Einstein's Connections

II. Nondegenerate Case

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Summary. In a recent paper [3], I gave the solution of EINSTEIN's first set of equations of unified field theory by the method of repeated substitution for the third and fourth classes and for the special case of the first class when $D = 0$. These solutions hold for any index of inertia for which the particular class exists. The object of this paper is to obtain the solutions for the first and second classes for all indices of inertia, by the method of repeated substitution, and to show that the general solution for the first class yields the solution for other classes. It has already been shown that Mme. TONNELAT's solution [4] holds only for the first class. HLAVATÝ's solution for the first class holds also for the second class, but not for the third class for which he found another form of solution. An open question (HLAVATÝ [1], p. 99), regarding the conditions that the form of the solution for the third class is also the form of the solution for other classes, is also solved.

1. Introduction. EINSTEIN's unified field theory is based on three principles (HLAVATÝ [1], pp. X, XI):

(a) The basic real tensor $g_{\lambda\mu}$ is non-symmetric and may be split into its symmetric part $h_{\lambda\mu}$ and its skew-symmetric part $k_{\lambda\mu}$:

$$(1.1) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} .$$

The tensor $h_{\lambda\mu}$ is assumed to be of rank 4. It is said to be of index of inertia 0, 2 or 4 according as the signature of $h_{\lambda\mu}$ is $(+ + - -)$, $(+ + + -)$ or $(+ + + +)$.*

(b) The basic connection $\Gamma_{\lambda\mu}^\alpha$ is given by

$$(1.2) \quad D_\omega g_{\lambda\mu} = 2S_{\omega\mu}^\alpha g_{\lambda\alpha} ,$$

where D_σ is the symbolic vector of the covariant derivative with respect to

*This includes the equivalent combinations of $+$ and $-$ signs also.

$\Gamma_{\lambda\mu}^\nu$ and

$$(1.3) \quad S_{\lambda\mu}{}^\nu \stackrel{\text{def}}{=} \Gamma_{[\lambda\mu]}^\nu \stackrel{\text{def}}{=} \frac{1}{2}(\Gamma_{\lambda\mu}^\nu - \Gamma_{\mu\lambda}^\nu).$$

The conditions imposed on the connection $\Gamma_{\lambda\mu}^\nu$ amount to

$$(1.4) \quad S_\lambda \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^\alpha = 0,$$

$$(1.5a) \quad R_{\mu\lambda} \stackrel{\text{def}}{=} R_{\omega\mu\lambda}{}^\omega = \partial_{[\mu} X_{\lambda]} ,$$

where X_λ is an arbitrary vector and

$$(1.5b) \quad R_{\omega\mu\lambda}{}^\nu \stackrel{\text{def}}{=} \partial_\mu \Gamma_{\lambda\omega}^\nu - \partial_\omega \Gamma_{\lambda\mu}^\nu + \Gamma_{\alpha\mu}^\nu \Gamma_{\lambda\omega}^\alpha - \Gamma_{\alpha\omega}^\nu \Gamma_{\lambda\mu}^\alpha .$$

Denoting by $\mathfrak{h}$, $\mathfrak{g}$ and $\mathfrak{f}$ the determinants of $h_{\lambda\mu}$, $g_{\lambda\mu}$ and $k_{\lambda\mu}$ and taking $h^{\lambda\mu}$ as a tensor inverse to $h_{\lambda\mu}$, let us put

$$(1.6a) \quad g \stackrel{\text{def}}{=} \frac{\mathfrak{g}}{\mathfrak{h}} ,$$

$$(1.6b) \quad k \stackrel{\text{def}}{=} \frac{\mathfrak{f}}{\mathfrak{h}} ,$$

$$(1.6c) \quad {}^{(0)}k_\lambda{}^\nu = \delta_\lambda{}^\nu, \quad {}^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \quad {}^{(p)}k_\lambda{}^\nu = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu \quad (p = 2, 3, \dots),$$

$$(1.6d) \quad 4K = k_{\alpha\beta} k_{\gamma\delta} h^{\alpha\gamma} h^{\beta\delta} = k_{\alpha\beta} k^{\alpha\beta} = -{}^{(2)}k_\alpha{}^\alpha .$$

We then have

$$(1.7a) \quad g = 1 + 2K + k$$

and

$$(1.7b) \quad k_{\omega\alpha} k_{\mu\beta} + k_{\alpha\mu} k_{\omega\beta} + k_{\mu\omega} k_{\alpha\beta} = \sqrt{\mathfrak{f}} \mathfrak{e}_{\omega\alpha\mu\beta} ,$$

where $\mathfrak{e}_{\omega\alpha\mu\beta}$ is a covariant indicator, being a tensor density of weight -1 , skew-symmetric in all indices, whose components have numerical values

- (a) $+1$ whenever $\omega\alpha\mu\beta$ is an even permutation of I II III IV,
- (b) -1 whenever $\omega\alpha\mu\beta$ is an odd permutation of I II III IV,
- (c) 0 in all remaining cases.

The tensor $g_{\lambda\mu}$ or $k_{\lambda\mu}$ is said to be

- (a) of the first class if $k \neq 0$,
- (b) of the second class if $K \neq 0$, $k = 0$,
- (c) of the third class if $K = 0$, $k = 0$, ${}^{(2)}k_\lambda{}^\nu \neq 0$,
- (d) of the fourth class if ${}^{(2)}k_\lambda{}^\nu = 0$,* $k = 0$.

For the first two classes, there are exactly four linearly independent vectors a_i^ν satisfying (HLAVATÝ [1], p. 11)

*This means that K always vanishes.

$$(1.8) \quad (a) \quad \lambda a^{\mu}_{x} = k^{\mu}_{x} a^{\lambda}_{x}, \quad (b) \quad h_{\lambda\mu} a^{\lambda}_{x} a^{\mu}_{x} = 0 *$$

where λ are solutions of

$$(1.9) \quad \lambda^4 + 2K\lambda^2 + k = 0.$$

The vectors a^{ν}_{x} are called eigenvectors of k^{ν}_{x} and the scalars λ are the corresponding eigenvalues.

For the first class we have

$$(1.10a) \quad \lambda_1 = -\lambda_2 = i\sqrt{\sqrt{D} + K} \neq 0,$$

$$(1.10b) \quad \lambda_3 = -\lambda_4 = \sqrt{\sqrt{D} - K} \neq 0,$$

where

$$(1.11) \quad D = K^2 - k;$$

and for the second class we have

$$(1.12a) \quad \lambda_1 = -\lambda_2 = \sqrt{-2K}, \quad \text{for } K > 0,$$

$$(1.12b) \quad \lambda_3 = -\lambda_4 = 0,$$

or

$$(1.13a) \quad \lambda_1 = -\lambda_2 = 0, \quad \text{for } K < 0.$$

$$(1.13b) \quad \lambda_3 = -\lambda_4 = \sqrt{-2K},$$

It has been shown by HLAVATÝ ([1], pp. 52-53) that if the system (1.2) admits a solution $\Gamma^{\nu}_{\lambda\mu}$, it must be of the form

$$(1.14) \quad \Gamma^{\nu}_{\lambda\mu} = \left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\} + S_{\lambda\mu}{}^{\nu} + U^{\nu}_{\lambda\mu},$$

where $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$ are the Christoffel symbols with respect to the tensor $h_{\alpha\beta}$ and

$$(1.15) \quad U^{\nu}_{\lambda\mu} \stackrel{\text{def}}{=} 2h^{\nu\alpha} S_{\alpha(\lambda} k_{\mu)\beta}{}^{\beta}.$$

The tensors $S_{\lambda\mu}{}^{\nu}$ and $U^{\nu}_{\lambda\mu}$ are related by

$$(1.16) \quad S_{\omega\mu\nu} = \frac{1}{2}K_{\omega\mu\nu} - 2U_{\alpha\nu[\mu} k_{\omega]}{}^{\alpha}$$

where

$$(1.17) \quad K_{\omega\mu\nu} \stackrel{\text{def}}{=} \nabla_{\omega} k_{\nu\mu} + \nabla_{\mu} k_{\omega\nu} + \nabla_{\nu} k_{\omega\mu},$$

*The indices x, y, z, t do not follow the summation convention. The summation for these indices is denoted by the sign $\sum$.

the operator ∇_μ being the symbolic vector of the covariant derivative with respect to $\left\{ \begin{smallmatrix} \nu \\ \lambda\mu \end{smallmatrix} \right\}$.

Since $U^\nu_{\lambda\mu}$ is symmetric in λ and μ , we have from (1.16)*

$$(1.18) \quad S_{[\omega\mu\nu]} = \frac{1}{2}K_{[\omega\mu\nu]}.$$

From (1.15) and (1.16) we have

$$(1.19) \quad S_{\omega\mu\nu} = \frac{1}{2}K_{\omega\mu\nu} + 2k_{[\omega}{}^\alpha k_{\mu]}{}^\beta S_{\nu\alpha\beta} + 2k_{[\omega}{}^\alpha S_{\mu]\alpha\beta} k_\nu{}^\beta.$$

If we can solve the equation (1.19) for $S_{\omega\mu\nu}$ in terms of $g_{\lambda\mu}$, we can substitute $S_{\omega\mu\nu}$ in (1.14) and then $\Gamma^\nu_{\lambda\mu}$ will be determined completely.

