Basic Principles of the Unified Field Theory of the Second Kind, II ***

V. HLAVATÝ

CHAPTER III. PRELIMINARY INFORMATION ABOUT $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$

14. Formal Solution of (A). In this chapter we shall find necessary conditions for the solution of the system (A), *i.e.*

$$(14.1a) \quad \partial_{\omega} g_{\lambda\mu} - \Gamma_{\lambda}{}^{\alpha}{}_{\omega} g_{\alpha\mu} - \Gamma_{\omega}{}^{\alpha}{}_{\mu} g_{\lambda\alpha} = 0.$$

The connection $\Gamma_{\lambda}{}^{\nu}{}_{\omega}$ gives rise to the *torsion tensor*

$$(14.2a) \quad S_{\lambda\mu}{}^{\nu} \stackrel{\text{def}}{=} \Gamma_{[\lambda}{}^{\nu}{}_{\mu]}$$

and to the *torsion vector*

$$(14.2b) \quad S_{\lambda} \stackrel{\text{def}}{=} S_{\lambda\alpha}{}^{\alpha}.$$

According to EINSTEIN the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ has to satisfy the condition

$$(14.1b) \quad S_{\lambda} = 0.$$

In this chapter we shall be concerned chiefly with the system (14.1a) and we shall denote by D_{ω} the symbol for the covariant derivative with respect to $\Gamma_{\lambda}{}^{\nu}{}_{\omega}$.

Theorem (14.1). *The system (14.1a) is equivalent to*

$$(14.3) \quad D_{\omega} g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}.$$

This system may be split into two, namely

$$(14.4a) \quad D_{\omega} h_{\lambda\mu} = 2S_{\omega(\mu}{}^{\alpha} g_{\lambda)\alpha},$$

$$(14.4b) \quad D_{\omega} k_{\lambda\mu} = 2S_{\omega[\mu}{}^{\alpha} g_{\lambda]\alpha}.$$

*This paper was prepared with the assistance of a grant from the Research Division (Air Force).

**This paper is a straightforward continuation of a paper published under the same title in the same journal (Journal of Mathematics and Mechanics). Therefore the numbering of sections starts with Section 14. The symbolism as well as the results of the first paper are also used in this second paper.

The tensors $h_{\lambda\mu}$ and $k_{\lambda\mu}$ satisfy the conditions

$$(14.5a) \quad D_{(\omega} h_{\lambda\mu)} = 0,$$

$$(14.5b) \quad D_{\lambda} h_{\omega\mu} + 2D_{(\omega} k_{\mu)\lambda} = 0.$$

Proof. Adding $2 S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha}$ to both sides of (14.1a) one obtains (14.3). The equations (14.4) follow at once from (14.3). The condition (14.5a) follows from (14.4a) and (14.2a). Using (14.4) we obtain

$$D_{\lambda} h_{\omega\mu} + 2D_{(\omega} k_{\mu)\lambda} = 2S_{\lambda(\omega}{}^{\alpha} g_{\mu)\alpha} + 2S_{\omega[\lambda}{}^{\alpha} g_{\mu]\alpha} + 2S_{\mu[\lambda} g_{\omega]\alpha} = 0.$$

In the next theorem we shall use the connection $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$ defined by (13.3).

Theorem (14.2). *Put*

$$(14.6a) \quad U^{\nu}{}_{\lambda\mu} \stackrel{\text{def}}{=} \Gamma_{(\lambda}{}^{\nu}{}_{\mu)} - \Lambda_{\lambda}{}^{\nu}{}_{\mu}.$$

Then

$$(14.7) \quad U^{\alpha}{}_{\mu\lambda} h_{\omega\alpha} = 2S_{\omega(\mu}{}^{\beta} k_{\lambda)\beta}$$

and

$$(14.8) \quad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = \Lambda_{\lambda}{}^{\nu}{}_{\mu} + S_{\lambda\mu}{}^{\nu} + U^{\nu}{}_{\lambda\mu}.$$

Proof. The formulæ (14.4a) and (14.5a) yield

$$(14.9a)^* \quad D_{\{\lambda} h_{\alpha\mu\}} = -D_{\alpha} h_{\lambda\mu} = -2S_{\alpha(\mu}{}^{\beta} g_{\lambda)\beta} = -2S_{\alpha(\mu}{}^{\beta} h_{\lambda)\beta} - 2S_{\alpha(\mu}{}^{\beta} k_{\lambda)\beta}.$$

On the other hand, we see from

$$D_{\alpha} h_{\beta\gamma} = \partial_{\alpha} h_{\beta\gamma} - \Gamma_{\beta}{}^{\delta}{}_{\alpha} h_{\delta\gamma} - \Gamma_{\gamma}{}^{\delta}{}_{\alpha} h_{\beta\delta}$$

that

$$(14.9b) \quad D_{\{\lambda} h_{\alpha\mu\}} = \gamma_{\alpha\lambda\mu} - 2S_{\alpha(\lambda}{}^{\alpha} h_{\mu)\alpha} - \Gamma_{(\mu}{}^{\gamma}{}_{\lambda)} h_{\gamma\alpha}.$$

Comparing (14.9a) and (14.9b) we obtain, by virtue of (13.2), the condition (14.7) for the tensor $U^{\nu}{}_{\lambda\mu}$ defined by (14.6a). The equation (14.8) is an immediate consequence of (14.6a).

Note. If $B_{\lambda\mu}$ is an arbitrary symmetric tensor and if we use the definition

$$(14.6b) \quad U^{\nu}{}_{\lambda\mu} \stackrel{\text{def}}{=} \Gamma_{(\lambda}{}^{\nu}{}_{\mu)} - \Lambda_{(\lambda}{}^{\nu}{}_{\mu)} + \underset{4}{a}{}^{\nu} B_{\lambda\mu}$$

instead of (14.6a) then we obtain again (14.7). However, the definition of $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$ involves an arbitrary tensor $b_{\lambda\mu}$, as is seen from (13.3). Therefore we may use (14.6a) and disregard (14.6b) without restriction of the generality.

Theorem (14.3). *Put*

$$(14.10a) \quad K_{\omega\mu\lambda} \stackrel{\text{def}}{=} 2\nabla_{(\mu} k_{\omega\lambda)}.$$

*See (8.1).

Then

$$(14.10b) \quad K_{\omega\mu\lambda} = \nabla_{\omega}k_{\lambda\mu} + \nabla_{\mu}k_{\omega\lambda} + \nabla_{\lambda}k_{\omega\mu} = 3\partial_{[\omega}k_{\lambda\mu]} + 2\nabla_{\lambda}k_{\omega\mu}$$

and

$$(14.11) \quad 2S_{\omega\mu}{}^{\alpha}h_{\alpha\lambda} = K_{\omega\mu\lambda} - 4U^{\alpha}{}_{\lambda[\mu}k_{\omega]\alpha}.$$

Proof. The equation (14.10b) follows at once from (14.10a) and (8.1). Using (14.3) and (14.5b) one gets

$$(14.12) \quad 2D_{(\omega}k_{\lambda)\mu} + D_{\mu}k_{\omega\lambda} = -(D_{\mu}h_{\lambda\omega} + D_{\mu}k_{\lambda\omega}) = -D_{\mu}g_{\lambda\omega} = -2S_{\mu\omega}{}^{\alpha}g_{\lambda\alpha}.$$

On the other hand, the substitution from (14.8) into $D_{\alpha}k_{\beta\gamma}$ gives

$$(14.13) \quad D_{\alpha}k_{\beta\gamma} = \nabla_{\alpha}k_{\beta\gamma} - (S_{\beta\alpha}{}^{\delta} + U^{\delta}{}_{\beta\alpha})k_{\delta\gamma} - (S_{\gamma\alpha}{}^{\delta} + U^{\delta}{}_{\gamma\alpha})k_{\beta\delta}.$$

Substituting from (14.13) into (14.12) one finds (14.11).

Note. We see from (14.8) that in order to know $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ we need $U^{\nu}{}_{\lambda\mu}$ and $S_{\lambda\mu}{}^{\nu}$. The equation (14.7) expresses $U^{\nu}{}_{\lambda\mu}$ in terms of $S_{\lambda\mu}{}^{\nu}$ so that finally (14.11) represents a set of 24 equations for $S_{\lambda\mu}{}^{\nu}$. Hence the problem of solving (14.1a) revolves around two problems, namely to express $U^{\nu}{}_{\lambda\mu}$ in terms of $S_{\lambda\mu}{}^{\nu}$ in accordance with (14.7) and to solve (14.11) for $S_{\lambda\mu}{}^{\nu}$. In subsequent considerations we shall deal with these problems by means of the tensors

$$(14.14) \quad \begin{aligned} (a) \quad s_{\lambda\mu\nu} &\stackrel{\text{def}}{=} S_{\lambda\mu}{}^{\alpha}a_{\alpha\nu}, \\ (b) \quad u_{\nu\lambda\mu} &\stackrel{\text{def}}{=} U^{\alpha}{}_{\lambda\mu}a_{\alpha\nu}, \\ (c) \quad K_{\lambda}{}^{\nu} &\stackrel{\text{def}}{=} k_{\lambda\mu}a^{\mu\nu}. \end{aligned}$$

None of these tensors is frame invariant. According to Theorem (7.5) we have

$$(14.15) \quad \begin{aligned} (a) \quad S_{\lambda\mu}{}^{\alpha}k_{\nu\alpha} &= s_{\lambda\mu\alpha}K_{\nu}{}^{\alpha}, \\ (b) \quad U^{\alpha}{}_{\lambda\mu}k_{\nu\alpha} &= u_{\alpha\lambda\mu}K_{\nu}{}^{\alpha}. \end{aligned}$$

We shall use these formulæ very often.

