i\nu_j + \lambda_j\nu_i) = \frac{2\mu}{K}(\sigma_{ab}\lambda_a\nu_b)\sigma_{ij}^*.$$

From (2.2) and (2.3) we may derive the following two sets of relations, namely

$$(2.4) \quad \xi_{ij}\nu_i\nu_j = 0$$

and

$$(2.5) \quad \sigma_{ij}^*\xi_{ij} = 0.$$

In fact, (2.4) is obtained merely by multiplying (2.2) by ν_i , summing on repeated indices, and making use of (2.1). Similarly, multiplying (2.3) by σ_{ij}^* , and making use of (1.5), we obtain (2.5) provided the speed $G - v_n$ of Σ is different from zero as we assume in this discussion.

If all $\xi_{ij} = 0$ then all $\lambda_i = 0$ from (2.2). Conversely if all $\lambda_i = 0$ we have

$$(2.6) \quad \xi_{ij}\nu_j = 0; \quad \xi_{ij} - \tfrac{1}{3}\xi_{kk}\delta_{ij} = 0,$$

from (2.2) and (2.3) respectively. But if we multiply the second relation (2.6) by ν_i and make use of the first relation (2.6) it follows that $\xi_{kk} = 0$. From the second relation (2.6) we then have $\xi_{ij} = 0$. *Hence in view of the first italicized statement in §1 it follows that not all quantities λ_i nor all quantities ξ_{ij} can vanish at any point of Σ .*

Let us next multiply (2.3) by $\lambda_i\nu_j$ and sum on the repeated indices. Taking account of the formula

$$\sigma_{ij}^* = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}$$

for the deviator components σ_{ij}^* and making use of (2.1) and (2.2) we now find that

$$(2.7) \quad \{\rho(G - v_n)^2 - \mu\}\lambda_i\lambda_i = -\frac{2\mu}{K}(\sigma_{ab}\lambda_a\nu_b)^2.$$

But $\lambda_i\lambda_i > 0$ since, as we have just observed, not all of the quantities λ_i can vanish on Σ . Hence the relation (2.7) permits us to make the following statement. *The inequality*

$$(G - v_n)^2 < \frac{\mu}{\rho}$$

holds if, and only if, the quantity $\sigma_{ab}\lambda_a\nu_b$ does not vanish. In other words, the speed of propagation of the discontinuity surface Σ is numerically less than the velocity of a shear wave if, and only if, $\sigma_{ab}\lambda_a\nu_b$ is different from zero on Σ . When $\sigma_{ab}\lambda_a\nu_b$ vanishes on Σ the speed of propagation of this surface will be equal to the velocity of a shear wave.

Remark 1. The above result can be given a simple dynamical interpretation. It is well known that the quantity $\sigma_{ab}\epsilon_{ab}$ represents the rate at which work is done by the stress in changing the *shape* of a unit volume of the material.³ Now consider the relation

$$[\sigma_{ab}\epsilon_{ab}] = \sigma_{ab}[\epsilon_{ab}] = \sigma_{ab}[v_{a,b}] = \sigma_{ab}\lambda_a\nu_b,$$

which follows from (1.7), the above definition of the rate of strain tensor ϵ , and the assumption that the stress is continuous across the surface Σ . Hence we can say that the speed of propagation of the surface Σ will be numerically less than the velocity of a shear wave if, and only if, the rate at which the stress does work in changing the shape of a unit volume of material is different at adjacent points on the two sides of Σ .

Remark 2. Since the density ρ is continuous across Σ by the result in §1 it follows that we must have relations of the form

$$(2.8) \quad [\rho_{,i}] = \zeta\nu_i; \quad \left[\frac{\partial\rho}{\partial t}\right] = -G\zeta.$$

Hence from (1.3), (1.4) and (2.8) we have

$$(G - v_n)\zeta = 0.$$

Hence $\zeta = 0$ since the speed $G - v_n$ is not equal to zero by assumption. In other words, the first coordinate derivatives $\rho_{,i}$ and the time derivative $\partial\rho/\partial t$ are continuous across the discontinuity surface Σ .

Remark 3. If we differentiate the relation (1.5) with respect to the coordinates x_k we obtain

$$\sigma_{ij}^*\sigma_{ij,k}^* = \sigma_{ij}^*\sigma_{ij,k} = 0.$$

Hence, applying (1.8), we find that

$$\sigma_{ij}^*\xi_{ij}\nu_k = 0.$$

Thus we again obtain (2.5) since not all of the quantities ν_k can vanish simultaneously. This derivation of the relation (2.5) does not involve the assumption that the speed $G - v_n$ is different from zero which was made in our previous derivation.

³ See, for example, W. PRAGER & P. G. HODGE, JR., *The Theory of Perfectly Plastic Solids*, John Wiley & Sons, New York, 1951, p. 35.