If the nonholonomic components of ${}^{(p)}k_\lambda{}^\nu$ in the first two classes are given by

$$(1.20) \quad {}^{(p)}k_x{}^i = (\lambda)^p \delta_x{}^i,$$

then the solution of (1.19) in the nonholonomic frame is given by (HLAVATÝ [1], p. 79)

$$(1.21a) \quad S_{xyz} = \frac{1}{2\lambda} (K_{xyz} + 3\lambda\lambda K_{[xyz]}).$$

where

$$(1.21b) \quad \lambda \stackrel{\text{def}}{=} 1 + \lambda\lambda + \lambda\lambda + \lambda\lambda \quad (xyz \text{ are not all equal}).$$

The nonholonomic components of ${}^{(p)}k_\lambda{}^\nu$ in the third class can be written as (HLAVATÝ [1], p. 87)

$$(1.22a) \quad k_1{}^2 = k_2{}^4 = k_{12} = -k_{21} = -k^{42} = k^{24} = 1,$$

$$(1.22b) \quad \text{the remaining } k_{ij}, k^{ij} \text{ and } k_i{}^j = 0,$$

$$(1.22c) \quad {}^{(p)}k_1{}^4 = -{}^{(p)}k_{11} = -{}^{(p)}k^{44} = \delta_2{}^p, \quad p > 1,$$

$$(1.22d) \quad \text{the remaining } {}^{(p)}k_{ij}, {}^{(p)}k^{ij} \text{ and } {}^{(p)}k_i{}^j = 0.$$

For the nonholonomic components of $K_{\omega\mu\nu}$ in the third class we have

$$(1.22e) \quad K_{i[24]} = 0,$$

$$(1.22f) \quad K_{434} = 0, \quad K_{343} = 0.$$

The solution of (1.19) in the third class is

$$(1.23) \quad S_{\omega\mu\nu} = \frac{1}{2}K_{\omega\mu\nu} + k_{[\omega}{}^\alpha K_{\mu]\alpha\beta} k_\nu{}^\beta.$$

*Here we follow the notation

$$6T_{[\omega\mu\nu]} = (T_{\omega\mu\nu} + T_{\mu\nu\omega} + T_{\nu\omega\mu}) - (T_{\mu\omega\nu} + T_{\nu\mu\omega} + T_{\omega\nu\mu}).$$

The nonholonomic components of the Christoffel symbols are given by

$$(1.24) \quad \left\{ \begin{smallmatrix} i \\ jk \end{smallmatrix} \right\} = -a^\lambda_{i} a^\mu_{k} \nabla_\mu a^i_\lambda$$

where a^i_λ are the vectors inverse to a^λ , such that

$$(1.25a) \quad a^i_\lambda a^\lambda = \delta^i_j,$$

$$(1.25b) \quad a^i_\lambda a^\mu = \delta^\mu_\lambda.$$

From (1.25) we have

$$(1.26a) \quad a^\lambda_1 = a^2_\lambda, \quad a^\lambda_2 = a^1_\lambda,$$

$$(1.26b) \quad a^\lambda_3 = a^4_\lambda, \quad a^\lambda_4 = a^3_\lambda.$$

2. Recurrence relations for ${}^{(p)}k_\lambda{}^\nu$. We know that (HLAVATÝ [1], p. 23) in the first class when $D \neq 0$

$$(2.1a) \quad {}^{(4)}k_\lambda{}^\nu + 2K {}^{(2)}k_\lambda{}^\nu + k \delta_\lambda{}^\nu = 0, \quad k \neq 0.$$

From this we can easily prove the relation

$$(2.1b) \quad {}^{(p+4)}k_\lambda{}^\nu + 2K {}^{(p+2)}k_\lambda{}^\nu + k {}^{(p)}k_\lambda{}^\nu = 0, \quad p = 0, 1, \dots$$

Theorem (2.1). *In the first class, when $D = 0$, the recurrence relations for ${}^{(p)}k_\lambda{}^\nu$ are*

$$(2.2a) \quad {}^{(2)}k_\lambda{}^\nu + K \delta_\lambda{}^\nu = 0,$$

$$(2.2b) \quad {}^{(p+2)}k_\lambda{}^\nu + K {}^{(p)}k_\lambda{}^\nu = 0.$$

Proof. When $D = 0$, the equation (1.9) assumes the form

$$(\lambda^2 + K)^2 = 0$$

or

$$(2.3) \quad \lambda^2 + K = 0.$$

From (1.20) we have

$$(2.4) \quad {}^{(2)}k_x{}^i + K \delta_x{}^i = ((\lambda)^2_x + K) \delta_x{}^i = 0$$

where λ_x are solutions of (2.3).

As (2.4) is a tensor equation in the nonholonomic frame, we have (2.2a) in the holonomic frame.

The equation (2.2b) follows by multiplying (2.2a) by ${}^{(p)}k_\nu{}^\alpha$ and summing on ν .

Theorem (2.2). *In the first class when $D \neq 0$, we have*

$$(2.5a) \quad ({}^{(3)}k_{\omega}^{\alpha}({}^{(2)}k_{\mu}^{\beta} + {}^{(3)}k_{\mu}^{\beta}({}^{(2)}k_{\omega}^{\alpha}))T_{\alpha\beta} = k(k_{\omega}^{\alpha}\delta_{\mu}^{\beta} + \delta_{\omega}^{\alpha}k_{\mu}^{\beta})T_{\alpha\beta},$$

$$(2.5b) \quad ({}^{(3)}k_{\omega}^{\alpha}\delta_{\mu}^{\beta} + {}^{(3)}k_{\mu}^{\beta}\delta_{\omega}^{\alpha})T_{\alpha\beta} = -2K(k_{\omega}^{\alpha}\delta_{\mu}^{\beta} + k_{\mu}^{\beta}\delta_{\omega}^{\alpha})T_{\alpha\beta} \\ - ({}^{(2)}k_{\omega}^{\alpha}k_{\mu}^{\beta} + {}^{(2)}k_{\mu}^{\beta}k_{\omega}^{\alpha})T_{\alpha\beta},$$

$$(2.5c) \quad ({}^{(3)}k_{\omega}^{\alpha}k_{\mu}^{\beta} + {}^{(3)}k_{\mu}^{\beta}k_{\omega}^{\alpha})T_{\alpha\beta} = kT_{[\omega\mu]} - 2Kk_{\omega}^{\alpha}k_{\mu}^{\beta}T_{\alpha\beta} - {}^{(2)}k_{\omega}^{\alpha}({}^{(2)}k_{\mu}^{\beta}T_{\alpha\beta} +$$

$$(2.5d) \quad {}^{(3)}k_{\omega}^{\alpha}({}^{(3)}k_{\mu}^{\beta}T_{\alpha\beta} = k k_{\omega}^{\alpha}k_{\mu}^{\beta}T_{\alpha\beta} + 2K({}^{(2)}k_{\omega}^{\alpha}({}^{(2)}k_{\mu}^{\beta}T_{\alpha\beta} \\ + k({}^{(2)}k_{\omega}^{\alpha}\delta_{\mu}^{\beta} + \delta_{\omega}^{\alpha}({}^{(2)}k_{\mu}^{\beta})T_{\alpha\beta}$$

where $T_{\alpha\beta}$ is an arbitrary skew-symmetric tensor.

Proof. With the help of (1.10), (1.11) and (1.20) we have

$$({}^{(3)}k_x^i({}^{(3)}k_y^j - k k_x^i k_y^j)T_{ij} = \lambda\lambda(\lambda^2\lambda^2 - k)T_{xy} = [k(\lambda^2 + \lambda^2) + 2K\lambda^2\lambda^2]T_{xy} \\ = [2K({}^{(2)}k_x^i({}^{(2)}k_y^j + k({}^{(2)}k_x^i\delta_y^j + \delta_x^i({}^{(2)}k_y^j)]T_{ij}.$$

Writing this equation in holonomic form we have (2.5d). The remaining equations (2.5a-c) can be proved similarly.

Remark 1. In the first class when $D = 0$, the equations (2.5) reduce to identities, in consequence of (2.2).

Remark 2. IKEDA & ABE ([2], p. 63) have proved that in the first class, when the index of inertia of $h_{\lambda\mu}$ is 2, the tensors

$$h_{\lambda\mu}, \quad k_{\lambda\mu}, \quad {}^{(2)}k_{\lambda\mu}, \quad {}^{(3)}k_{\lambda\mu}$$

are linearly independent with scalar coefficients. This happens because when the index of inertia of $h_{\lambda\mu}$ is 2, then D is never zero. $D = 0$ when the index of inertia of $h_{\lambda\mu}$ is 0 or 4 (MISHRA [3]). In that case the above result does not hold.

In the second class (IKEDA & ABE [2], p. 63), we have

$$(2.6a) \quad {}^{(3)}k_{\lambda}^{\nu} = -2Kk_{\lambda}^{\nu}, \quad K \neq 0.$$

This equation follows immediately from (1.9) and (1.20).

Multiplying (2.6a) by ${}^{(p)}k_{\nu}^{\alpha}$, we obtain the recurrence relation

$$(2.6b) \quad {}^{(3+p)}k_{\lambda}^{\nu} = -2K{}^{(p+1)}k_{\lambda}^{\nu}, \quad p = 0, 1, \dots$$

Theorem (2.3). *In the second class we have*

$$(2.7a) \quad ({}^{(2)}k_{\omega}^{\alpha}k_{\mu}^{\beta} + {}^{(2)}k_{\mu}^{\beta}k_{\omega}^{\alpha})T_{\alpha\beta} = 0,$$

$$(2.7b) \quad ({}^{(2)}k_{\omega}^{\alpha}({}^{(2)}k_{\mu}^{\beta} - 2Kk_{\omega}^{\alpha}k_{\mu}^{\beta})T_{\alpha\beta} = 0.$$

Proof. Putting $k = 0$, and using (2.6a) in (2.5a) or (2.5b), we have (2.7a). Similarly putting $k = 0$, and using (2.6a) in (2.5c) or (2.5d), we have (2.7b).