15. The Tensors $U^{\nu}{}_{\lambda\mu}$, $K_{\omega\mu\lambda}$ and $L_{\omega\mu\lambda}$.

Theorem (15.1). If (14.3) admits at least one solution $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, then it admits every connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ defined by

$$(15.1) \quad \Gamma'_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{\lambda}{}^{\nu}{}_{\mu} + T_{\lambda\mu}{}^{\nu},$$

where $T_{\lambda\mu}{}^{\nu}$ is an arbitrary tensor satisfying the condition

$$(15.2) \quad T_{\lambda\omega}{}^{\alpha}g_{\alpha\mu} + T_{\omega\mu}{}^{\alpha}g_{\lambda\alpha} = 0.$$

Proof. Put

$$(15.3) \quad Q_{\omega\lambda\mu} \stackrel{\text{def}}{=} D_{\omega}g_{\lambda\mu} - 2S_{\omega\mu}{}^{\alpha}g_{\lambda\alpha}$$

and disregard for a moment the validity of (14.3). We shall prove that the components of $Q_{\omega\lambda\mu}$ are not independent. Using (1.4) we obtain easily

$$(15.4a) \quad 0 = D_{\omega} g_{\lambda\mu} a^{\lambda} a^{\mu} = D_{\omega} h_{\lambda\mu} a^{\lambda} a^{\mu} = (D_{\omega} h_{\lambda\mu}) a^{\lambda} a^{\mu} = (D_{\omega} g_{\lambda\mu}) a^{\lambda} a^{\mu}.$$

On the other hand, we see from (14.2a) that

$$(15.4b) \quad 2a^{\omega} a^{\mu} a^{\lambda} S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} = 0.$$

Therefore we have from (15.3), by virtue of (15.4),

$$(15.5) \quad a^{\omega} a^{\mu} a^{\lambda} Q_{\omega\lambda\mu} = 0$$

and this equation holds even for $Q_{\omega\mu\lambda} \neq 0$.* Consequently the equations (14.3) are not independent; they satisfy the condition (15.5) where $Q_{\omega\lambda\mu}$ is defined by (15.3). If (14.3) (and (15.5)) are incompatible, then (14.1a) does not admit any solution. If (14.3) (and (15.5)) are compatible and if (14.1a) admits at least one solution, then it admits more than one solution. If $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is one such solution, then any other solution must be of the form (15.1). The condition (15.2) is an immediate consequence of the fact that $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ as well as $\Gamma'_{\lambda}{}^{\nu}{}_{\mu}$ satisfy (14.1a).

In the next theorem we shall use the equations (11.4), (13.3) and (13.4).

Theorem (15.2). *Put*

$$(15.6) \quad \begin{aligned} (a) \quad K_{\omega\mu\lambda} &\stackrel{\text{def}}{=} 2\nabla_{[\mu}^0 k_{\omega\lambda]} , \\ (b) \quad K'_{\omega\mu\lambda} &\stackrel{\text{def}}{=} 2\nabla_{[\mu}^0 k'_{\omega\lambda]} , \\ (c) \quad K'_{\omega\mu\lambda} &\stackrel{\text{def}}{=} 2\nabla_{[\mu}^0 k'_{\omega\lambda]} . \end{aligned}$$

Then

$$(15.7) \quad \begin{aligned} (a) \quad K_{\omega\mu\lambda} &= K_{\omega\mu\lambda}^0 - 4a_{[\omega}^3 b_{\mu]\lambda} , \\ (b) \quad K'_{\omega\mu\lambda} &= K_{\omega\mu\lambda}^0 + 4\epsilon a_{[\omega}^3 t_{\mu]\lambda} , \\ (c) \quad K'_{\omega\mu\lambda} &= K_{\omega\mu\lambda}^0 + 4\epsilon a_{[\omega}^3 t_{\mu]\lambda} . \end{aligned}$$

Proof. A simple inspection based on (13.3) and (14.10a) leads to

$$K_{\omega\mu\lambda} = 2\nabla_{[\mu}^0 k_{\omega\lambda]} - 2a_{\alpha}^{\omega} k_{\alpha(\lambda} b_{\omega\mu)} + 2a_{\alpha}^{\omega} k_{\alpha(\omega} b_{\mu\lambda]} .$$

Substituting into this equation from (2.3) one obtains (15.7a). The equations (15.7b,c) may be proved in a similar manner by means of (11.4).

Theorem (15.2) will be used later very often.

*In other words (15.5) is an identity for all possible connections.

Theorem (15.3). *A necessary and sufficient condition for (14.7) to admit at least one solution $U^\nu_{\lambda\mu}$ is*

$$(15.8a) \quad a^\alpha (S_{\alpha\mu}{}^\beta k_{\lambda\beta} + S_{\alpha\lambda}{}^\beta k_{\mu\beta}) = 0.$$

This condition is equivalent to

$$(15.8b) \quad a^\alpha (s_{\alpha\mu\beta} K_\lambda{}^\beta + s_{\alpha\lambda\beta} K_\mu{}^\beta) = 0,$$

where $s_{\alpha\beta\gamma}$ and $K_\gamma{}^\beta$ are defined by (14.14a,c).

Proof. Multiplying (14.7) by a^ω one obtains (15.8a) by virtue of (1.4). The equivalence of (15.8a) and (15.8b) follows from Theorem (7.5). The sufficiency of (15.8) will be proved together with the next theorem. In it we shall denote by $u_{\lambda\mu}$ an arbitrary symmetric tensor which transforms by (6.2) according to

$$(15.9) \quad u'_{\lambda\mu} = \epsilon u_{\lambda\mu}.$$

Theorem (15.4). *Put*

$$(15.10a) \quad u_{\nu\lambda\mu} \stackrel{\text{def}}{=} s_{\nu\lambda\beta} K_\mu{}^\beta + s_{\nu\mu\beta} K_\lambda{}^\beta + \overset{4}{d}_\nu u_{\lambda\mu}.$$

If (15.8) is satisfied then

$$(15.10b) \quad U^\nu_{\lambda\mu} \stackrel{\text{def}}{=} a^{\nu\alpha} u_{\alpha\lambda\mu}$$

is the most general solution of (14.7).

Proof. We have according to (15.10b)

$$(15.11) \quad U^\nu_{\lambda\mu} h_{\nu\omega} = a^{\nu\alpha} (a_{\nu\omega} - \overset{4}{d}_\nu \overset{4}{d}_\omega) u_{\alpha\lambda\mu} = u_{\omega\lambda\mu} - \overset{4}{d}_\omega u_{\alpha\lambda\mu} \overset{4}{d}_\alpha.$$

If we define $u_{\nu\lambda\mu}$ by (15.10a) then (15.8) is equivalent to

$$\overset{4}{d}_\alpha u_{\alpha\lambda\mu} = u_{\lambda\mu}$$

so that according to (15.10a) and (15.11)

$$U^\nu_{\lambda\mu} h_{\nu\omega} = u_{\omega\lambda\mu} - \overset{4}{d}_\omega u_{\lambda\mu} = s_{\omega\lambda\beta} K_\mu{}^\beta + s_{\omega\mu\beta} K_\lambda{}^\beta = S_{\omega\lambda}{}^\beta k_{\mu\beta} + S_{\omega\mu}{}^\beta k_{\lambda\beta}.$$

If we put

$$\overset{0}{U}^\nu_{\lambda\mu} = 2a^{\nu\alpha} S_{\alpha(\lambda}{}^\beta k_{\mu)\beta}$$

then $\overset{0}{U}^\nu_{\lambda\mu}$ is also a solution of (14.7) and the most general solution $U^\nu_{\lambda\mu}$ of (14.7) must be of the form

$$(15.12) \quad U^\nu_{\lambda\mu} = \overset{0}{U}^\nu_{\lambda\mu} + \overset{4}{d}_\alpha u_{\alpha\lambda\mu}$$

where $u_{\lambda\mu}$ is an arbitrary symmetric tensor. An easy inspection shows that (15.12) and (15.10b) are equivalent.

In the next lemma we shall combine the two arbitrary tensors introduced earlier, namely the tensor $b_{\lambda\mu}$ involved in (13.3) and the tensor $u_{\lambda\mu}$ used in (15.10a).

Lemma (15.1). *Put*

$$(15.13) \quad \begin{aligned} (a) \quad & W_{\mu\lambda} \stackrel{\text{def}}{=} u_{\mu\lambda} + b_{\mu\lambda} (= W_{\lambda\mu}), \\ (b) \quad & L_{\omega\mu\lambda} \stackrel{\text{def}}{=} K_{\omega\mu\lambda} - 4\overset{0}{a}_{[\omega}^3 W_{\mu]\lambda}, \\ (c) \quad & L'_{\omega\mu\lambda} \stackrel{\text{def}}{=} K'_{\omega\mu\lambda} - 4\overset{0}{a}'_{[\omega}^3 W'_{\mu]\lambda}. \end{aligned}$$

Then

$$(15.14) \quad \begin{aligned} (a) \quad & L'_{\omega\mu\lambda} = L_{\omega\mu\lambda} + 4\epsilon\overset{3}{a}_{[\omega} t_{\mu]\lambda}, \\ (b) \quad & L'_{[\omega\mu\lambda]} = L_{[\omega\mu\lambda]}, \\ (c) \quad & L_{[\omega\mu\lambda]} = \overset{0}{K}_{[\omega\mu\lambda]}. \end{aligned}$$

Proof. According to (13.1) and (15.9) we obtain

$$(15.15) \quad W_{\mu\lambda}' = \epsilon W_{\mu\lambda}.$$

Then (15.14a) follows from (15.7b). The equations (15.14b,c) follow from (15.13b,c) and from the symmetry of $W_{\mu\lambda}$.

The tensor $W_{\mu\lambda}$ is completely arbitrary. We shall restrict its arbitrariness by an agreement which is *not* imposed by our problem.

Agreement (15.1). *Throughout this paper we shall assume that*

$$(15.16) \quad W_{xx} \stackrel{\text{def}}{=} a_{\underset{x}{x}}^{\lambda} a_{\underset{x}{x}}^{\mu} W_{\lambda\mu} = 0.$$

Note. The requirement (15.16) is not a substantial restriction of generality. We may choose, for instance, the tensor $u_{\lambda\mu}$ ($b_{\lambda\mu}$) completely arbitrarily and then the choice of the remaining tensor $b_{\lambda\mu}$ ($u_{\lambda\mu}$) is restricted only by (15.16). The geometrical meaning of the free choice of $u_{\lambda\mu}$ ($b_{\lambda\mu}$) is that the choice of $u_{r\lambda\mu}$ ($\Lambda_{\lambda}^r{}_{\mu}$) is not restricted while $\Lambda_{\lambda}^r{}_{\mu}$ ($u_{r\lambda\mu}$) is restricted by the condition

$$\Lambda_{\underset{x}{x}}^4 = -u_{xx} \quad (u_{4xx} = -b_{xx})$$

where of course u_{xx} (b_{xx}) is completely arbitrary.