3. Plane stress. Consider a thin rectangular plate subjected to uniform tension at its ends. Choosing the x_2 axis of our rectangular system in the direction of the applied tension and the x_3 axis perpendicular to the flat sides of the plate, we may assume a state of *plane stress* in the plate such that

$$(3.1) \quad \begin{cases} \sigma_{i3} = 0 & (i = 1, 2, 3), \\ \sigma_{ij} = \phi_{ij}(x_1, x_2, t) & (i, j = 1, 2), \end{cases}$$

where the ϕ 's are functions only of the two coordinates x_1, x_2 and the time t . The plane stress condition (3.1) will be imposed throughout the following discussion.

Under the condition (3.1) it follows from (1.8) that

$$(3.2) \quad [\sigma_{i3}, k] = \xi_{i3}\nu_k = 0; \quad [\sigma_{ij}, 3] = \xi_{ij}\nu_3 = 0.$$

Hence $\xi_{i3} = 0$ from the first relation (3.2) since not all ν_k can vanish. From the second relation (3.2) we have $\nu_3 = 0$ since not all ξ_{ij} can be equal to zero from the result in §2. Hence, putting $i = 3$ in (2.2), we have

$$\xi_{3i}\nu_i = -\rho(G - v_n)\lambda_3 = 0.$$

Hence $\lambda_3 = 0$ for surfaces Σ whose speed $G - v_n$ is different from zero.

Remark. Since $\nu_3 = 0$ we see that the surface Σ must be a cylinder perpendicular to the flat sides of the plate.

4. Further consequences of the conditions. To simplify our discussion of the conditions (2.1), (2.2) and (2.3) let us translate the coordinate axes so that the origin of the system falls on an arbitrary point P of the surface Σ and let us then rotate the x_1, x_2 axes about the x_3 axis through such an angle that the x_1 axis will be normal to Σ and its positive direction will coincide with the direction of the vector ν . All preceding relations will of course be left invariant by a coordinate transformation of this character.

Relative to the new coordinate system we now have $\nu_1 = 1, \nu_2 = \nu_3 = 0$ at P . Hence from (2.1) we see that $\lambda_1 = 0$ at P . Hence $\lambda_2 \neq 0$ at P ; otherwise we would have all $\lambda_i = 0$ (see §3) and this would contradict the result in §2.

We now proceed to consider the relations (2.2) and (2.3) at the origin P of our transformed system. Thus, expanding (2.2), we have

$$(4.1) \quad \lambda_1 = \frac{-\xi_{11}}{\rho(G - v_n)} = 0; \quad \lambda_2 = \frac{-\xi_{12}}{\rho(G - v_n)},$$

in which account is taken of the fact that $\lambda_1 = 0$. From the first of these relations we see that $\xi_{11} = 0$ and from the second relation that $\xi_{12} \neq 0$ since $\lambda_2 \neq 0$ by the above result.

If we take $i = 1, j = 3$ or $i = 2, j = 3$ in (2.3) we find that the relation is satisfied identically as a consequence of preceding conditions. Moreover, if we put $i = j$ in (2.3) and sum on the repeated indices it follows that the resulting

equation is satisfied, in view of (2.1). It is clear therefore that we can confine our attention to the values of the indices

$$i = 1, j = 1; \quad i = 1, j = 2; \quad i = 3, j = 3$$

in treating the system (2.3). But for these values of the indices we obtain

$$(G - v_n)\xi_{22} = -\frac{6\mu}{K}(\sigma_{12}\lambda_2)\sigma_{11}^* ,$$

$$(G - v_n)\xi_{12} + \mu\lambda_2 = \frac{2\mu}{K}(\sigma_{12}\lambda_2)\sigma_{12} ,$$

$$(G - v_n)\xi_{22} = \frac{2\mu}{K}(\sigma_{12}\lambda_2)\sigma_{kk} ,$$

where use has been made of the fact that $\xi_{11} = 0$, $\lambda_1 = \lambda_2 = 0$ and $\nu_1 = 1$, $\nu_2 = \nu_3 = 0$. Hence, replacing λ_2 by the expression in the second equation (4.1) and taking account of the fact that $\xi_{12} \neq 0$, we have

$$(4.2) \quad (G - v_n)^2\xi_{22} = \frac{6\mu}{K\rho}\sigma_{12}\sigma_{11}^*\xi_{12} ,$$

$$(4.3) \quad (G - v_n)^2 = \left(1 - \frac{2\sigma_{12}^2}{K}\right)\frac{\mu}{\rho} ,$$

$$(4.4) \quad (G - v_n)^2\xi_{22} = -\frac{2\mu}{K\rho}\sigma_{12}\sigma_{kk}\xi_{12} .$$