These equations can be proved independently as follows: In consequence

of (1.20) and (1.12) or (1.13), we have

$$(2.8a) \quad ({}^{(2)}k_x^i k_y^j + {}^{(2)}k_y^i k_x^j) T_{ij} = \lambda \lambda_{xy} (\lambda + \lambda_y) T_{xy} = 0,$$

$$(2.8b) \quad ({}^{(2)}k_x^i {}^{(2)}k_y^j T_{ij} = \lambda \lambda_{xy} (\lambda \lambda) T_{xy} = 2K \lambda \lambda T_{xy} = 2K k_x^i k_y^j T_{ij}.$$

Writing (2.8) in holonomic form we have (2.7).

Remark. The equations (2.1), (2.2), (2.5) and (2.7) can all be proved without the use of the nonholonomic frame. But the proof becomes extremely simple when we use the nonholonomic frame. As an example we shall prove (2.8a) without using the nonholonomic frame:

$$({}^{(2)}k_\omega^\alpha k_\mu^\beta + k_\omega^\alpha {}^{(2)}k_\mu^\beta) T_{\alpha\beta} = (k_{\omega\gamma} k_{\delta\theta} k_{\mu\varphi} + k_{\mu\gamma} k_{\delta\varphi} k_{\omega\theta}) h^{\gamma\delta} h^{\theta\alpha} h^{\varphi\beta} T_{\alpha\beta}.$$

Interchanging ϕ and θ , α and β in the second term on the right hand side and remembering that

$$k_{\mu\omega} k_{\gamma\varphi} k_{\delta\theta} h^{\gamma\delta} h^{\theta\alpha} h^{\varphi\beta} T_{\alpha\beta} = k_{\omega\mu} {}^{(2)}k_\mu^\beta T_{\alpha\beta} = 0$$

(since ${}^{(2)}k^{\beta\alpha}$ is symmetric in β, α), we get

$$\begin{aligned} ({}^{(2)}k_\omega^\alpha k_\mu^\beta + k_\omega^\alpha {}^{(2)}k_\mu^\beta) T_{\alpha\beta} &= (k_{\omega\gamma} k_{\mu\varphi} + k_{\gamma\mu} k_{\omega\varphi} + k_{\mu\omega} k_{\gamma\varphi}) k^{\gamma\alpha} h^{\varphi\beta} T_{\alpha\beta} \\ &= \sqrt{t} e_{\omega\gamma\mu\varphi} k^{\gamma\alpha} h^{\varphi\beta} T_{\alpha\beta} = 0 \end{aligned}$$

in the second class.

The other equations can similarly be proved.

Theorem (2.4). *In the third class we have*

$$(2.9a) \quad ({}^{(p)}k_\lambda^p = 0, \quad p \geq 3,$$

$$(2.9b) \quad ({}^{(2)}k_\omega^\alpha k_\mu^\beta + {}^{(2)}k_\mu^\beta k_\omega^\alpha) T_{\alpha\beta} = 0,$$

$$(2.9c) \quad ({}^{(2)}k_\omega^\alpha {}^{(2)}k_\mu^\beta T_{\alpha\beta} = 0.$$

Proof. Putting $K = 0$ in (2.6) and (2.7), we have (2.9).

All three equations (2.9) can be proved independently using the nonholonomic frame of the third class. For example, in consequence of (1.22c) we have

$$({}^{(2)}k_x^i k_y^j + {}^{(2)}k_y^i k_x^j) T_{ij} = \delta_{xy}^{11} (T_{42} + T_{24}) + (\delta_{xy}^{12} + \delta_{xy}^{21}) T_{44} = 0$$

since T_{xy} is skew-symmetric in x and y . Hence we have (2.9b). The other two equations can be proved similarly.

Theorem (2.5). *The necessary and sufficient condition for the*

$$\left\{ \begin{array}{l} \text{second} \\ \text{third} \end{array} \right. \text{ class is } \left\{ \begin{array}{ll} ({}^{(3)}k_\lambda^p + 2K k_\lambda^p = 0, & K \neq 0 \\ ({}^{(3)}k_\lambda^p = 0, & ({}^{(2)}k_\lambda^p \neq 0. \end{array} \right.$$

Proof. From remark 2, at the end of Theorem (2.2), it is clear that, when $D \neq 0$, the lowest order recurrence relation in the first class is (2.1a).

Multiplying (2.6a) by k_{α}^{α} and substituting in (2.1a), we get $k = 0$, which proves the sufficiency of the condition (2.6a), in the second class.

That the condition is necessary was proved by IKEDA & ABE ([2], p. 63).

The statement about the third class can be proved similarly, with the consideration that in this case the tensors $h_{\lambda\mu}$, $k_{\lambda\mu}$ and ${}^{(2)}k_{\lambda\mu}$ are linearly independent with scalar coefficients, and from (1.22).

Theorem (2.6). Put $k = 0$ in the recurrence relation (2.1a) of the first class when $D \neq 0$. We then have

$$(2.10a) \quad {}^{(4)}k_{\lambda}^{\nu} + 2K {}^{(2)}k_{\lambda}^{\nu} \equiv k_{\lambda}^{\alpha} ({}^{(3)}k_{\alpha}^{\nu} + 2K k_{\alpha}^{\nu}) = 0,$$

which holds in the second class. Conversely, if (2.10a) holds then $k_{\lambda\mu}$ is of the second class and the equation (2.10a) is equivalent to

$$(2.10b) \quad {}^{(3)}k_{\lambda}^{\nu} + 2K k_{\lambda}^{\nu} \equiv k_{\lambda}^{\alpha} ({}^{(2)}k_{\alpha}^{\nu} + 2K \delta_{\alpha}^{\nu}) = 0.$$

But in this case

$${}^{(2)}k_{\alpha}^{\nu} + 2K \delta_{\alpha}^{\nu} \neq 0$$

necessarily.

Theorem (2.7). The relationship (2.2a) (which holds for the first class with $D = 0$) is not satisfied for the third class, although we have in the third class $D = 0$. However, it is satisfied for the fourth class (where again $D = 0$).

Theorem (2.8). Put $K = 0$ in the recurrence relation (2.6a) of the second class. We have

$$(2.11) \quad {}^{(3)}k_{\lambda}^{\nu} \equiv k_{\lambda}^{\alpha} {}^{(2)}k_{\alpha}^{\nu} = 0$$

which holds for the third class. Conversely, if (2.11) holds then necessarily

$${}^{(2)}k_{\alpha}^{\nu} \neq 0$$

and $k_{\lambda\mu}$ is of the third class.

Proof. The proof of these theorems is obvious.

From the recurrence relations (2.1), (2.2) and (2.6), we have the following theorem:

Theorem (2.9). In the first class when $D \neq 0$,

$$(2.12) \quad {}^{(4)}k_{\lambda}^{\lambda} = 4(D + K^2).$$

In the first class when $D = 0$,

$$(2.13a) \quad {}^{(3+2p)}k_{\lambda}^{\lambda} = 0, \quad p = 0, 1, \dots$$

$$(2.13b) \quad {}^{(2+2p)}k_{\lambda}^{\lambda} = 4(-K)^{p+1},$$

In the second class

$$(2.14a) \quad {}^{(3+2p)}k_{\lambda}^{\lambda} = 0, \quad p = 0, 1, \dots$$

$$(2.14b) \quad {}^{(4+2p)}k_{\lambda}^{\lambda} = 8K^2(-2K)^p,$$

Proof. Putting $\nu = \lambda$ in (2.1a), and remembering that

$${}^{(2)}k_{\lambda}^{\lambda} = -4K,$$

we have (2.12).

Putting $p = 1$, $\nu = \lambda$ in (2.2b), we get in consequence of

$$k_{\lambda}^{\lambda} = 0$$

the equation

$${}^{(3)}k_{\lambda}^{\lambda} = 0.$$

Setting $\lambda = \nu$ in (2.6a), we get

$${}^{(3)}k_{\lambda}^{\lambda} = 0.$$

The remaining part of the proof is obvious.

3. The operator A . If we put

$$(3.1a) \quad A_{\omega\mu\nu}^{\alpha\beta\gamma} \stackrel{\text{def}}{=} {}^{(x)}k_{\omega}^{\alpha} {}^{(y)}k_{\mu}^{\beta} {}^{(z)}k_{\nu}^{\gamma},$$

then we know that (MISHRA [3])

$$(3.1b) \quad A_{\omega_1\mu_1\nu_1}^{\alpha_1\beta_1\gamma_1} A_{\omega_2\mu_2\nu_2}^{\alpha_2\beta_2\gamma_2} \cdots A_{\omega_n\mu_n\nu_n}^{\alpha_n\beta_n\gamma_n} = A_{\omega_n\mu_n\nu_n}^{\alpha_n\beta_n\gamma_n}$$

where

$$(3.1c) \quad x = \sum x_i, \quad y = \sum y_i, \quad z = \sum z_i.$$

If we write the right hand side of (3.1a) in the nonholonomic frame and make use of (1.20), we get

$$A_{xyz}^{pqr} = (\lambda)^p (\lambda)^q (\lambda)^r \delta_{xyz}^{ijk}.$$

The sum of the powers of λ , λ , λ is $p + q + r$, which will be defined as the weight of $A_{\omega\mu\nu}^{\alpha\beta\gamma}$.

Definition (3.1). The quantity $x + y + z$ is defined as the weight of $A_{\omega\mu\nu}^{\alpha\beta\gamma}$. If a tensorial equation has the same weight in all terms it is called isobaric.

We shall use this definition later on.