16. Conditions for $s_{\omega\mu\lambda}$.

Theorem (16.1). *The tensor $s_{\omega\mu\lambda}$ satisfies the following set of conditions:*

$$(16.1) \quad \begin{aligned} E_{\omega\mu\lambda} \stackrel{\text{def}}{=} s_{\omega\mu\lambda} + (s_{\alpha\lambda\beta} K_{\mu}^{\beta} + s_{\alpha\mu\beta} K_{\lambda}^{\beta}) K_{\omega}^{\alpha} \\ - (s_{\alpha\lambda\beta} K_{\omega}^{\beta} + s_{\alpha\omega\beta} K_{\lambda}^{\beta}) K_{\mu}^{\alpha} - \frac{1}{2} L_{\omega\mu\lambda} - s_{\omega\mu 4} \overset{4}{a}_{\lambda} = 0 \\ (s_{\omega\mu 4} \stackrel{\text{def}}{=} s_{\omega\mu\lambda} \overset{4}{a}_{\lambda}). \end{aligned}$$

Proof. One finds easily

$$S_{\omega\mu}{}^\alpha \overset{4}{a}_\alpha \overset{4}{a}_\lambda = s_{\omega\mu\lambda} \overset{4}{a}_\lambda$$

so that

$$(16.2) \quad S_{\omega\mu}{}^\alpha h_{\alpha\lambda} = S_{\omega\mu}{}^\alpha (a_{\alpha\lambda} - \overset{4}{a}_\alpha \overset{4}{a}_\lambda) = s_{\omega\mu\lambda} - s_{\omega\mu\lambda} \overset{4}{a}_\lambda.$$

On the other hand, we obtain, by virtue of (15.10) and (2.3),

$$(16.3) \quad 2U^\alpha{}_{\lambda[\mu} k_{\omega]\alpha} = 2u_{\alpha\lambda[\mu} K_{\omega]}{}^\alpha = (s_{\alpha\lambda\beta} K_\mu{}^\beta + s_{\alpha\mu\beta} K_\lambda{}^\beta) K_\omega{}^\alpha \\ - (s_{\alpha\lambda\beta} K_\omega{}^\beta + s_{\alpha\omega\beta} K_\lambda{}^\beta) K_\mu{}^\alpha + 2\overset{3}{a}_{[\omega} u_{\mu]\lambda}.$$

Substituting from (16.2), (16.3), (15.13b) and (15.7a) into (14.11) one obtains (16.1).

Besides (16.1) we have some other conditions for $s_{\omega\mu\lambda}$. These conditions depend on (16.1) and some of them will be very important in subsequent considerations. It is advisable to express these conditions in F -components

$$s_{ijk} = a_i{}^\omega a_j{}^\mu a_k{}^\lambda s_{\omega\mu\lambda}.$$

Theorem (16.2). *We have*

$$(16.4) \quad s_{j[x^*x]} = (\partial_j \lambda) / 2g,$$

where

$$\partial_j \stackrel{\text{def}}{=} a_i{}^\lambda \partial_\lambda$$

and g is defined by (5.4c).

Proof. We shall use (5.5a), i.e.

$$(16.5) \quad k_{xi} = \lambda h_{xi}$$

and denote by $\Gamma_i{}^\iota{}_k$ the F -components of $\Gamma_\lambda{}^\nu{}_\mu$:

$$(16.6) \quad \Gamma_i{}^\iota{}_k \stackrel{\text{def}}{=} -a_i{}^\lambda a^\mu D_\mu \overset{\iota}{a}_\lambda.$$

Then

$$D_i k_{xx^*} = \partial_i k_{xx^*} - \Gamma_x{}^x{}_i k_{xx^*} - \Gamma_{x^*}{}^{x^*}{}_i k_{xx^*} = \partial_i \lambda - \lambda \Gamma_x{}^x{}_i h_{xx^*} - \lambda \Gamma_{x^*}{}^{x^*}{}_i h_{xx^*} \\ = \partial_i \lambda + \lambda D_i h_{xx^*}$$

and, by virtue of (4.3), (14.4) and (16.5),

$$(16.7a) \quad \partial_i \lambda = D_i k_{xx^*} - \lambda D_i h_{xx^*} = 2[S_{j[x^*}{}^\iota g_{x]}{}^\iota - \lambda S_{j(x^*}{}^\iota g_{x)}{}^\iota] \\ = 2(s_{j[x^*x]} - \lambda s_{j(x^*x)}) + s_{jx^*x} \lambda - s_{jxx^*} \lambda - \lambda (s_{jxx^*} \lambda + s_{jxx^*} \lambda) \\ = 2s_{j[x^*x]}(1 - \lambda^2).$$

This equation is equivalent to (16.4).

Note. Letting $j = x$ in (16.4) we obtain

$$(16.7b) \quad s_{xx^*x} = (\partial_x \lambda)/g.$$

Later we shall see that (16.7b) is a special case of a *tensorial formula* for s_{ijk} .

We shall turn now to the equations (15.8) whose importance is stressed in Theorem (15.3).

Theorem (16.3). *The equations (15.8a) are equivalent to*

$$(16.8) \quad \begin{aligned} (a) \quad & s_{4\{12\}} = 0, \\ (b) \quad & s_{4xx}\lambda = 0, \\ (c) \quad & s_{43x} = 0, \\ (d) \quad & s_{4x3} = 0, \\ (e) \quad & s_{4x4} + \lambda s_{43x} = 0. \end{aligned}$$

The equation (16.8a) is equivalent to

$$(16.9) \quad \partial_4 \lambda = 0.$$

Proof. The equations (15.8a) are equivalent to (15.8b). The latter may be rewritten in F -components:

$$(16.10) \quad s_{4pj}K_q^j + s_{4qj}K_p^j = 0.$$

Substituting into (16.10) the values

$$\begin{aligned} p = q = x; \quad p = q = 3; \quad p = 4, q = 3; \\ p = 4, q = x; \quad p = 3, q = x \end{aligned}$$

and recalling (7.16) we obtain (16.8b–e). The values $p = x, q = x^*$ lead to

$$(16.11) \quad \lambda s_{4\{x^*x\}} = 0.$$

For the first class ($\lambda \neq 0$) this equation is equivalent to (16.8a); for both classes this latter equation is equivalent to (16.9), as (16.4) shows for $j = 4$. This equation shows also that (16.8a) must hold also for the second class ($\lambda = 0$).

Note I. According to (16.9), we have in every preferred y -system

$$(16.11) \quad \lambda = \lambda(y^I, y^{II}, y^{III}).$$

Note II. We do not know the behavior of $s_{\omega\mu\lambda}$ with regard to the change of frame (6.2). On the other hand, it is reasonable to insist on the solvability of (14.7) in *every* frame. This requirement is formulated in the following

Agreement (16.1). *Throughout this paper we shall assume that (16.8) hold and are reproductive.*

Note. According to this agreement we have besides (16.8)

$$(16.8)' \quad \begin{aligned} & \text{(a) } s'_{4'[1'2']} = 0, \quad \text{(b) } s'_{4'x'x'} \lambda'_{x'} = 0, \quad \text{(c) } s'_{4'3'x'} = 0, \\ & \text{(d) } s'_{4'x'3'} = 0, \quad \text{(e) } s'_{4'x'4'} + \lambda'_{x'} s'_{4'3'x'} = 0 \end{aligned}$$

for

$$\lambda'_{1'} \stackrel{\text{def}}{=} \lambda_1, \quad \lambda'_{2'} \stackrel{\text{def}}{=} \lambda_2.$$

The consequences of Agreement (16.1) will be investigated in the next chapter.

CHAPTER IV. CONDITIONS IMPOSED ON $s_{\omega\mu\lambda}$ BY THE PROBLEM

17. Auxiliary Formulæ. In this chapter we shall investigate in detail the conditions (16.8) in conjunction with Agreement (16.1). We shall find necessary and sufficient conditions for (16.8) to be satisfied and reproductive. *Each of these conditions will be assumed to hold without further agreement.*

In order to facilitate the investigation, we shall display in this section some auxiliary formulæ we shall need.

Lemma (17.1). *We have*

$$(17.1) \quad \begin{aligned} & \text{(a) } \nabla_i^0 k_{xx^*} = \partial_i \lambda_x, \\ & \text{(b) } \nabla_i^0 k_{xx^*} = -w_i^0 \lambda_x, \\ & \text{(c) } \nabla_i^0 k_{xx} = -\delta_x^3 (\Lambda_x^3 \lambda_x + \Lambda_x^4 \delta_x^u) + \delta_x^4 \Lambda_x^3 \quad (\text{sum for } u \neq x). \end{aligned}$$

Proof. The formulæ (17.1) are based on

$$\nabla_i^0 k_{ih} = \partial_i k_{ih} - \Lambda_i^l k_{lh} - \Lambda_h^l k_{il}.$$

Substituting into it from (5.3b) and using (12.4) one obtains easily (17.1).

Other useful formulæ are displayed in the following

Lemma (17.2). *We have*

$$(17.2) \quad \begin{aligned} & \text{(a) } K_{ijh} = \nabla_i^0 k_{hj} + \nabla_j^0 k_{ih} + \nabla_h^0 k_{ij}, \\ & \text{(b) } L_{ijh} = K_{ijh} - 4\delta_{[i}^3 W_{j]h} \end{aligned}$$

so that in particular

$$(17.2c) \quad L_{uvh} = K_{uvh}, \quad u, v \neq 3,$$

$$(17.2d) \quad L_{ixx} = \overset{0}{K}_{ixx} ,$$

$$(17.2e) \quad L_{[ijk]} = \overset{0}{K}_{[ijk]} .$$

Proof. (17.2a) ((17.2b)) is obtained from (15.6a) ((15.13b)) in the F -frame. (17.2c,e) are obvious. (17.2d) follows from (17.2b) and (15.16).

Another relationship needed later is stated in the following

Lemma (17.3). *We have*

$$(17.3) \quad \begin{aligned} -3\overset{0}{K}_{[xx^*x]} &= \delta_x^3 \{ -2\overset{0}{\Lambda}_{(x \ x^*)}^3 \lambda + 2\overset{0}{\Lambda}_{[x^* \ x]}^4 + \partial_3 \lambda \} + \delta_x^4 (2\overset{0}{\Lambda}_{[x \ x^*]}^3 + \partial_4 \lambda), \\ -3\overset{0}{K}_{[3 \ 4 \ x]} &= -2\overset{0}{w}_x + \overset{0}{\Lambda}_x^3{}^3 + \overset{0}{\Lambda}_x^3{}^4 \lambda. \end{aligned}$$

The proof follows from (17.2) and (17.1).