Now $\sigma_{12} \neq 0$ since otherwise the speed $G - v_n$ would not be numerically less than the velocity $\sqrt{\mu/\rho}$ as we see from (4.3). Using the fact that $\sigma_{12} \neq 0$ and eliminating the ξ 's between the two equations (4.2) and (4.4) we find that $\sigma_{11} = 0$. Expressed invariantly, *i.e.* independently of the rectangular coordinate system, this latter condition becomes

$$(4.5) \quad \sigma_{ij}\nu_i\nu_j = 0.$$

5. Invariant equation for the speed of the disturbance. The equation (4.3) gives the speed $G - v_n$ of the disturbance or discontinuity surface Σ in terms of the stress components relative to the special coordinate system employed in §4. To deduce an invariant relation, corresponding to (4.5), for the speed we observe first that (1.5) becomes

$$K = \frac{2}{3}(\sigma_{11}^2 - \sigma_{11}\sigma_{22} + \sigma_{22}^2) + 2\sigma_{12}^2 ,$$

under the plane stress condition (3.1). But in the special coordinate system of §4 this relation reduces to

$$(5.1) \quad K = \frac{2}{3}\sigma_{22}^2 + 2\sigma_{12}^2 ,$$

since $\sigma_{11} = 0$. Substituting this latter expression for K into (4.3) we now have

$$(G - v_n)^2 = \frac{2\sigma_{22}^2}{3K} \frac{\mu}{\rho}.$$

Or we can write

$$(5.2) \quad (G - v_n)^2 = \frac{2(\sigma_{kk})^2}{3K} \frac{\mu}{\rho},$$

again using the fact that $\sigma_{11} = 0$ in the special system. Equation (5.2) is the desired invariant formulation since this equation obviously remains unchanged under orthogonal transformations of the rectangular coordinates.

Remark 1. Equations (4.5) and (5.2) are the two main consequences of the general conditions in §2 in that these equations do not involve the quantities λ_i and ξ_{ij} defined on Σ . Otherwise the conditions of §2 involve relations in the quantities λ_i and ξ_{ij} which, as we have seen, can be expressed very simply by means of the special coordinate system introduced in §4.

Remark 2. From the relation (5.1) we see that σ_{12}^2 must lie between zero and $\frac{1}{2}K$. In the limiting case $\sigma_{12} = 0$ we have

$$(G - v_n)^2 = \frac{\mu}{\rho},$$

and for $\sigma_{12}^2 = \frac{1}{2}K$ we have $G - v_n = 0$ from (4.3). Conceivably $G - v_n$ can have any value satisfying (1.10) under the assumptions of this note.

6. Application to the symmetrical fracture of flat plates under tension. When fracture of a flat plate, subjected to a uniform load at its ends, is caused by a crack which runs across the plate in a direction perpendicular to the tension (symmetrical fracture) the velocity of the crack is found to be less than the shear velocity $\sqrt{\mu/\rho}$ (see §1). Due to the high concentration of stress in the immediate neighborhood of the tip of the crack one is led to assume that the material is plastic within this region. Suppose, now, that the disturbance associated with such a crack can be represented by a wave surface Σ . If the wave Σ is perpendicular to the direction of propagation of the crack, as illustrated in Fig. 1, the speed of propagation $G - v_n$ of Σ at the tip or vertex V of the crack is given by the special formula (4.3) and the value of G in this formula can be identified with the velocity of the crack.

Neglecting the normal flow velocity v_n in comparison with G in the formula (4.3) the problem of the determination of the velocity of the crack is reduced, on the basis of our model, to the problem of determining the stress distribution in the plastic neighborhood surrounding the tip of the crack, when due consideration is taken of the angle of the crack opening at V and the nature of the applied load at the ends of the flat plate. Now in formula (4.3) we have

$$\frac{2\sigma_{12}^2}{K} = \frac{2\sigma_{12}^2}{2\sigma_{12}^2 + \frac{2}{3}\sigma_{22}^2} = \frac{1}{1 + \frac{1}{3}\left(\frac{\sigma_{22}}{\sigma_{12}}\right)^2},$$

FIG. 1

since $\sigma_{12} \neq 0$ as observed in §4. Hence the determination of the velocity G of the crack reduces specifically to the determination of the ratio σ_{22}/σ_{12} at V . But from symmetry $\sigma_{12} = 0$ along the x_1 axis in front of the running crack (see Fig. 1). Hence the stress field σ_{ij} must be singular at V in the sense that, at any instant, the values of some of the components (including σ_{12}) at V must depend on the direction along which V is approached. Obviously the proper value of the ratio σ_{22}/σ_{12} in this consideration is that obtained by approaching V along a direction perpendicular to the x_1 axis.

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