Theorem (3.1). Put

$$(3.2a) \quad 2 A_{\omega\mu\nu}^{\alpha\beta\gamma} \stackrel{\text{def}}{=} A_{\omega\mu\nu}^{\alpha\beta\gamma} + A_{\omega\mu\nu}^{\alpha\beta\gamma},$$

$$(3.2b) \quad 3! A_{\omega\mu\nu}^{\alpha\beta\gamma} = (A_{\omega\mu\nu}^{\alpha\beta\gamma} + A_{\omega\mu\nu}^{\alpha\beta\gamma} + A_{\omega\mu\nu}^{\alpha\beta\gamma}) + (A_{\omega\mu\nu}^{\alpha\beta\gamma} + A_{\omega\mu\nu}^{\alpha\beta\gamma} + A_{\omega\mu\nu}^{\alpha\beta\gamma}) *.$$

*This leads to

$$3 A_{[\omega\mu]\nu}^{\alpha\beta\gamma} S_{\alpha\beta\gamma} = (A_{[\omega\mu]\nu}^{\alpha\beta\gamma} + A_{[\omega\mu]\nu}^{\alpha\beta\gamma} + A_{[\omega\mu]\nu}^{\alpha\beta\gamma}) S_{\alpha\beta\gamma}.$$

Then the equation (1.19) can be written as

$$(3.3a) \quad S_{\omega\mu\nu} = B_{\omega\mu\nu} - (A_{\omega\mu\nu}^{110\alpha\beta\gamma} + 2 A_{\omega\mu\nu}^{(10)1\alpha\beta\gamma}) S_{\alpha\beta\gamma}$$

or

$$(3.3b) \quad S_{\omega\mu\nu} = B_{\omega\mu\nu} - 3 A_{\omega\mu\nu}^{(110)\alpha\beta\gamma} S_{\alpha\beta\gamma}$$

where

$$(3.3c) \quad B_{\omega\mu\nu} = \frac{1}{2}(K_{\omega\mu\nu} + 3 A_{\omega\mu\nu}^{110\alpha\beta\gamma} K_{[\alpha\beta\gamma]}) = \frac{1}{2} K_{\alpha\beta\gamma} (\delta_{[\omega\mu]\nu}^{\alpha\beta\gamma} + 3 A_{[\omega\mu]\nu}^{110\alpha\beta\gamma}).$$

Proof. Adding the term $k_{\omega}^{\alpha} k_{\mu}^{\beta} S_{\alpha\beta\nu}$ to $2k_{[\omega}^{\alpha} k_{\mu]}^{\beta} S_{\nu\alpha\beta}$ and subtracting it from $2k_{[\omega}^{\alpha} S_{\mu]\alpha\gamma} k_{\nu}^{\gamma}$ of (1.19), we get (3.3) with the use of (1.18) and (3.2).

We shall use the following abbreviations: We shall write T for the tensor $T_{\omega\mu\nu}$. When A ($\equiv A_{\omega\mu\nu}^{pqr\alpha\beta\gamma}$) acts as an operator on the tensor T , we shall denote it by $A \times T$ or T .

$$(3.4a) \quad T \stackrel{\text{def}}{=} A \times T \stackrel{\text{def}}{=} A_{\omega\mu\nu}^{pqr\alpha\beta\gamma} T_{\alpha\beta\gamma},$$

$$(3.4b) \quad T \stackrel{\text{def}}{=} A \times T \stackrel{\text{def}}{=} T_{\omega\mu\nu},$$

$$(3.4c) \quad B \times A \stackrel{\text{def}}{=} B_{\omega\mu\nu}^{\nu_1 q_1 r_1} A_{\alpha\beta\gamma}^{\nu_2 q_2 r_2} \stackrel{\text{def}}{=} B_{\omega\mu\nu}^{\alpha\beta\gamma} A_{\alpha\beta\gamma}^{\omega_1 \mu_1 \nu_1}.$$

With the above abbreviations the equation (3.3a) assumes the form

$$(3.5a) \quad S = B - S - 2 S$$

or

$$(3.5b) \quad S = B - 3 S.$$

Replacing $T_{\alpha\beta}$ by $S_{\alpha\beta\gamma}$ in (2.5), (2.7) and (2.9), we have

$$(3.6a) \quad S = k S,$$

$$(3.6b) \quad S = -2K S - S,$$

$$(3.6c) \quad 2 S = k S - 2K S - S,$$

$$(3.6d) \quad S = k S + 2K S + 2k S$$

in the first class when $D \neq 0$ and

$$(3.7a) \quad S = 0,$$

$$(3.7b) \quad S = 2K S$$

in the second and third classes, when in the third class $K = 0$. In the first class when $D = 0$, the equations (3.6) reduce to identities and in the fourth class the equations (3.7) reduce to identities.

Theorem (3.2). *We have in the first class when $D \neq 0$*

$$\begin{aligned} A^{p+4\ qr} &= -2K A^{p+2\ qr} - k A^{pqr}, \\ A^{p+4\ q+4\ r} &= 4K^2 A^{p+2\ q+2\ r} + 2Kk(A^{p\ q+2\ r} + A^{p+2\ qr}) + k^2 A^{pqr} \\ &\quad + A^{p\ q+4\ r} + A^{p\ q+4\ r+4}. \end{aligned}$$

and similar equations for A , A .

Proof. These equations may be proved with the help of (3.1) and (2.1).

Definition (3.2). A set of tensors $P^{(pq)r}$ will be termed a complete set if all tensors $P^{(pq)r} \times A^{(110)}$ can be expressed in terms of the tensors $P^{(pq)r}$.

Remark. It is easy to see that the repeated substitution for S from (3.5) on the right hand side of (3.5) always yields $S^{(pq)r} (= A^{(pq)r} \times S)$, where the weight of $S^{(pq)r}$ is

$$p + q + r = 2N,$$

N being an integer. Hence we shall consider only the tensors $S^{(pq)r}$ of weight $2N$.

Theorem (3.3). *In the first class when $D \neq 0$, every tensor $A^{pqr} \times A^{110}$ for $p, q, r > 3$ can be expressed as a linear combination of A^{pqr} for $p, q, r \leq 3$.*

Proof. When $p, q, r \geq 4$, A^{pqr} can be written in any one of the following ways

$$A^{s+4\ t\ u}, \quad A^{s+4\ t+4\ u}, \quad A^{s+4\ t+4\ u+4}.$$

In consequence of Theorem (3.2), all these tensors can be reduced to linear combinations of A^{xyz} where x, y, z are each ≤ 3 . Hence we have the theorem.

Theorem (3.4). *In the first class when $D \neq 0$, the set of all tensors $S^{(pq)r} \times A^{(110)}$ for $p, q \leq 2, r \leq 3, p + q + r = 2N$ may be expressed as linear combinations of the set*

$$(3.8) \quad S, \quad S^{002}, \quad S^{(10)1}, \quad S^{(10)3}, \quad S^{110}, \quad S^{112}, \quad S^{(12)1}, \quad S^{(12)3}, \quad S^{(20)0}, \quad S^{(20)2}, \quad S^{220}, \quad S^{222}$$

in the form

$$(3.9a) \quad 3S \times A^{000} = 2S^{(110)} + S^{(10)1} + S^{110},$$

$$(3.9b) \quad 3 S \times A = S + 2 S,$$

$$(3.9c) \quad 3 S \times A = S + S + S,$$

$$(3.9d) \quad 3 S \times A = S - 2K S - k S - 2K S - k S,$$

$$(3.9e) \quad 3 S \times A = 2 S + S,$$

$$(3.9f) \quad 3 S \times A = 2 S + S,$$

$$(3.9g) \quad 6 S \times A = 2k S + k S - 2K S + S,$$

$$(3.9h) \quad 6 S \times A = 2k S - 2Kk S + 4K^2 S - 2K S - k^2 S + 2Kk S - k S,$$

$$(3.9i) \quad 6 S \times A = k S - 2K S - S - 4K S,$$

$$(3.9j) \quad 6 S \times A = k S - 2K S - S - 4K S,$$

$$(3.9k) \quad 3 S \times A = 2k S + 2K S + 2k S + k S,$$

$$(3.9l) \quad 3 S \times A = 2k S + 2K S + 2k S + k S.$$

Proof. For $p, q \leq 2, r \leq 3, p + q + r = 2N$, the tensors $S \times A$ are all represented by the left hand side of (3.9), except for the factor 3 or 6. I shall prove one of the equations (3.9), say (3.9h). The remaining may be proved similarly. In consequence of (3.1)

$$3 S \times A = S + S + S.$$

With the help of (2.1) this equation takes the form

$$3 S \times A = S - 2K S - k S - 2K S - k S.$$

By virtue of (3.6) this equation assumes the form

$$6 S \times A = 2k S - 2Kk S + 4K^2 S - 2K S - k^2 S + 2Kk S - k S$$

which is the equation (3.9h).

The remaining equations may be proved similarly.

Theorem (3.5). *The equations (2.1), (3.6) and (3.9) of the first class when $D \neq 0$ are isobaric.*

Proof. Since

$$\lambda\lambda\lambda\lambda = k,$$

$$\lambda\lambda + \lambda\lambda + \lambda\lambda + \lambda\lambda + \lambda\lambda + \lambda\lambda = 2K$$

we will take the weights of k , k^2 and $2K$ as 4, 8 and 2 respectively. Then it is easy to see that the equations (2.1) and (3.6) are isobaric and the equations (3.9), which have been obtained with the help of (2.1) and (3.6), are also isobaric.