In subsequent sections we shall apply these formulæ to (16.1) which we shall consider in the F -frame:

$$(17.4) \quad \begin{aligned} E_{pqr} \stackrel{\text{def}}{=} s_{pqr} - s_{pa4} \delta_r^4 - \frac{1}{2} L_{pqr} + (s_{ai} K_r^i + s_{irj} K_a^j) K_p^i \\ - (s_{ipj} K_r^j + s_{irj} K_p^j) K_a^i = 0. \end{aligned}$$

18. Condition (16.8a).

Theorem (18.1). *Assume that $g_{\lambda\mu}$ is of the first class. A necessary and sufficient condition for (16.8a) to hold is that the coefficients involved in (12.6) satisfy the condition*

$$(18.1) \quad c_{xy} = c_{yx} , \quad \text{i.e.} \quad \Lambda_{[xy]}^3 = 0.$$

This condition is frame invariant and reproductive. If it is satisfied then the following equations hold and are reproductive:

$$(18.2) \quad \begin{aligned} (a) \quad s_{4[xx^*]} &= 0, \\ (b) \quad L_{xx^*4} &= 0. \end{aligned}$$

Proof. We have from (17.4)

$$(18.3a) \quad E_{xx^*4} \equiv 2\lambda^2 s_{4[xx^*]} - \frac{1}{2} L_{xx^*4} = 0.$$

On the other hand, the equations (17.2c), (17.1) and (16.4) for $j = 4$ yield

$$(18.4) \quad L_{xx^*4} = \overset{0}{K}_{xx^*4} = \partial_4 \lambda + 2\overset{0}{\Lambda}_{[x^* \ x]}^3 = 2g s_{4[xx^*]} + 2\overset{0}{\Lambda}_{[x^* \ x]}^3 .$$

Therefore (18.3a) reduces to

$$(18.3b) \quad (\lambda^2 + 1) s_{4[xx^*]} = \overset{0}{\Lambda}_{[x^* \ x]}^3 .$$

Assume now that (18.1) is satisfied. Then

$$(18.5a) \quad \overset{0}{\Lambda}_{[x^* x]}^3 = 0$$

so that (18.3b) reduces to

$$(18.5b) \quad (\lambda^2 + 1)s_{4[xx^*]} = 0.$$

If $\lambda^2 + 1 \neq 0$, then (18.5) and (18.4) yield (18.2). If $\lambda^2 + 1 = 0$ then $\partial_4 \lambda = 0$, and Theorem (16.3) (second part) yields (18.2a). This equation, together with (18.5a), leads again to (18.2b).

Assume now that (18.2a) holds. Then (18.3b) yields (18.5a) and this latter condition leads to (18.1) by virtue of (12.6). Taking into account the formula

$$(18.6) \quad \overset{0}{\nabla}_\mu^3 a_\lambda' = \epsilon \overset{0}{\nabla}_\mu^3 a_\lambda = \epsilon \overset{0}{\nabla}_\mu^3 a_\lambda,$$

we see from (6.2) that

$$(18.7) \quad \begin{aligned} c_{x'y'}' &= \epsilon A_x^x A_y^y c_{xy}, \\ c_{x'y'} &= A_x^x A_y^y c_{xy} \end{aligned}$$

and these equations prove the frame invariance and reproductivity of (18.1).

On the other hand, we obtain, according to (15.14a) and (6.2),

$$(18.8) \quad L'_{x'x^{**}4'} = L_{x'x^{**}4'} = a_{x'}^\omega a_{x^{**}4'}^\mu a^\lambda L_{\omega\mu\lambda} = \epsilon L_{xx^{**}4}.$$

The equations (18.8) show that (18.2b) is frame invariant and reproductive. The frame invariance and reproductivity of (18.2a) follow then from (18.2b), (18.4), (18.5a) and (18.8).

Theorem (18.2). Assume that $g_{\lambda\mu}$ is of the second class. Then the equations (18.1) and (18.2) hold and have the same properties as stated in Theorem (18.1).

Proof. For the second class ($\lambda = 0$) the equation (18.2a) is a consequence of (16.4) for $j = 4$. The frame invariance and reproductivity of (18.2a) follow also from (16.4). The equation (18.3a) reduces to (18.2b). The remaining part of the proof follows the pattern of the corresponding part of the proof of Theorem (18.1).

19. Condition (16.8b). The following theorems will be proved simultaneously for the first class.

Theorem (19.1). Assume that $g_{\lambda\mu}$ is of the first class. Then (16.8b) is equivalent to

$$(19.1) \quad s_{4xx} = 0.$$

Theorem (19.2). A necessary and sufficient condition for (19.1) to hold is that the coefficients c_{xy} involved in (18.1) satisfy the equation

$$(19.2) \quad c_{xx} = 0, \quad \text{i.e.} \quad \Lambda_{xx}^3 = 0.$$

This condition is frame invariant and reproductive. If it is satisfied, then the following equations hold, are frame invariant and are reproductive:

$$(19.3) \quad \begin{aligned} (a) \quad s_{xxx} &= 0, \\ (b) \quad L_{xxx} &= 0. \end{aligned}$$

Theorem (19.3). The condition (19.2) is equivalent to the following requirement: The curves defined by

$$(19.4) \quad a_x^v = \frac{dx^v}{dp}$$

are a_x^v -paths (with respect to Λ_{λ}^v):

$$(19.5) \quad a_x^\mu \nabla_\mu^0 a_x^v = f a_{xx}^v.$$

This condition is frame invariant.

Proof. Theorem (19.1) is obvious. From (17.4) we have

$$(19.6) \quad \begin{aligned} (a) \quad E_{4xx} &= s_{4xx}(1 + \lambda^2) - 2\lambda s_{3xx} - \frac{1}{2}L_{4xx} = 0, \\ (b) \quad E_{3xx} &= s_{3xx}(1 + \lambda^2) + 2\lambda s_{4xx} - \frac{1}{2}L_{3xx} = 0. \end{aligned}$$

Let us first assume

$$\Delta \stackrel{\text{def}}{=} (1 + \lambda^2)^2 + 4\lambda^2 \neq 0.$$

Then (19.6) yields

$$(19.7) \quad \begin{aligned} (a) \quad 2\Delta s_{4xx} &= L_{4xx}(1 + \lambda^2) + 2\lambda L_{3xx}, \\ (b) \quad 2\Delta s_{3xx} &= L_{3xx}(1 + \lambda^2) - 2\lambda L_{4xx}. \end{aligned}$$

On the other hand, we have from (17.2d) and (17.1)

$$(19.8) \quad \begin{aligned} (a) \quad L_{4xx} &= 2\Lambda_x^3, \\ (b) \quad L_{3xx} &= -2\Lambda_x^3 \lambda, \end{aligned}$$

so that (19.7a) reduces to

$$(19.9a) \quad \Delta s_{4xx} = g \Lambda_x^3.$$

Hence a necessary and sufficient condition for (19.1) to hold is

$$(19.9b) \quad \Lambda_x^3 = 0.$$

If this condition is satisfied, then we obtain, by virtue of (19.8) and (19.7),

the equations (19.3), while (19.9b) is obviously equivalent to (19.2). Assume now $\Delta = 0$, *i.e.*

$$1 + \lambda^2 = 2\epsilon i \lambda_x$$

where $\epsilon = \pm 1$. The equations (19.6a,b) reduce to

$$(19.6) \quad \begin{aligned} (c) \quad & 2\lambda(s_{4xx}i\epsilon - s_{3xx}) = \frac{1}{2}L_{4xx}, \\ (d) \quad & 2\lambda(s_{4xx}i\epsilon - s_{3xx})i\epsilon = -\frac{1}{2}L_{3xx}, \end{aligned}$$

so that, according to (19.8a,b),

$$(19.8c) \quad i\epsilon\Lambda_x^3 = \lambda\Lambda_x^3.$$

Using the obvious inequality $\lambda \neq i\epsilon$ we obtain (19.9b) from (19.8c). Therefore (19.3b) holds and, moreover, according to (19.6c,d),

$$(19.9c) \quad s_{4xx}i\epsilon - s_{3xx} = 0.$$

The vectors a_1^v, a_2^v being complex conjugate we obtain from the real tensor $S_{\omega\mu\lambda}$

$$\bar{s}_{4xx} = s_{4x^*x^*}, \quad \bar{s}_{3xx} = s_{3x^*x^*}.$$

On the other hand, the equation (19.9c) holds for $x = 1, 2$. Therefore we have also

$$s_{4x^*x^*}i\epsilon - s_{3x^*x^*} = 0,$$

or according to previous equations

$$(19.9d) \quad \bar{s}_{4xx}i\epsilon - \bar{s}_{3xx} = 0.$$

The equations (19.9c,d) lead at once to (19.3a).

The frame invariance and reproductivity of (19.2) may be proved in the same manner as we proved these properties for (18.1). On the other hand, we obtain from (15.14a)

$$(19.10) \quad L'_{4'x'x'} = L_{4'x'x'} = (A_{x'}^x)^2 L_{4xx}.$$

This equation proves the reproductivity of (19.3b) for $z = 4$. The reproductivity of the remaining equations (19.3) will be proved later. According to (19.2) and (18.1) we may put

$$(19.11) \quad \nabla_\mu^0 a_\lambda^3 = 2\alpha a_{(\mu}^1 a_{\lambda)}^2 + \sum c_{zx} a_\mu^z a_\lambda^x,$$

so that

$$(19.2a) \quad 0 = a_x^\mu a^\lambda \nabla_\mu^0 a_\lambda^3 = -a_x^\mu (\nabla_\mu^0 a^\lambda) a_\lambda^3.$$

On the other hand, the equation (12.4f)* yields

$$(19.12b) \quad 0 = a^\mu a^\lambda \overset{0}{\underset{x}{\nabla}}_\mu \overset{4}{\underset{x}{a}}_\lambda = -a^\mu (\overset{0}{\underset{x}{\nabla}}_\mu a^\lambda) \overset{4}{\underset{x}{a}}_\lambda.$$

Therefore

$$(19.12c) \quad a^\mu \overset{0}{\underset{x}{\nabla}}_\mu a^\nu = f a^\nu + \overset{xx}{f} a^\nu.$$

In order to derive some consequences from (19.12c) let us observe that, by virtue of (10.6),

$$0 = \overset{0}{\underset{x}{\nabla}}_\mu a^\lambda a^\nu h_{\lambda\nu} = 2(\overset{0}{\underset{x}{\nabla}}_\mu a^\lambda) a^\nu h_{\lambda\nu} = 2(\overset{0}{\underset{x}{\nabla}}_\mu a^\lambda) \overset{x}{a}^{\nu*}.$$

Hence if we multiply (19.12c) by $\overset{x}{a}^{\nu*}$, we obtain $f = 0$, so that (19.12c) reduces to (19.5) (with $f \overset{def}{=} f$). The frame invariance of (19.5) is obvious from the equivalence of (19.5) and (19.2) and from the frame invariance of the latter equations.**

In the next theorem we shall deal with the tensor $t_{\lambda\mu}$ defined by (11.3).

Theorem (19.4). *We have for the first class†*

$$(19.13a) \quad \overset{0}{\underset{x}{\nabla}}_\mu \overset{3}{\underset{x}{a}}_\lambda = 2\alpha \overset{1}{\underset{x}{a}}_{(\mu} \overset{2}{\underset{x}{a}}_{\lambda)} + \sum c_{xx} \overset{2}{\underset{x}{a}}_\mu \overset{x}{a}^{\lambda*}, \ddagger$$

$$(19.13b) \quad t_{\lambda\mu} = \overset{3}{\underset{x}{a}}_{(\lambda} \partial_{\mu)} \rho + \rho \{ 2\alpha \overset{1}{\underset{x}{a}}_{(\mu} \overset{2}{\underset{x}{a}}_{\lambda)} + \sum c_{xx} \overset{3}{\underset{x}{a}}_{(\mu} \overset{x}{a}^{\lambda*} \}$$

so that, in particular,

$$(19.13) \quad \begin{aligned} (c) \quad t_{xx} &= a^\lambda \overset{x}{a}^\mu t_{\lambda\mu} = 0, \\ (d) \quad t_{44} &= a^\lambda \overset{x}{a}^\mu t_{\lambda\mu} = 0, \\ (e) \quad 2t_{43} &= a^\lambda \overset{x}{a}^\mu t_{\lambda\mu} = a^\mu \partial_\mu \rho. \end{aligned}$$

Proof. The equation (19.13a) is identical with (19.11). The equations (19.13b) follow from (19.13a) and (11.3). The equations (19.13c, d, e) are obvious.