Theorem (3.6). *In the first class when $D \neq 0$ the set (3.8) is a complete set.*

Proof. Assume $p, q \leq 2, r \leq 3$; then we have exhausted all combinations of S in the set (3.8). We have seen from Theorem (3.4) that the set of all tensors $S \times A$ can be expressed as linear combinations of S .

Assume $p, q > 2, r \leq 3$; then S can be expressed as linear combinations of S , $p, q \leq 2$, by virtue of (3.6) and (2.1).

If $p, q \leq 2, r > 3$, then S can be expressed as linear combinations of S , $p, q \leq 2, r < 3$ by virtue of (2.1).

Hence S is a complete set.

Theorem (3.7). *In the first class when $D = 0$, the repeated application of the tensor A to the tensor S ($p + q + r = 2N$) yields only the three tensors S , S , S .*

Proof.

$$(3.10a) \quad 3 S \times A = S + 2 S,$$

$$(3.10b) \quad 3 S \times A = k^2 S - 4K S,$$

$$(3.10c) \quad 3 S \times A = -k S - K^2 S - K S,$$

in consequence of (2.2), which proves the statement for $p, q, r \leq 1$.

When $p, q, r \geq 1$ then, by virtue of (2.2), $S \times A$ can be expressed as a linear combination of S , S , S . Hence we have the theorem.

Theorem (3.8). *In the second class, we have*

$$(3.11a) \quad A = -2K A,$$

$$(3.11b) \quad A = 4K^2 A.$$

Theorem (3.9). In the second class, every tensor $A \times_{pq\bar{r}}^{110}$ for $p, q, r \geq 2$, $p + q + r = 2N$ may be expressed as a linear combination of A for $p, q, r \leq 2$.

Theorem (3.10). In the second class, the set of all tensors $S \times_{pq\bar{r}}^{(pq)r} A$ for $p + q, r \leq 2$, $p + q + r = 2N$ may be expressed as linear combinations of the set

$$(3.12) \quad S, \quad S^{(10)1}, \quad S^{(20)2}, \quad S^{110}, \quad S^{112}, \quad S^{002}, \quad S^{(20)2}$$

in the form

$$(3.13a) \quad 3 S^{000} \times A^{(110)} = 2 S^{(10)1} + S^{110},$$

$$(3.13b) \quad 3 S^{(10)1} \times A^{(110)} = S^{112} + S^{(20)2},$$

$$(3.13c) \quad 3 S^{(20)2} \times A^{(110)} = -2K S^{112} + 4K^2 S^{(10)1},$$

$$(3.13d) \quad 3 S^{110} \times A^{(110)} = 2K S^{112},$$

$$(3.13e) \quad 3 S^{112} \times A^{(110)} = 2K S^{112},$$

$$(3.13f) \quad 3 S^{002} \times A^{(110)} = S^{112} - 4K S^{(10)1},$$

$$(3.13g) \quad 3 S^{(02)0} \times A^{(110)} = -2K S^{110} - 2K S^{(10)1}.$$

Theorem (3.11). In the second class, the equations (2.6), (3.7) and (3.13) are isobaric.

Theorem (3.12). In the second class, the set (3.12) is a complete set.

Proof. The proofs of Theorems (3.8), (3.9), (3.10), (3.11) and (3.12) follow the patterns of the proofs of Theorems (3.2), (3.3), (3.4), (3.5) and (3.6) respectively.

Remark. The equations (3.13) cannot be obtained from the equations (3.9) only by putting $k = 0$. To obtain them one has also to substitute the relation (2.6) for the second class.

4. Solution of (1.19) in the first class by repeated substitution. We shall now find the solution of (1.19). For this we have the following theorems:

Theorem (4.1). In the first class we have

$$(4.1a) \quad B = S + S^{110} + 2 S^{(10)1},$$

$$(4.1b) \quad B^{(10)1} = S^{(10)1} + S^{(12)1} + S^{(20)2} + S^{112},$$

$$(4.1c) \quad 2 B^{(12)1} = 2 S^{(12)1} + 2k S^{(10)1} + k S^{002} - 2K S^{112} + S^{222},$$

$$\begin{aligned}
(4.1d) \quad {}^{(20)2} 2 B &= {}^{(20)2} 2 S + {}^{002} k S - {}^{112} 2K S - {}^{(10)3} 4K S - {}^{222} S, \\
(4.1e) \quad B &= {}^{(10)3} S + {}^{(10)3} S - {}^{(12)3} 2K S - {}^{(20)2} k S - {}^{(20)0} 2K S - {}^{112} k S - {}^{110} k S, \\
(4.1f) \quad {}^{(12)3} 2 B &= {}^{(12)3} 2 S + {}^{(10)3} 2k S - {}^{002} 2Kk S \\
&\quad + {}^{112} 4K^2 S - {}^{222} 2K S - {}^{110} k^2 S + {}^{110} 2Kk S - {}^{220} k S, \\
(4.1g) \quad {}^{(20)0} 2 B &= {}^{(20)0} 2 S + {}^{110} k S - {}^{220} 2K S - {}^{(10)1} S - {}^{(10)1} 4K S, \\
(4.1h) \quad B &= {}^{110} 2 S + {}^{(12)1} S + {}^{110} S + {}^{220} S, \\
(4.1i) \quad B &= {}^{112} 2 S + {}^{(12)3} S + {}^{112} S + {}^{222} S, \\
(4.1j) \quad B &= (1 + 2K) S + {}^{220} 2k S + {}^{(10)1} 2k S + {}^{(20)0} k S + {}^{110} k S, \\
(4.1k) \quad B &= (1 + 2K) S + {}^{222} k S + {}^{112} 2k S + {}^{(10)3} 2k S + {}^{(20)2} 2k S, \\
(4.1l) \quad B &= {}^{002} S + {}^{002} S + {}^{112} 2 S + {}^{(10)3} 2 S.
\end{aligned}$$

Proof. The equation (4.1a) is the same as the equation (3.5). Applying*

the tensors ${}^{(pq)r} A$ ($p, q \leq 2, r \leq 3, p + q + r = 2N$) to B and using (3.4) and (3.9) we have (4.1).

Theorem (4.2). *The equations (4.1) are also obtained by the repeated substitution of S on the right hand side of (3.5).*

Proof. The equation (3.5) is

$$(4.2) \quad S_{\omega\mu\nu} = B_{\omega\mu\nu} - A_{\omega\mu\nu}^{\alpha\beta\gamma} S_{\alpha\beta\gamma} - 2 A_{\omega\mu\nu}^{\alpha\beta\gamma} S_{\alpha\beta\gamma}.$$

Substituting for $S_{\alpha\beta\gamma}$ from (4.2) in the term $A_{\omega\mu\nu}^{\alpha\beta\gamma} S_{\alpha\beta\gamma}$ of (4.2) we get

$$(4.3) \quad S_{\omega\mu\nu} = B_{\omega\mu\nu} - 2 A_{\omega\mu\nu}^{\alpha\beta\gamma} S_{\alpha\beta\gamma} - A_{\omega\mu\nu}^{\alpha\beta\gamma} B_{\alpha\beta\gamma} + 2 A_{\omega\mu\nu}^{\alpha\beta\gamma} S_{\alpha\beta\gamma} + A_{\omega\mu\nu}^{\alpha\beta\gamma} S_{\alpha\beta\gamma}.$$

Subtracting (4.3) from (4.2) we have (4.1h).

In this way all the equations (4.1) can be obtained.

Remark. The equations (4.1) are not isobaric, since the original equation (3.5) is not isobaric. But if we consider all these equations with ${}^{(pq)r} B - {}^{(pq)r} S$ on the left hand side, then the terms on the right hand side have weight 2 more than the weight of the terms on the left hand side.

*Applying ${}^{(pq)r} A$ to B means ${}^{(pq)r} A \times B = {}^{(pq)r} B = A_{\omega\mu\nu}^{\alpha\beta\gamma} B_{\alpha\beta\gamma}.$

Theorem (4.3). *The equations (4.1) can be reduced in the nonholonomic frame of the first and second classes to the form*

$$(4.4) \quad \lambda_{xyz} S_{xyz} = B_{xyz} = \frac{1}{2}(K_{xyz} + 3\lambda_{xy} \lambda_{xz}).$$

Hence (4.4) is the solution of (1.19) in the first two classes in nonholonomic form (HLAVATÝ [1], p. 79).

Proof. Writing (4.1a) in nonholonomic form and using (1.20), we have (4.4). Writing the equations (4.1b-l) in the nonholonomic frame of the first two classes and using (1.9) and (1.20), we find that all these equations reduce to (4.4) multiplied on both sides by homogeneous functions of $\lambda_x, \lambda_y, \lambda_z$:

$$(\lambda_x^p \lambda_y^q + \lambda_x^q \lambda_y^p) \lambda_z^r, \quad p, q \leq 2, r \leq 3.$$

In consequence of (1.10), this expression vanishes only in four cases when

$$xy = 12 \quad \text{or} \quad 34$$

and

$$pq = 10 \quad \text{or} \quad 12.$$

Hence when $xy = 12$ or 34 only four equations (4.1b, c, e, f) reduce to identities. All other equations yield (4.4) for all indices including $xy = 12$ or 34 . But since the equations (4.1b, c, e, f), as well as all other equations (4.1), have been derived from (4.1a), we have (4.4) as the nonholonomic form for all equations (4.1).

Theorem (4.4). *The tensor $B_{\omega\mu\nu}^{(pq)r} (\equiv A_{\omega\mu\nu}^{\alpha\beta\gamma} B_{\alpha\beta\gamma})$ may be written as*

$$(4.5a) \quad B_{\omega\mu\nu}^{(pq)r} = \frac{1}{2}(K_{\omega\mu\nu}^{(pq)r} + 3k_{\omega}^{\alpha} k_{\mu}^{\beta} K_{[\alpha\beta\nu]}^{(pq)r})$$

or

$$(4.5b) \quad B_{\omega\mu\nu}^{(pq)r} = \frac{1}{2}K_{\alpha\beta\gamma}^{(pq)r} (A_{\omega\mu\nu}^{\alpha\beta\gamma} + 3 A_{\omega\mu\nu}^{\theta\varphi\psi} A_{[\alpha\beta\gamma]}^{110}).$$

Hence $B_{\omega\mu\nu}^{(pq)r}$ is a linear combination of $K_{\omega\mu\nu}^{(pq)r}$.

Proof. In consequence of (3.3c), we have (4.5b). Using (3.1b) and (3.3c), we have

$$B_{\omega\mu\nu}^{(pq)r} = \frac{1}{2}(K_{\omega\mu\nu}^{(pq)r} + 3 A_{\omega\mu\nu}^{\alpha\beta\gamma} K_{[\alpha\beta\gamma]}^{(p+1q+1)r}) = \frac{1}{2}(K_{\omega\mu\nu}^{(pq)r} + 3 A_{\omega\mu\nu}^{\alpha\beta\gamma} K_{[\alpha\beta\gamma]}^{110})$$

which proves (4.5a).

Theorem (4.5). *The solution $S_{\omega\mu\nu}$ of (1.19) is given as a linear combination of $K_{\omega\mu\nu}$ in the form*

$$(4.6a) \quad \begin{vmatrix} (S_{\omega\mu\nu} - B_{\omega\mu\nu})g & -1 & -1 & 0 \\ B_{\omega\mu\nu} & -k(1+2K) & 2k(1+K) & 1-2K \\ B_{\omega\mu\nu} & 2K-1+k & 1-2K-3k & 2 \\ B_{\omega\mu\nu} & 3k & k-1 & 1 \end{vmatrix} = 0$$

where*

$$(4.6b) \quad \begin{aligned} B_1 = & k^2 B + 2 B^{(12)3} + 2Kk B^{002} + 2(k - 2K^2) B^{112} \\ & - 2k B^{(12)1} + 2KB^{222} + 2k B^{(20)2} \\ & + 2k(1 + 2K + 2K^2) B^{110} - 2k(1 + 2K) B^{(10)3} \\ & - k(1 + 2K) B^{220} - 2k(1 + 2K)^2 B^{(10)1}, \end{aligned}$$

$$(4.6c) \quad \begin{aligned} B_2 = & 2(4K^2 - 1 + k + 4Kk) B^{(10)1} + 2(1 + K) B^{112} - B^{222} - k B^{002} + 2 B^{(20)2} \\ & + 4(K + k) B^{(10)3} + 2k B^{(20)0} - (4K^2 - 1 + k + 4Kk) B^{110} \\ & - k^2 B + (2K - 1 + 2k) B^{220}, \end{aligned}$$

$$(4.6d) \quad \begin{aligned} B_3 = & 2k^2 B + 2 B^{(12)3} - k B^{002} + 2KB^{112} + 2(1 + 2K) B^{(12)1} - B^{222} + 2k B^{(10)3} \\ & - (1 + k)(1 + 2K) B^{110} + (1 + k) B^{220} + 2k(1 + 2K) B^{(10)1} - 2k B^{(20)0}. \end{aligned}$$

Proof. Eliminating S , S , S , S , S , S , S , S , S from the equations (4.1) we get

$$(4.7a) \quad B = S + 2 S^{(10)1} + S^{110},$$

$$(4.7b) \quad B_1 = -2kg(1 + 2K) S^{(10)1} + 2kg(1 + K) S^{110} + 2g(1 - 2K) S^{(12)3},$$

$$(4.7c) \quad B_2 = 2g(2K - 1 + k) S^{(10)1} - g(2K - 1 + 3k) S^{110} + 4g S^{(12)3},$$

$$(4.7d) \quad B_3 = 6kg S^{(10)1} + g(k - 1) S^{110} + 2g S^{(12)3}.$$

Eliminating S , S and S from these equations, we obtain (4.6a).

*Here the indices are dropped, but B_r ($r = 1, 2, 3$) always stands for $B_{\omega\mu\nu}$.

From the form of the solution (4.6) it is clear that $S_{\omega\mu\nu}$ is given as a linear combination of $B_{\omega\mu\nu}^{(pa)r}$, $B_{\omega\mu\nu}^{(r)}$ ($r = 1, 2, 3$), which are themselves given as linear

combinations of $B_{\omega\mu\nu}$, as can be seen from (4.6b-d). Hence, in consequence of Theorem (4.4), we have the important result (HLAVATÝ [1], pp. 75-78):

The solution $S_{\omega\mu\nu}$ is expressible as a linear combination of $K_{\omega\mu\nu}$.

From (4.6) it is apparent that the necessary and sufficient conditions for the existence and uniqueness of the solution $S_{\omega\mu\nu}$ of (1.19) are

$$(4.8) \quad g \begin{vmatrix} -k(1+2K) & 2k(1+K) & 1-2K \\ 2K-1+k & 1-2K-3k & 2 \\ 3k & k-1 & 1 \end{vmatrix} \neq 0.$$

The inequality (4.8) can be simplified to

$$(4.9) \quad g(a^2 - 4kb^2) \neq 0,$$

where

$$(4.10a) \quad a \stackrel{\text{def}}{=} 1 - 2K + 5k = g - 2 - 6k,$$

$$(4.10b) \quad b \stackrel{\text{def}}{=} -2(1 - K) = g - 3 - k.$$

Theorem (4.6). *Put*

$$(4.11a) \quad L_{\omega\mu\nu} = \frac{1}{g(a^2 - 4kb^2)} \begin{vmatrix} K_{\omega\mu\nu} g & 1 & 1 & 0 \\ K_{\omega\mu\nu}^{(1)} & -k(1+2K) & 2k(1+K) & 1-2K \\ K_{\omega\mu\nu}^{(2)} & 2K-1+k & 1-2K-3k & 2 \\ K_{\omega\mu\nu}^{(3)} & 3k & k-1 & 1 \end{vmatrix}$$

where $K_{\omega\mu\nu}^{(p)}$ are the same linear algebraic functions of $K_{\omega\mu\nu}$ as $B_{\omega\mu\nu}^{(p)}$ are of $B_{\omega\mu\nu}$, as given by (4.6b-d). Then the solution $S_{\omega\mu\nu}$ of (1.19) is given by

$$(4.11b) \quad 2S_{\omega\mu\nu} = L_{\omega\mu\nu} + 3k_{\omega}^{\alpha} k_{\mu}^{\beta} L_{[\alpha\beta\nu]}.$$

Proof. With the help of (4.9), the solution (4.6a) can be written as

$$(4.12) \quad g(a^2 - 4kb^2)S_{\omega\mu\nu} = \begin{vmatrix} gB_{\omega\mu\nu} & 1 & 1 & 0 \\ B_{\omega\mu\nu}^{(1)} & -k(1+2K) & 2k(1+K) & 1-2K \\ B_{\omega\mu\nu}^{(2)} & 2K-1+k & 1-2K-3k & 2 \\ B_{\omega\mu\nu}^{(3)} & 3k & k-1 & 1 \end{vmatrix}.$$

With the help of (4.5a) and (4.6b-d), it is easy to verify that

$$(4.13) \quad B_{\omega\mu\nu} = \frac{1}{2}(K_{\omega\mu\nu} + 3k_{\omega}^{\alpha}k_{\mu}^{\beta}K_{[\alpha\beta\nu]}).$$

Substituting (4.13) in (4.12) and using (4.11a), we have (4.11b).

Remark 1. The solution in the form (4.11b) has already been given by HLAVATÝ([1], p. 78), where $L_{\omega\mu\nu}$ is expressed for the index of inertia 2, in a form (different from (4.11a) which exists only for the first two classes).

Remark 2. As in HLAVATÝ's solution ([1], p. 78), it is clear again from the form (4.11) that $S_{\omega\mu\nu}$ is expressible as a linear combination of $K_{\omega\mu\nu}$.

We now come back to the conditions (4.9). We have the following theorem:

Theorem (4.7). The conditions (4.9) for the existence and uniqueness of the solution $S_{\omega\mu\nu}$ of (1.19) are

- (a) $g(a^2 - 4kb^2) \neq 0 \quad (I/0/-)^*$,
- (b) $g \neq 0 \quad (I/2/-)$,
- (c) $a^2 - 4kb^2 \neq 0 \quad (I/4/-)$,
- (d) $(1 + K)(1 - 3K) \neq 0 \quad (I/0/0)$,
- (e) $1 - 3K \neq 0 \quad (I/4/0)$,
- (f) $g(g - 2) \equiv 4K^2 - 1 \neq 0 \quad (II/2^0/-)$,
- (g) $g - 2 \equiv 2K - 1 \neq 0 \quad (II/4/-)$.

In the first and second classes, the cases for which the conditions (a-g) are not given do not exist.

The conditions (4.9) are always satisfied for the third and fourth classes (both of which exist when the index of inertia of $h_{\lambda\mu}$ is 0 and only the third class exists when the index of inertia is 2, whereas neither of them exists when the index of inertia is 4).

Proof. In the first class, when the index of inertia is 0, k is always positive. Hence we have both the conditions

$$g \neq 0, \quad a^2 - 4kb^2 \neq 0.$$

In this case, when $D = 0$, these conditions reduce to the conditions (d).