Note I. We are now able to prove the reproductivity of all equations (19.3). Using (15.14) we obtain, by virtue of (6.2) and (19.13c),

$$(19.14) \quad L'_{3'x'x'} = L_{3'x'x'} = a^\omega a^\mu a^\lambda L_{\omega\mu\lambda} = (A_{x'}^z)^2 [\epsilon L_{3xx} + \rho L_{4xx}].$$

The frame invariance and the reproductivity of (19.3b) follow from (19.10)

*This equation has nothing to do with the conditions (16.8) and is always satisfied.

**Incidentally, the frame invariance of (19.5) results as a byproduct of the next theorem.

†Cf. the note at the end of this section.

‡We see from (18.6) that α transforms according to $\alpha' = \epsilon\alpha$.

and (19.14). These properties, together with (19.7) and (19.3b), show that (19.3a) are frame invariant and reproductive for $\Delta \neq 0$. If $\Delta = 0$ then the reproductivity of (19.3a) follows at once from the manner in which these equations were obtained via (19.9c,d).

Note II. Using (19.13b) and (11.4) we find

$$a^\mu \nabla'_\mu a^\nu = A_{x'}^x a^\mu [(\nabla'_\mu a^\nu) A_{x'}^x + a^\nu \partial_\mu A_{x'}^x] = (A_{x'}^x)^2 [a^\mu \nabla_\mu a^\nu + a^\nu \partial_x \ln A_{x'}^x].$$

Therefore, by virtue of (19.5),

$$a^\mu \nabla'_\mu a^\nu = a^\nu f'$$

where

$$f' = A_{x'}^x f + \partial_x A_{x'}^x.$$

This proves the frame invariance stated in Theorem (19.3).

Theorem (19.5). Assume that $g_{\lambda\mu}$ is of the second class. Then (16.8b) is satisfied and, moreover,

$$(19.15) \quad 2s_{xxx} = L_{xxx} = 2 \delta_x^4 \Lambda_x^3.$$

Proof. For the second class we have $\lambda = 0$ and therefore (16.8b) is satisfied. Substituting $\lambda = 0$ ($\Delta = 1$) into (19.7) and using (19.8a,b) we obtain at once (19.15).

Note. The equation (19.15) splits into two sets

$$(19.16) \quad \begin{aligned} (a) \quad 2s_{4xx} &= L_{4xx}(2\Lambda_x^3), \\ (b) \quad 2s_{3xx} &= L_{3xx} = 0. \end{aligned}$$

In the next theorem we shall deal with (19.16b).

Theorem (19.6). Assume that $g_{\lambda\mu}$ is of second class. A necessary and sufficient condition that

$$(19.17a) \quad L_{3xx} = 0$$

be reproductive is that the requirement of Theorem (19.2) be satisfied. If so, then Theorems (19.2)—(19.4) hold.

Proof. We see from (19.14) that a necessary and sufficient condition for (19.17a) to be reproductive is

$$(19.7b) \quad L_{4xx} = 0.$$

The remaining part of the proof follows the pattern of the proofs of Theorems (19.2)—(19.4) and is based on (19.16) and (19.17).

Note. This theorem shows that (19.13) hold also for the second class.

20. Conditions (16.8c). The conditions (16.8c) split into two

$$(20.1) \quad (a) \quad s_{434} = 0, \quad (b) \quad s_{433} = 0.$$

We shall consider these conditions in the following two theorems.

Theorem (20.1). *A necessary and sufficient condition for (20.1) to be satisfied is*

$$(20.2a) \quad w_z^0 = -W_{4z}.$$

This condition is equivalent to

$$(20.2b) \quad L_{43z} = 0.$$

The conditions (20.2) for $z = 4$ are reproductive.

Theorem (20.2). *The condition (20.1a) is reproductive. A necessary and sufficient condition for (20.1b) and (20.2) for $z = 3$ to be reproductive is that the change of frame be restricted by*

$$(20.3) \quad a^\lambda \partial_{\lambda\rho} = 0.$$

Proof. The equation (17.4) yields

$$(20.4) \quad E_{43z} = s_{43z} - s_{434}\delta_z^4 - s_{3z4} + s_{4z3} - \frac{1}{2}L_{43z} = 0.$$

On the other hand, we obtain from (10.2a), (17.2) and (17.1)

$$(20.5) \quad L_{43z} = 2(W_{4z} + \delta_z^3 w_3^0).$$

Therefore we may write for (20.4)

$$(20.6) \quad (a) \quad s_{434} = \frac{1}{2}L_{434} = W_{44}, \\ (b) \quad 2s_{433} = \frac{1}{2}L_{433} = W_{43} + w_3^0.$$

Using (15.14) we obtain, by virtue of (19.13c, d),

$$(20.7) \quad (a) \quad L'_{4'3'4'} = L_{4'3'4'} = a^\omega a^\mu a^\lambda L_{\omega\mu\lambda} = \epsilon L_{434}, \\ (b) \quad L'_{4'3'3'} = a^\omega a^\mu a^\lambda (L_{\omega\mu\lambda} + 4\epsilon a_{[\omega} t_{\mu]\lambda}) \\ = \epsilon(L_{433} - 2\epsilon t_{43}) + \rho(L_{434} - 2t_{44}) \\ = \epsilon L_{433} + \rho L_{434} - a^\lambda \partial_{\lambda\rho}.$$

The proof follows from (10.2b), (20.6) and (20.7).

Note. It is clear that Theorems (20.1) and (20.2) hold for both classes. In contradistinction to all previous conditions *the requirement (20.3) restricts the generality of the frame change.* This fact suggests the following

Definition (20.1). The change (6.2) of frame will be termed an admissible change if ρ is eventually restricted by the requirement

$$(20.8) \quad a^\lambda \partial_\lambda \rho = 0 \quad \text{for some } u$$

and otherwise is completely arbitrary. Throughout this paper we shall deal only with admissible changes (6.2) of frame.

21. Auxiliary Results. In order to investigate properly the conditions (16.8d,e) we shall need some preliminary results which we establish in this section.

Lemma (21.1). We have

$$(21.1) \quad \begin{aligned} (a) \quad 2t_{x3} &= a^\lambda \partial_\lambda \rho - \overset{0}{\Lambda}_{x \ x}^3 \rho, \\ (b) \quad 2t_{x4} &= -\overset{0}{\Lambda}_{x \ 4}^3 \rho. \end{aligned}$$

Proof. Using (19.13b) we obtain, by virtue of (12.4d),

$$2t_{x3} = 2a^\mu a^\lambda [\overset{3}{a}_{(\lambda} \partial_{\mu)} \rho + \rho \overset{0}{\nabla}_{(\mu} \overset{3}{a}_{\lambda)}] = a^\mu \partial_\mu \rho - 2\overset{0}{\Lambda}_{(3 \ x)}^3 \rho = a^\mu \partial_\mu \rho - \overset{0}{\Lambda}_{x \ 3}^3 \rho.$$

The equation (21.1b) may be proved in a similar way.

Lemma (21.2). We have

$$(21.2) \quad \begin{aligned} (a) \quad L_{x44} &= -2\overset{0}{\Lambda}_{x \ 4}^3, \\ (b) \quad L_{x33} &= 2W_{x3} - 2\overset{0}{\Lambda}_{3 \ x}^4 + 2\overset{0}{\Lambda}_{x \ 3}^3 \lambda, \\ (c) \quad L_{x34} &= 2W_{x4} - \overset{0}{\Lambda}_{x \ 3}^3 + \overset{0}{\Lambda}_{x \ 4}^3 \lambda, \\ (d) \quad L_{34x} &= -2W_{x4} - \overset{0}{\Lambda}_{x \ 3}^3 - \overset{0}{\Lambda}_{x \ 4}^3 \lambda, \\ (e) \quad L_{4x3} &= 2w_x + \overset{0}{\Lambda}_{x \ 3}^3 - \overset{0}{\Lambda}_{x \ 4}^3 \lambda. \end{aligned}$$

Proof. Using (15.13b) we get

$$L_{34x} = \overset{0}{K}_{34x} - 2W_{4x} = -2W_{4x} + \overset{0}{\nabla}_3 k_{x4} + \overset{0}{\nabla}_4 k_{3x} + \overset{0}{\nabla}_x k_{34}.$$

Substituting into this equation from (17.1) one obtains (21.2d). The remaining equations (21.2) may be obtained in a similar way.

Lemma (21.3). The L 's involved in (21.2) satisfy the following rule of transformation:

$$\begin{aligned}
(21.3) \quad (a) \quad L'_{x'4'4'} &= A_x^x L_{x44} , \\
(b) \quad L'_{x'3'3'} &= A_x^x [L_{x33} + 2\epsilon \rho L_{x(34)} + \rho^2 L_{x44} \\
&\quad - \epsilon a^\lambda \partial_{\lambda\rho} + \Lambda_x^3 \rho \epsilon + \rho^2 \Lambda_x^3 \epsilon], \\
(c) \quad L'_{x'3'4'} &= A_x^x [L_{x34} + \rho \epsilon L_{x44} + \epsilon \rho \Lambda_x^3 \epsilon], \\
(d) \quad L'_{3'4'x'} &= A_x^x [L_{34x} - \Lambda_x^3 \rho \epsilon], \\
(e) \quad L'_{4'x'3'} &= A_x^x [L_{4x3} - \epsilon \rho L_{x44}].
\end{aligned}$$

Proof. The equation (15.14a) yields, by virtue of (21.1b),

$$\begin{aligned}
L'_{3'4'x'} &= a^\omega_{3'} a^\mu_{4'} a^\lambda_{x'} (L_{\omega\mu\lambda} + 4\epsilon a^3_{[\omega} t_{\mu]\lambda}) = A_x^x \epsilon (\epsilon a^\omega_{3'} + a^\omega_{4'} \rho) a^\mu_{4'} a^\lambda_{x'} [L_{\omega\mu\lambda} + 4\epsilon a^3_{[\omega} t_{\mu]\lambda}] \\
&= A_x^x \epsilon [\epsilon L_{34x} + 2t_{4x}] = A_x^x [L_{34x} - \Lambda_x^3 \rho \epsilon].
\end{aligned}$$

The remaining equations (21.3) may be proved in a similar manner.

These results will be applied in the next section to conditions (16.8d,e).