In the first class, when the index of inertia is 2, k is always negative and $D \neq 0$ (hence $a^2 - 4kb^2$ is always positive). Therefore in this case we have only one condition:

$$g \neq 0.$$

*Here the (first) Roman numeral denotes the class of $k_{\lambda\mu}$; the (second) Arabic numeral denotes the index of inertia of $h_{\lambda\mu}$; and the third number 0 denotes the condition $D = 0$. If the third number does not appear then $D \neq 0$.

In the first class, when the index of inertia is 4, both k and K are always positive and hence g is positive. Therefore we have only one condition:

$$a^2 - 4kb^2 \neq 0, \quad D \neq 0.$$

which reduces to

$$1 - 3K \neq 0$$

when $D = 0$.

When the index of inertia is 4, K is always positive. Hence we do not have third and fourth classes in this case.

In the second class, when the index of inertia is 0 or 2, K may be positive or negative and the conditions reduce to (f). But when the index of inertia is 4, K is always positive and hence $g \equiv 1 + 2K$ is positive. Therefore in this case we have only one condition (g).

Putting $K = k = 0$, $g = 1$ in the conditions, we find that they are always satisfied (in the third and fourth classes).

When the index of inertia is 2, ${}^{(2)}k_\alpha{}^\beta$ is never zero. Hence the fourth class does not exist in this case.

5. Deductions of solutions. Putting $K^2 = k$, $g = (1 + K)^2$ in (4.6) we can obtain the solution of (1.19) in the first class when $D = 0$. The solution can be simplified if we also use the recurrence relations (2.2) for this case. It may, however, be noted that the solution has been obtained directly and more simply by the method of repeated substitution, and using (2.2) in MISHRA ([3]).

The solution (4.6) is continuous in k and K and does not contain derivatives of k and K . Hence, by putting $k = 0$, we can obtain the solution in the second class. It may be pointed out that Mme. TONNELAT's solution contains derivatives of k and K . The solution will be simplified if we also use the recurrence relation (2.6a) and the relations (3.7). We will now show this. Putting $k = 0$ in (4.6b-d) and using (2.6a) and (3.7), we have

$$B_1 = 0,$$

$$B_2 = -2 \overset{(10)1}{B} + 2 \overset{112}{B} + 2 \overset{(20)2}{B} + (1 - 2K) \overset{110}{B},$$

$$B_3 = -\overset{110}{B}.$$

Hence the solution (4.6a) assumes the form

$$(5.1) \quad \begin{vmatrix} g(S - B) & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 - 2K \\ -2 \overset{(10)1}{B} + 2 \overset{112}{B} + 2 \overset{(20)2}{B} + (1 - 2K) \overset{110}{B} & 2K - 1 & 1 - 2K & 2 \\ -\overset{110}{B} & 0 & -1 & 1 \end{vmatrix} = 0$$

which can be simplified to

$$(5.2) \quad (S - B)(1 - 4K^2) = 2B + 2 \overset{112}{B} - (1 - 2K) \overset{110}{B} - 2 \overset{(10)1}{B}.$$

(5.2) is the solution in the second class.

The solution can, however, be obtained directly as follows:

Theorem (5.1). *In the second class we have the following set of seven linear equations in $\overset{(pq)r}{S}$:*

$$(5.3a) \quad B = S + 2 \overset{(10)1}{S} + \overset{110}{S},$$

$$(5.3b) \quad B = \overset{(10)1}{S} + \overset{(10)1}{S} + \overset{112}{S} + \overset{(20)2}{S},$$

$$(5.3c) \quad B = \overset{(20)2}{S} - 2K \overset{(20)2}{S} + 4K^2 \overset{112}{S} + \overset{(10)1}{S},$$

$$(5.3d) \quad B = \overset{110}{S}(1 + 2K),$$

$$(5.3e) \quad B = \overset{112}{S}(1 + 2K),$$

$$(5.3f) \quad B = \overset{(20)0}{S} - 2K \overset{(20)0}{S} - 2K \overset{110}{S} - 2K \overset{(10)1}{S},$$

$$(5.3g) \quad B = \overset{002}{S} + \overset{002}{S} - 4K \overset{112}{S} - 4K \overset{(10)1}{S}.$$

Proof. Applying the tensors $\overset{(10)1}{A}$, $\overset{(20)2}{A}$, $\overset{110}{A}$, $\overset{112}{A}$, $\overset{(20)0}{A}$, and $\overset{002}{A}$ to B and using (2.6) and (3.7), we obtain the equations (5.3).

Remark. Putting $k = 0$ and using the recurrence relation (2.6) in (4.1), we find that 12 equations (4.1) of the first class reduce to seven equations (5.3) in the second class since $K \neq 0$.

The equations (5.3) are seven equations in the seven unknowns $\overset{(pq)r}{S}$, ($pq = 00, 10, 11, 20$; $r = 0, 1, 2$, $p + q + r = 2N$). Eliminating six unknown quantities $\overset{(pq)r}{S}$ ($(pq)r \neq 000$) from (5.3), we can obtain the solution S .

Theorem (5.2). *The solution $S_{\omega\mu\nu}$ of (1.19) in the second class is (5.2).*

Proof. Eliminating $\overset{(pq)r}{S}$ ($(pq)r \neq 000$) from (5.3a-e), we obtain (5.2).

Remark. In the second class S is obtained by eliminating $\overset{(pq)r}{S}$ ($(pq)r \neq 000$) from only five equations (5.3a-e) out of seven equations (5.3a-g), whereas, in the first class one has to use all the 12 equations (4.1) to obtain S by eliminating $\overset{(pq)r}{S}$ ($(pq)r \neq 000$).

Theorem (5.3). *If the solution $S_{\omega\mu\nu}$ of (1.19) in the second class is of the form*

(4.11b), then we have for this class

$$(5.4) \quad L_{\omega\mu\nu} = K_{\omega\mu\nu} + \frac{1}{1-4K^2} [(1-2K)K_{\omega\mu\nu}^{(110)} - 2K_{\omega\mu\nu}^{(10)1} + 2K_{\omega\mu\nu}^{(112)} + 2K_{\omega\mu\nu}^{(20)2}].$$

Proof. Putting $k = 0$ in (4.11a) and using (3.7), we obtain (5.4).

Remark. From the form of the solution (5.2) and from (5.4), it is again clear that $S_{\omega\mu\nu}$ can be expressed as a linear combination of $K_{\omega\mu\nu}$.

Putting $K = 0$ in (5.2) we can obtain the solution in the third class. The solution will be simplified if we use the following lemma.

Lemma (5.1). *In the third class we have*

$$(5.5a) \quad B_{\omega\mu\nu}^{(112)} = 0,$$

$$(5.5b) \quad B_{\omega\mu\nu}^{(20)2} = 0.$$

Proof. Writing $B_{\omega\mu\nu}^{(20)2}$ in nonholonomic form and using (3.3c), we have

$$B_{pqr}^{(20)2} = \frac{1}{2}K_{ijk}^{(20)2} [A_{pqr}^{ijk} + 3A_{pqr}^{(20)2} A_{lmn}^{(110)2} A_{lmn}^{[ijk]}],$$

which vanishes in consequence of (1.22) and the fact that B_{pqr} is skew-symmetric in p and q .

The equation (5.5a) is proved similarly.

Theorem (5.4). *The solution $S_{\omega\mu\nu}$ of (1.19) in the third class is*

$$(5.6a) \quad S_{\omega\mu\nu} = B_{\omega\mu\nu} - 3B_{\omega\mu\nu}^{(110)}$$

which reduces to the form

$$(5.6b) \quad S_{\omega\mu\nu} = \frac{1}{2}K_{\omega\mu\nu} + k_{[\omega}^{\alpha} K_{\mu]\alpha\beta} k_{\nu]}^{\beta}.$$

Proof. The equation (5.6a) follows from (5.2) by putting in it $K = 0$ and using (5.5).

Putting the value of B from (3.3c) in (5.6a) and using (1.22), we have (5.6b).

Remark. The solution in the third class can be obtained directly as follows:

Applying to (4.1a) the tensor $A_{\omega\mu\nu}^{(110)}$ and using the fact that for the third class $S_{\omega\mu\nu}^{(211)}$, $S_{\omega\mu\nu}^{(202)}$ and $S_{\omega\mu\nu}^{(222)}$ all vanish, we have

$$B_{\omega\mu\nu}^{(110)} = S_{\omega\mu\nu}^{(110)}.$$

Eliminating $S_{\omega\mu\nu}^{(110)}$ between this equation and (4.1a), we get the result.

*This form of the solution has already been obtained by HLAVATÝ ([1], p. 88), by a different method.

Theorem (5.5). *The solution (5.6b) of the third class is also the solution in the fourth class.*

Proof. The fourth class is characterized by the conditions of the third class together with the additional condition

$${}^{(2)}k_{\alpha}{}^{\beta} = 0.$$

But the solution (5.6b) does not contain the terms ${}^{(2)}k_{\alpha}{}^{\beta}$, hence it is also the solution in the fourth class.

Remark. In obtaining the solution of (1.19), HLAVATÝ considered the physically admissible index of inertia of $h_{\lambda\mu}$ as 2. In this case D is never zero in the first two classes. In his solution D occurs in the denominator always. But in the third class $D = 0$, hence his solution cannot yield the solution for the third class, unless certain conditions were satisfied. HLAVATÝ ([1], p. 97) obtained the conditions under which the term $B_{\omega\mu\nu}$ of the solution in the first two classes is the solution (5.6b) in the third class. The conditions were in the form

$$(5.7) \quad \alpha^{\lambda} \alpha^{\mu} \nabla_{\mu} a_{\lambda}^u = 0 \quad (u = 1, 2; v = 2, 4).$$

We will find these conditions in an alternative form.