22. Conditions (16.8d,e) (First Class). In this section we shall confine ourselves to the first class only; moreover, we assume that λ ($\neq 0$) defined by (4.3) is not constant.*

Theorem (22.1). *Necessary and sufficient conditions that (16.8d,e) hold and be reproductive are the following:*

(1) *The vector $\overset{3}{a}_\lambda$ is a gradient and*

$$(22.1) \quad \overset{0}{\nabla}_\mu \overset{3}{a}_\lambda = 2\alpha \overset{1}{a}_{(\mu} \overset{2}{a}_{\lambda)} = \alpha (h_{\mu\lambda} - \overset{3}{a}_\mu \overset{3}{a}_\lambda).$$

(2) *The components $\overset{0}{w}_x$ of $\overset{0}{w}_\mu$ vanish:*

$$(22.2a) \quad \overset{0}{w}_x = 0.$$

(3) *The components W_{xz} of $W_{\mu\lambda}$ satisfy the condition*

$$(22.2b) \quad W_{xz} = -\delta_z^3 a^\lambda a^\mu \partial_{[\mu} \overset{4}{a}_{\lambda]}.$$

(4) *The admissible change of frame (6.2) is restricted by*

$$(22.3) \quad a^\lambda \partial_{\lambda\rho} = 0.$$

These conditions are frame invariant. If they are satisfied, then the following equations hold and are reproductive:

*As a matter of fact, we exclude only two numerical values for λ . See the footnote on p. 852.

$$(22.4) \quad \begin{aligned} (a) \quad & L_{x33} = L_{x44} = L_{34x} = L_{4x3} = L_{x34} = 0, \\ (b) \quad & s_{x33} = s_{x44} = s_{34x} = s_{4x3} = s_{x34} = 0. \end{aligned}$$

Proof. The equations

$$(22.5a) \quad E_{x33}, \quad E_{x44}, \quad E_{34x}, \quad E_{4x3}, \quad E_{x34}$$

are linear in the unknowns

$$(22.5b) \quad s_{x33}, \quad s_{x44}, \quad s_{34x}, \quad s_{4x3}, \quad s_{x34}$$

and contain only these unknowns, while their right hand terms are in turn

$$(22.5c) \quad \frac{1}{2}L_{x33}, \quad \frac{1}{2}L_{x44}, \quad \frac{1}{2}L_{34x}, \quad \frac{1}{2}L_{4x3}, \quad \frac{1}{2}L_{x34}.$$

The determinant Δ of the system (22.5a) is

$$(22.6) \quad \Delta = g \neq 0.$$

Hence the system (22.5a) admits only one solution for the unknowns (22.5b). Substituting the computed values of s_{4x3} , s_{4x4} and s_{43x} into (16.8d,e) we obtain

$$(22.7) \quad \begin{aligned} (a) \quad & 3\lambda^2 L_{[x34]} + \lambda(L_{x33} - 2L_{x44}) - L_{x34} = 0, \\ (b) \quad & \lambda^3(-18L_{x34} - 14L_{x43} - 6L_{34x}) \\ & + \lambda^2(4L_{x33} - L_{x44}) + 2\lambda L_{4[3x]} + L_{x44} - L_{x33} = 0. \end{aligned}$$

These equations have to be reproductive, *i.e.* we must have also

$$(22.7)' \quad \begin{aligned} (a) \quad & 3\lambda^2 L'_{[x'3'4']} + \lambda(L'_{x'3'3'} - 2L'_{x'4'4'}) - L'_{x'3'4'} = 0, \\ (b) \quad & \lambda^3(-18L'_{x'3'4'} - 14L'_{x'4'3'} - 6L'_{3'4'x'}) \\ & + \lambda^2(4L'_{x'3'3'} - L'_{x'4'4'}) + 2\lambda L'_{4'[3'x']} + L'_{x'4'4'} - L'_{x'3'3'} = 0. \end{aligned}$$

From (21.3) we see that the transformation of the L 's does not involve any λ 's and that all transformation formulæ contain the same factor $A_x{}^\pi$. Hence one obtains (22.7)' from (22.7) if and only if the coefficient of $(\lambda)^\pi$ ($\pi = 0, \dots, 3$) in (22.7)' transforms into the product of $A_x{}^\pi$ and of the corresponding coefficient of $(\lambda)^\pi$ in (22.7). In particular, we must have

$$(22.8) \quad \begin{aligned} (a) \quad & 2L'_{4'[3'x']} = 2A_x{}^\pi L_{4[3x]}, \\ (b) \quad & L'_{x'3'3'} - L'_{x'4'4'} = A_x{}^\pi [L_{x33} - L_{x44}]. \end{aligned}$$

Using (21.3d,e) we see, by virtue of (21.2), that the requirement (22.8a) is equivalent to

$$(22.9) \quad L_{x44} = -2\Lambda_x{}^3 = 0.$$

Therefore the requirement (22.8b) reduces, according to (21.3b) and (21.2), to

$$(22.10) \quad \epsilon\rho[2(W_{x^4} - \overset{0}{w}_x) - \overset{0}{\Lambda}_{x^3}^3] - \underset{x}{a}^\lambda \partial_\lambda \rho = 0.$$

Because we are dealing with admissible changes of frame only the equation (22.10) splits into two:

$$(22.11) \quad \begin{aligned} (a) \quad & 2(W_{x^4} - \overset{0}{w}_x) = \overset{0}{\Lambda}_{x^3}^3, \\ (b) \quad & \underset{x}{a}^\lambda \partial_\lambda \rho = 0. \end{aligned}$$

The λ 's involved in (22.7) satisfy the equation

$$(22.12) \quad \mathfrak{G}(\lambda) = 0$$

where $\mathfrak{G}(\lambda)$ is defined by (3.8). The equations (22.7) and (22.12) must yield the same λ 's. Therefore we must have

$$(22.13) \quad L_{x^{33}} = 0.$$

Substituting into (22.7) from (17.2e), (17.3), (21.2), (22.9), (22.11a) and (22.13) and recalling that $\lambda \neq 0$ we obtain

$$(22.14) \quad \begin{aligned} \lambda^2(\dot{W}_{x^4} - 2\overset{0}{w}_x) + \overset{0}{w}_x &= 0, \\ \lambda^2(52W_{x^4} - 48\overset{0}{w}_x) + 2(W_{x^4} - \overset{0}{w}_x) &= 0. \end{aligned}$$

These two homogeneous linear equations in W_{x^4} and $\overset{0}{w}_x$ have a non-trivial solution if and only if their determinant Δ is equal to zero. The roots λ of $\Delta = 0$ are constant and according to our assumption this possibility is excluded.* Therefore we have only a trivial solution of (22.14), *i.e.*

$$(22.15) \quad \overset{0}{w}_x = W_{x^4} = 0,$$

so that (22.11a) reduces to

$$(22.16) \quad \overset{0}{\Lambda}_{x^3}^3 = 0.$$

Moreover, the equation (22.13) reduces to

$$(22.17) \quad W_{x^3} = \overset{0}{\Lambda}_{x^3}^4 = -\underset{x^3}{a}^\mu \overset{0}{a}^\lambda \nabla_\mu^4 a_\lambda = -\underset{x^3}{a}^\mu \overset{0}{a}^\lambda \partial_{[\mu} a_{\lambda]}$$

and the L 's involved in (22.5c) vanish. Hence the equations (22.5a) are homogeneous so that according to (22.6) they admit only one solution, namely (22.4b).

So far we have found the necessary and sufficient conditions in the form

*Two of these roots λ are real and therefore *a priori* excluded according to (4.3) and (3.14b). The remaining two roots are of the form $\pm im$, where m is a real number. The values $\pm im$ are the only ones excluded by our assumption.

(22.2), (22.3) and (22.9), (22.16). The latter two equations in conjunction with (19.13a) yield (22.1). Hence $\overset{3}{a}_\lambda$ is a gradient.

The frame invariance of (22.1)—(22.4) is obvious (cf. (10.3)). The reproducibility of (22.4) is evident from the manner in which these equations have been obtained.

Theorem (22.2). *Among the components $L_{[ijk]}$ (i, j, k distinct) only one is different from zero:*

$$(22.18) \quad \begin{aligned} (a) \quad & -3L_{[xx^*3]} = -2\overset{0}{\Lambda}_{(x \ x^*)} \overset{3}{\lambda}_x + 2\overset{0}{\Lambda}_{[x^* \ x]} + \partial_3 \overset{3}{\lambda}_x, \\ (b) \quad & L_{[xx^*4]} = 0, \\ (c) \quad & L_{[34x]} = 0. \end{aligned}$$

The proof follows easily from (16.9), (17.2e) and (17.3), (22.16) and (22.9).

This theorem will be needed in the next chapter.

Note. We know that the curves defined by

$$(22.19) \quad \overset{u}{a}' = \frac{dx^u}{dp}, \quad u = 1, 2, 4,$$

are $\overset{u}{a}'$ -paths with respect to $\overset{0}{\Lambda}_\lambda{}^u$ (cf. Theorems (19.3) and (10.3)).

In the following theorem we shall state some properties of the vectors $\overset{u}{a}'$.

Theorem (22.3). *The following statements hold:*

- (1) *The vector fields $\overset{u}{a}'$, $u = 1, 2, 4$ are X_3 -forming.*
- (2) *Every one of these subspaces X_3 is enveloped by the vectors of the field $\overset{3}{a}_\mu$.*
- (3) *The vector field $\overset{u}{a}'$ consists of parallel vectors (with respect to $\overset{0}{\Lambda}_\lambda{}^u$) along each of the X_3 's.*
- (4) *The function ρ is constant along each of the X_3 's.*
- (5) *The vector field $\overset{u}{a}'$ consists of parallel vectors (with respect to $\overset{0}{\Lambda}_\lambda{}^u$) along the $\overset{u}{a}'$ -paths.*

Proof. The vector $\overset{3}{a}_\mu$ is a gradient:

$$(22.20) \quad \partial_{[\mu} \overset{3}{a}_{\lambda]} = 0,$$

and therefore

$$(22.21) \quad \overset{u}{a}^\mu \overset{v}{a}^\lambda \partial_{[\mu} \overset{3}{a}_{\lambda]} = 0, \quad u, v = 1, 2, 4.$$

Hence the vector fields $\overset{u}{a}'$, $\overset{v}{a}'$, $\overset{w}{a}'$ are X_3 -forming, and every one of these X_3 's is enveloped by the field $\overset{3}{a}_\mu$. If v^λ is an arbitrary linear combination of $\overset{1}{a}'$, $\overset{2}{a}'$, $\overset{4}{a}'$ then we have, according to (10.5a), (10.2b) and (22.2a),

$$v^\mu \overset{0}{\nabla}_\mu a^\nu = (v^\mu w_\mu^0) a^\nu = 0.$$

This equation proves the statement (3). The statement (4) follows from

$$v^\lambda \partial_\lambda \rho = 0 \quad (v^\lambda \overset{3}{a}_\lambda = 0)$$

which is a consequence of (20.3) and (22.3).

Using (10.2a) we obtain, according to (22.2a),

$$(22.22a) \quad 0 = \overset{0}{w}_x = -a^\mu a^\lambda \overset{0}{\nabla}_\mu \overset{4}{a}_\lambda = a^\mu a^\lambda \overset{0}{\nabla}_\lambda \overset{4}{a}_\mu = -(\overset{0}{a}^\lambda \overset{0}{\nabla}_\lambda a^\nu) a_\nu.$$

On the other hand, we have, by virtue of (22.9),

$$(22.22b) \quad 0 = \overset{0}{\Lambda}_x^3 = -a^\mu a^\lambda \overset{0}{\nabla}_\mu \overset{3}{a}_\lambda = (\overset{0}{a}^\mu \overset{0}{\nabla}_\mu a^\nu) \overset{3}{a}_\nu.$$

Consequently

$$(22.23) \quad \overset{0}{a}^\mu \overset{0}{\nabla}_\mu a^\nu = F a^\nu + F a^\nu_{x^* x^*}.$$

The proof of statement (5) follows then the same pattern as the proof of Theorem (19.3).

Theorem (22.4). *We have*

$$(22.24a) \quad t_{\lambda\mu} = \rho \{ \alpha h_{\lambda\mu} + \overset{3}{\sigma} \overset{3}{a}_\lambda \overset{3}{a}_\mu \}$$

where

$$(22.24b) \quad \overset{3}{\sigma} \stackrel{\text{def}}{=} \overset{3}{a}^\lambda (\partial_\lambda \ln \rho) - \alpha.$$

Proof. Using (22.1) we obtain

$$(22.25a) \quad t_{\lambda\mu} = \overset{0}{\nabla}_{(\mu} \overset{3}{a}_{\lambda)} \rho = \overset{3}{a}_{(\lambda} \partial_{\mu)} \rho + \alpha \rho (h_{\lambda\mu} - \overset{3}{a}_\lambda \overset{3}{a}_\mu).$$

On the other hand, according to (22.3) and (20.3),

$$(22.25b) \quad \partial_\mu \rho = \delta_\mu^\lambda \partial_\lambda \rho = \overset{3}{a}_\mu a^\lambda \partial_\lambda \rho = \overset{3}{a}_\mu a^\lambda \partial_\lambda \rho.$$

The equations (22.24) follow from (22.25).

23. Conditions (16.8d,e) (Second Class). In this section we shall investigate the conditions (16.8d,e) for the second class ($\lambda = 0$), *i.e.* the equations

$$(23.1) \quad (a) \quad s_{4x3} = 0, \quad (b) \quad s_{4x4} = 0.$$

These conditions are weaker than the corresponding conditions for the first class ($\lambda \neq 0$). Therefore we may prescribe here some conditions which in the case of the first class are imposed by the problem itself.

Agreement (23.1). In dealing with the second class we choose the component W_{x4} of the arbitrary tensor $W_{\lambda\mu}$ according to (22.2b) for $z = 4$, i.e. we assume

$$(23.1) \quad (c) \quad W_{x4} = 0.$$

Note. Our problem reduces to investigating the equations (23.1a,b,c). We shall see that (23.1c) is compatible with (23.1a,b). Moreover, it is obvious that (23.1c) is not only reproductive but also frame invariant. This follows easily from (15.15) and (6.2).

Theorem (23.1). Theorem (22.1) without (22.2b) for $z = 4$ holds also for the second class.

Proof. We start again with the equations (22.5a) for the unknowns (22.5b). The determinant of this system is given by (22.6) which for the second class reduces to

$$(23.2) \quad \Delta = 1.$$

Substituting the solutions s_{x43} , s_{4x4} into (23.1a,b) one obtains the equations (22.7a,b) with $\lambda = 0$, i.e.

$$(23.3) \quad L_{x34} = 0,$$

$$(23.4) \quad L_{x44} - L_{x33} = 0.$$

According to (21.3c) the equation (23.3) is reproductive if and only if

$$L_{x44} + \overset{0}{\Lambda}_{x4}^3 = 0,$$

or, by virtue of (21.2a), if

$$(23.5) \quad -2\overset{0}{\Lambda}_{x4}^3 = L_{x44} = 0.$$

The equation (23.3) is, according to (21.2c) and (23.1c), equivalent to

$$(26.6) \quad 2W_{x4} = \overset{0}{\Lambda}_{x3}^3 = 0$$

while the equation (23.4) reduces to

$$(23.7) \quad L_{x33} = 0.$$

This equation reduces, according to (21.2b), to (22.2b) for $z = 3$. The requirement of reproductivity of (23.7) yields again (22.11a,b). In particular, (22.11a) is reduced by means of (23.6) to (22.2a). The remaining part of the proof follows the pattern of the corresponding part of the proof of Theorem (22.1).

Theorem (23.2). Theorem (22.2) holds also for the second class and, in particular, (22.18a) reduces to

$$(23.8) \quad -3L_{[xx^*3]} = 2\overset{0}{\Lambda}_{[x^*x]}^4.$$

The proof is obvious.

Theorem (23.3). *Theorems (22.3) and (22.4) hold also for the second class.*

The proof is obvious.

This concludes the investigation of the condition (16.8). In the next chapter we shall use these results for solving (14.3) for $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$.

CHAPTER V. THE CONNECTION $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$.

24. Prerequisites. So far we used 17 of the equations (17.4), namely

$$(24.1) \quad \begin{array}{ccccc} E_{xx^*4} & , & E_{xxx} & , & E_{43x} & , & E_{xzt} & , & E_{zz^*x} & , & * \\ 1 & & 4 & & 2 & & 8 & & 2 & & \end{array}$$

and found for both classes that

$$(24.2) \quad \begin{array}{ccccc} L_{xx^*4} & = & L_{xxx} & = & L_{43x} & = & L_{xzt} & = & L_{zz^*x} & = & 0 \\ 1 & & 4 & & 2 & & 8 & & 2 & & \end{array}$$

and

$$(24.3) \quad \begin{array}{ccccccc} s_{4\{xx^*\}} & = & s_{xxx} & = & s_{43x} & = & s_{xzt} & = & s_{zz^*x} & = & 0. \\ 1 & & 4 & & 2 & & 8 & & 2 & & \end{array}$$

As far as the tensor $W_{\mu\lambda}$ is concerned we found the conditions**

$$(24.4) \quad W_{xx} = 0, \quad W_{4x} = -\overset{0}{w}_x, \quad W_{xx} = -\delta_{\underset{3}{x}}^3 a^{\lambda} a^{\mu} \partial_{[\mu} \overset{3}{a}_{\lambda]}$$

while W_{xx^*} is still arbitrary. The (admissible) change of frame is restricted by

$$(24.5) \quad \underset{u}{a}^{\lambda} \partial_{\lambda} \rho = 0, \quad u = 1, 2, 4.$$

The equations (17.4) reduce, by virtue of (15.8), to

$$(24.6) \quad \begin{aligned} E_{pq^r} &= s_{pq^r} - s_{pq^4} \delta_r^4 - \frac{1}{2} L_{pq^r} + (s_{uqi} K_r^i + s_{uri} K_q^i) K_p^u \\ &\quad - (s_{upj} K_r^j + s_{urj} K_p^j) K_q^u = 0 \quad (u = 1, 2, 3). \end{aligned}$$

Theorem (24.1). *The equations*

$$(24.7) \quad \begin{array}{cccc} E_{xx^*3} & , & E_{x^*3x} & , & E_{x^*4x} & , & E_{xx^*x} \\ 1 & & 2 & & 2 & & 2 \end{array}$$

*The figure under each individual E shows the number of independent equations represented by the corresponding symbol. The figures under the s 's and L 's have a similar significance.

**See Agreements (15.1) and (23.1).

yield

$$\begin{aligned}
 (24.8) \quad & \text{(a)} \quad 2gs_{xx^*3} = L_{xx^*3} - 3\lambda^2 L_{[xx^*3]} , \\
 & \text{(b)} \quad 2gs_{3xx^*} = L_{3xx^*} + 2\lambda^2 s_{xx^*4} , \\
 & \text{(c)} \quad 2gs_{x4x^*} = L_{x4x^*} - 3\lambda L_{[x3x^*]} , \\
 & \text{(d)} \quad 2gs_{xx^*x} = L_{xx^*x}
 \end{aligned}$$

for the first class and

$$(24.9) \quad 2s_{uvw} = L_{uvw} \quad (uvw = xx^*3, 3xx^*, x4x^*, xx^*x)$$

for the second class.

The proof of (24.8) follows by straightforward computation. The equations (24.9) follow from (24.8) for $\lambda = 0$.

Theorem (24.2). *The first equation (24.3) is a consequence of (24.8c).*

Proof. Using (18.2b) we obtain from (22.18b)* and (24.8c)

$$4gs_{4[xx^*]} = 2L_{4[xx^*]} = -L_{xx^*4} = 0.$$

Theorem (24.3). *The component s_{xx^*4} is not defined by the system (24.6).*

Proof. The equations (24.1) and (24.7) exhaust all 24 equations (24.6). From (24.3) and (24.8) we see that they do not define s_{xx^*4} .

Note. According to this theorem our problem does not yield the component s_{xx^*4} and therefore we have a free choice for it. We shall see that the following agreement defines s_{xx^*4} in a unique way. This agreement is obviously suggested by (10.1).

Agreement (24.1). *Throughout this paper we shall assume that $D_{[\mu}\hat{a}_{\lambda]}$ is independent of the connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ defined by (14.8).*

Theorem (24.4). *Agreement (24.1) is equivalent to either one of the following conditions:*

$$\begin{aligned}
 (24.10) \quad & \text{(a)} \quad D_{[\mu}\hat{a}_{\lambda]} = \partial_{[\mu}\hat{a}_{\lambda]} , \\
 & \text{(b)} \quad s_{xx^*4} = 0.
 \end{aligned}$$

Proof. The connection $\Lambda_{\lambda}{}^{\nu}{}_{\mu}$ and the tensor $U^{\nu}{}_{\lambda\mu}$ are symmetric in subscripts. Therefore we obtain from (14.8)

$$(24.11) \quad D_{[\mu}\hat{a}_{\lambda]} = \overset{0}{\nabla}_{[\mu}\hat{a}_{\lambda]} - S_{\lambda\mu}{}^{\nu}{}_{\nu}\hat{a}_{\nu} = \partial_{[\mu}\hat{a}_{\lambda]} - S_{\lambda\mu}{}^{\nu}{}_{\nu}\hat{a}_{\nu} .$$

On the other hand, we find from (14.14a) and (7.4)

*See also Theorem (23.2).

$$S_{\lambda\mu}{}^{\rho} \overset{4}{a}_{\rho} = S_{\lambda\mu}{}^{\alpha} a_{\alpha\rho}{}^{\rho} = s_{\lambda\mu\rho} \overset{4}{a}_{\rho} = s_{\lambda\mu 4} ,$$

so that (24.11) may be written

$$(24.12) \quad D_{[\mu} \overset{4}{a}_{\lambda]} = \partial_{[\mu} \overset{4}{a}_{\lambda]} - s_{\lambda\mu 4} .$$

Hence a necessary and sufficient condition for Agreement (24.7) to be fulfilled is $s_{\lambda\mu 4} = 0$, or in F -components

$$(24.13) \quad s_{i4} = 0.$$

According to (24.3) the condition (24.13) reduces to (24.10b). The remaining part of the proof is obvious.