Theorem (5.6). *The necessary and sufficient conditions that $B_{\omega\mu\nu}$ be the solution $S_{\omega\mu\nu}$ in the third class are*

$$(5.8a) \quad 3 B_{\omega\mu\nu}^{(110)} - 2 B_{\omega\mu\nu}^{(20)2} - 2 B_{\omega\mu\nu}^{112} = 0$$

which, in the third class, reduces to

$$(5.8b) \quad A_{\omega\mu\nu}^{\alpha\beta\gamma} K_{\alpha\beta\gamma}^{(110)} = 0.$$

Proof. The solution in the third class is

$$S = B - 3 B^{(110)} + 2 B^{(20)2} + 2 B^{112}.$$

Putting $S = B$ in this equation we get (5.8a).

Using (5.5), (5.8a) becomes

$$B_{\omega\mu\nu}^{(110)} = 0.$$

This equation reduces to (5.8b) with the help of (1.22e, f).

To prove the equivalence of the conditions (5.7) and (5.8b), we observe that (5.7) have been deduced from (HLAVATÝ [1], p. 97)

$$(5.9) \quad 2k_{[\omega}{}^{\alpha} K_{\mu]\alpha\beta} k_{\nu}{}^{\beta} = 3K_{[\alpha\beta\gamma]} k_{\omega}{}^{\alpha} k_{\mu}{}^{\beta}.$$

The conditions (5.9) are the same as (5.8b), in consequence of (1.22).

6. Solution of the open question. We have seen that the general solution (4.6) in the first class yields the solutions in all other classes. We will now find the conditions under which the form of the solution for the third (fourth) class is also the form of the solution for the first or second class. This question has been left open by HLAVATÝ ([1], p. 99).

Theorem (6.1). *If $K_{\omega\mu\nu} \neq 0$,* it is not possible for the form of the solution in the third class to be put in the form of the solution in the first class.*

Proof. The necessary and sufficient condition for the form of the solution in the third class to be the form of the solution in the first (second) class is

$$(6.1) \quad S_{\omega\mu\nu}^{1(2)} = S_{\omega\mu\nu}^3$$

where $S_{\omega\mu\nu}^{1(2)}$ and $S_{\omega\mu\nu}^3$ are respectively the values of $S_{\omega\mu\nu}$ given by (4.6) ((5.2)) and (5.6b). Writing (6.1) in the nonholonomic frame of the first class, we have, in consequence of (4.4) and (1.20), the equation

$$(6.2a) \quad \frac{1}{\lambda} (K_{xyz} + 3\lambda\lambda K_{[xyz]}) = K_{xyz} (1 - \lambda\lambda_{xz} - \lambda\lambda_{yz})$$

which with the help of (1.20) reduces to

$$(6.2b) \quad K_{ijk} Y_{[pq]r}^{[ij]k} = 0$$

where

$$(6.2c) \quad Y_{[pq]r}^{[ij]k} = k_{[p}^k k_{q]}^{[i} \delta_r^{j]} + \delta_r^{[i} k_{[p}^{j]} k_{q]}^k + {}^{(2)}k_{[p}^{[i} k_{q]}^{j]} {}^{(2)}k_r^k + {}^{(2)}k_{[q}^{[i} \delta_p^{j]} {}^{(2)}k_r^k + 2k_{[p}^{[i} k_{q]}^{j]} {}^{(2)}k_r^k + {}^{(2)}k_{[p}^{[i} k_{q]}^{j]} k_r^k + k_{[p}^{[i} {}^{(2)}k_{q]}^{j]} k_r^k.$$

By actual computation it has been found that the 24×24 determinant $|Y_{[pq]r}^{[ij]k}|$ is of rank 24. Therefore it is not possible for (6.2) to yield a non-trivial solution. Hence we have the theorem.

However, for the second class we have

Theorem (6.3). *The necessary and sufficient conditions that the form of the solution in the third (fourth) class is the form of the solution in the second class are*

- (a) $K = \text{constant}$,
- (b) the basic vectors a^{ν}_1, a^{ν}_2 are X_2 -forming
- (c) the basic vectors are such that

$$(6.3) \quad a^{\lambda} a^{\mu} \nabla_{\mu} a^{\nu}_{\lambda} = 0, \quad f = 3, 4; \begin{cases} x = 1, 2 \\ y = 2, 1 \end{cases}$$

*The case $K_{\omega\mu\nu} = 0$ has been dealt with in HLAVATÝ ([1], p. 111). It leads to a product Riemannian space.

when $K > 0$, and

(a') $K = \text{constant}$,

(b') the basic vectors a^λ, a^μ are X_2 -forming,

(c') the basic vectors are such that

$$(6.3') \quad a^\lambda a^\mu \nabla_\mu a_\lambda = 0, \quad x = 1, 2; \quad \begin{cases} f = 3, 4 \\ g = 4, 3 \end{cases}$$

when $K < 0$.

Proof. When $K > 0$, the equations (6.2) reduce to

$$(6.4) \quad K_{121} = K_{122} = K_{131} = K_{132} = K_{141} \\ = K_{142} = K_{231} = K_{232} = K_{241} = K_{242} = 0,$$

in consequence of (1.12). These equations simplify to

$$(6.5a) \quad \nabla_i k_{12} = 0, \quad i = 1, 2, 3, 4,$$

$$(6.5b) \quad \begin{Bmatrix} x \\ fx \end{Bmatrix} = \begin{Bmatrix} y \\ fy \end{Bmatrix},$$

$$(6.5c) \quad \nabla_x k_{xf} = 0 \quad \text{or} \quad \begin{Bmatrix} x \\ fy \end{Bmatrix} = 0, \quad \begin{matrix} f = 3, 4, \\ x \neq y = 1, 2. \end{matrix}$$

The equation (6.5a) is equivalent to $K = \text{constant}$. The equation (6.5b) is equivalent to

$$(6.6) \quad a^\lambda a^\mu \nabla_{[\mu} a_{\lambda]} = 0,$$

in consequence of (1.24) and (1.26).

The equation (6.6) is the condition for a^λ, a^μ to be X_2 -forming.

The equations (6.5c) are obtained with the help of (1.24).

The results about $K < 0$ can be proved similarly.

Remark. The question now arises: Under what conditions is the form of the solution in the second class also the form of the solution in the first class?

As in Theorem (4.3), the equations (5.3), which yield the solutions of the second class, can be reduced in the nonholonomic frame of the first class to the form

$$\lambda S_{xyz} = \frac{1}{2}(K_{xyz} + 3\lambda\lambda K_{[xyz]}).$$

But this is also the form to which the solution in the first class reduces in the nonholonomic frame.

Now if we compare the solutions for the two classes for different indices of S_{xyz} , we find that the corresponding forms for S_{xyz} in the two classes cannot be the same unless $K_{xyz} = 0$, since in the first class the coefficients of k can all vanish only when $K_{xyz} = 0$. When $K_{xyz} = 0$, we have a product Riemannian space (HLAVATÝ [1], p. 111) which we neglect. Hence we have the theorem

Theorem (6.4). *If $K_{\omega\mu\nu} \neq 0$, it is not possible for the form of the solution in the second class to be put in the form of the solution in the first class.*

7. Appendix. The tensor $U_{\omega\mu\nu}$. Having obtained $S_{\omega\mu\nu}$, we can substitute for $S_{\omega\mu\nu}$ from (4.6) in (1.15) and get $U_{\omega\mu\nu}$. However, it can be computed directly as follows: The equation in $U_{\omega\mu\nu}$ can be written in the form

$$(7.1) \quad U_{\omega\mu\nu} + 3 A_{\omega\mu\nu}^{(110)\alpha\beta\gamma} U_{\alpha\beta\gamma} = -2k_{(\nu}{}^{\alpha} K_{\mu)\omega\alpha}.$$

Comparing this with (3.3), we find that the two equations are of the same form, with the exception that (3.3) is skew-symmetric in the first two indices and (7.1) is symmetric in the last two indices. Hence the method of solution $S_{\omega\mu\nu}$, which is based on (3.6), cannot be used, as it is, for the solution of $U_{\omega\mu\nu}$.

We therefore use the following device: If we put

$$(7.2a) \quad C_{\omega\mu\nu} \stackrel{\text{def}}{=} K_{\omega\mu\alpha} k_{\nu}{}^{\alpha}$$

and

$$(7.2b) \quad V_{\omega\mu\nu} \stackrel{\text{def}}{=} S_{\omega\mu\alpha} k_{\nu}{}^{\alpha},^*$$

then $C_{\omega\mu\nu}$ and $V_{\omega\mu\nu}$ are both skew-symmetric in the first two indices and

$$(7.2c) \quad U_{\omega\mu\nu} = 2V_{\omega(\mu\nu)}.$$

We can now solve for $V_{\omega\mu\nu}$ the equation

$$(7.3) \quad V_{\omega\mu\nu} + 3 A_{\omega\mu\nu}^{(110)\alpha\beta\gamma} V_{\alpha\beta\gamma} = C_{\omega\mu\nu}$$

which is skew-symmetric in the first two indices exactly as (3.3).

Since the equation (7.3) is equivalent to

$$V_{\omega(\mu\nu)} + 3 A_{\omega(\mu\nu)}^{(110)\alpha\beta\gamma} V_{\alpha\beta\gamma} = C_{\omega(\mu\nu)},$$

which is the same as (7.1), we get $U_{\omega\mu\nu}$ from (7.2c).

In conclusion, I wish to thank Professor HLAVATÝ for his suggesting the problem and for his kind guidance in the preparation of this paper.

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*We remind the reader that since $S_{\omega\mu\alpha}$ is not previously determined, we are interested in finding $V_{\omega\mu\nu}$.