25. The Tensors $S_{\lambda\mu}{}^{\rho}$ and $U^{\rho}{}_{\lambda\mu}$. We shall first prove the following

Lemma (25.1). We have

$$(25.1) \quad 3K_i{}^{\rho} K_i{}^{\alpha} L_{[\rho q k]} = 6L_{[123]} \{ \lambda^2 \delta_k^3{}^{21}{}_{[ij]} + \lambda \delta_k^2{}^{41}{}_{[ij]} + \lambda \delta_k^1{}^{24}{}_{[ij]} \} .$$

The proof follows at once from (7.16) and Theorems (22.2) and (23.2).

Theorem (25.1). The solution $s_{\omega\mu\lambda}$ of (15.8b) and (16.1) is given by

$$(25.2) \quad 2gs_{\omega\mu\lambda} = L_{\omega\mu\lambda} + 3K_{[\omega}{}^{\alpha} K_{\mu]}{}^{\beta} L_{[\alpha\beta\lambda]}$$

for the first class and by

$$(25.3) \quad 2s_{\omega\mu\lambda} = L_{\omega\mu\lambda}$$

for the second class.

Proof. Let us start with the second class first. We see from (24.2), (24.3), (24.9) and (24.10b) that the F -components s_{ijk} are given by

$$(25.4) \quad 2s_{ijk} = L_{ijk} .$$

This is the nonholonomic form of (25.3). In the case of the first class we see from (24.2) that, by virtue of (25.1), the equations (24.3), (24.8) and (24.10b) may be condensed as follows:

$$(25.5) \quad 2gs_{ijk} = L_{ijk} + 3K_i{}^{\rho} K_i{}^{\alpha} L_{[\rho q k]} .$$

This is the nonholonomic form of (25.2).

Theorem (25.2). The tensor $S_{\lambda\mu}{}^{\rho}$ involved in (14.14a) is given by

$$(25.6a) \quad S_{\lambda\mu}{}^{\rho} = \frac{1}{2g} [L_{\lambda\mu\alpha} + K_{\lambda}{}^{\beta} K_{\mu}{}^{\gamma} L_{[\beta\gamma\alpha]}] a^{\alpha\rho}$$

for the first class and by

$$(25.6b) \quad S_{\lambda\mu}{}^{\rho} = \frac{1}{2} L_{\lambda\mu\alpha} a^{\alpha\rho}$$

for the second class.

The proof is obvious.

Theorem (25.3). The tensors $u_{\nu\lambda\mu}$ and $U'_{\lambda\mu}$ defined by (15.10) are given by

$$(25.7) \quad \begin{aligned} (a) \quad u_{\nu\lambda\mu} &= -\frac{1}{g} K_{(\mu}{}^{\beta} L_{\lambda)\nu\beta} + \frac{1}{4} a_{\nu} u_{\lambda\mu},^* \\ (b) \quad U'_{\lambda\mu} &= -\frac{1}{g} K_{(\mu}{}^{\beta} L_{\lambda)\alpha\beta} a^{\alpha\nu} + \frac{1}{4} a^{\nu} u_{\lambda\mu}. \end{aligned}$$

Proof. We have by virtue of (25.2)

$$(25.8) \quad 2gK_{(\mu}{}^{\beta} s_{\lambda)\nu\beta} = K_{(\mu}{}^{\beta} L_{\lambda)\nu\beta} + 3K_{(\mu}{}^{\beta} K_{\lambda)}{}^{\gamma} K_{\nu}{}^{\delta} L_{[\gamma\delta\beta]} = K_{(\mu}{}^{\beta} L_{\lambda)\nu\beta}.$$

The proof follows easily from (15.10) and (25.8).

26. The Transformation Rule for $S_{\lambda\mu}{}^{\nu}$. In the next lemma we shall use the tensors $L'_{\omega\mu\lambda}$ and $K'_{\lambda}{}^{\nu}$ defined by (15.13c) and (7.15).

Lemma (26.1). We have

$$(26.1) \quad 3L'_{[\alpha\beta\nu]} K'_{\omega}{}^{\alpha} K_{\mu}{}^{\beta} = 3L_{[\alpha\beta\nu]} K_{\omega}{}^{\alpha} K_{\mu}{}^{\beta} + 2\epsilon\rho\lambda L h_{\nu[\mu} a_{\omega]}{}^3 \quad (L \stackrel{\text{def}}{=} 3L_{[123]}).$$

Proof. According to (7.15) we have

$$(26.2) \quad K_{\omega}{}^{\alpha} K_{\mu}{}^{\beta} = K_{\omega}{}^{\alpha} K_{\mu}{}^{\beta} + \epsilon\rho(K_{\omega}{}^{\alpha} a_{\mu}{}^3 a^{\beta} + K_{\mu}{}^{\beta} a_{\omega}{}^3 a^{\alpha}) + p_{\omega\mu}{}^{\alpha\beta}$$

where $p_{\omega\mu}{}^{\alpha\beta}$ is a tensor containing the product $a^{\alpha} a^{\beta}$. In order to evaluate the left hand term in (26.1) we recall the fact that according to Theorems (22.2) and (23.2) the only one of the expressions $L_{[ijk]}$ which survives is $L_{[123]}$. Therefore we have, according to (26.2) and (15.14a).

$$(26.3) \quad 3L'_{[\alpha\beta\nu]} K_{\omega}{}^{\alpha} K_{\mu}{}^{\beta} = 3L_{[\alpha\beta\nu]} K_{\omega}{}^{\alpha} K_{\mu}{}^{\beta} + 3\epsilon\rho L_{[\alpha\beta\nu]} (K_{\omega}{}^{\alpha} a_{\mu}{}^3 a^{\beta} + K_{\mu}{}^{\beta} a_{\omega}{}^3 a^{\alpha}).$$

Using again the above mentioned fact about $L_{[ijk]}$ in conjunction with (7.14), we obtain easily (26.1) from (26.3).

In the subsequent theorems we shall investigate the tensors

$$(26.4) \quad \begin{aligned} (a) \quad s'_{\omega\mu\lambda} &= \frac{1}{2g} \{L'_{\omega\mu\lambda} + 3L'_{[\alpha\beta\nu]} K_{\omega}{}^{\alpha} K_{\mu}{}^{\beta}\}, \\ (b) \quad S'_{\omega\mu}{}^{\nu} &= s'_{\omega\mu\lambda} a^{\lambda\nu}. \end{aligned}$$

Theorem (26.1). We have

$$(26.5a) \quad s'_{\omega\mu\lambda} = s_{\omega\mu\lambda} + \epsilon\rho a_{[\omega} h_{\mu]\lambda} F \quad (F \stackrel{\text{def}}{=} (2\alpha + \lambda L)/g),^{**}$$

$$(26.5b) \quad S'_{\omega\mu}{}^{\nu} = S_{\omega\mu}{}^{\nu} + \epsilon\rho F a_{[\omega} \delta_{\mu]}{}^{\nu} + \epsilon\rho a_{\omega}{}^{\nu} S_{\omega\mu}$$

*These formulæ hold for both classes. For the second class we have of course $g = 1$.

**An easy inspection shows that $L' = \epsilon L$, so that $F' = \epsilon F$.

where

$$(26.6) \quad S_{\omega\mu} = -F\overset{3}{a}_{[\omega}\overset{4}{a}_{\mu]} + s_{\omega\mu\lambda}\overset{\lambda}{a}_{\mathfrak{z}}.$$

Proof. Using (15.14a), (22.24a), (26.4a) and (26.1) we get

$$\begin{aligned} 2gs'_{\omega\mu\lambda} &= L_{\omega\mu\lambda} + 4\epsilon\alpha\rho\overset{3}{a}_{[\omega}h_{\mu]\lambda} + 3L_{[\alpha\beta\lambda]}K_{\omega}{}^{\alpha}K_{\mu}{}^{\beta} + 2\epsilon\rho\lambda La_{[\omega}h_{\mu]\lambda} \\ &= 2gs_{\omega\mu\lambda} + 2\epsilon\rho(2\alpha + \lambda L)\overset{3}{a}_{[\omega}h_{\mu]\lambda} \end{aligned}$$

and this equation is equivalent to (26.5a). On the other hand, the equations (24.3), (24.8) and (24.10b) show that

$$(26.7a) \quad s_{\omega\mu\lambda}\overset{\lambda}{a}_{\mathfrak{z}} = 0.$$

Moreover, according to (1.4)

$$(26.7) \quad (b) \quad \overset{3}{a}_{[\omega}h_{\mu]\lambda}(2\epsilon\rho\overset{(\lambda}{a}_{\mathfrak{z}}^{\nu)} + \overset{\rho^2}{\mathfrak{z}}\overset{\lambda}{a}_{\mathfrak{z}}^{\nu}) = 0,$$

$$(c) \quad \overset{3}{a}_{[\omega}h_{\mu]\lambda}\overset{\lambda}{a}^{\nu} = \overset{3}{a}_{[\omega}(\delta_{\mu]}^{\nu} - \overset{4}{a}_{\mu]}\overset{\nu}{a}^{\lambda}).$$

Using (26.7) we obtain, by virtue of (26.5a) and (7.7a),

$$(26.8) \quad s'_{\omega\mu\lambda}\overset{\lambda}{a}^{\nu} = s_{\omega\mu\lambda}\overset{\lambda}{a}^{\nu} + \epsilon\rho F\overset{3}{a}_{[\omega}(\delta_{\mu]}^{\nu} - \overset{4}{a}_{\mu]}\overset{\nu}{a}^{\lambda}) + \epsilon\rho\overset{\nu}{a}_{\mathfrak{z}}s_{\omega\mu\lambda}\overset{\lambda}{a}_{\mathfrak{z}}.$$

This equation is equivalent to (26.5b).

In the next theorem we shall use the vectors

$$(26.9) \quad (a) \quad S_{\omega} \stackrel{\text{def}}{=} S_{\omega\alpha}{}^{\alpha}, \quad (b) \quad S'_{\omega} \stackrel{\text{def}}{=} S'_{\omega\alpha}{}^{\alpha}.$$

Theorem (26.2). *A necessary and sufficient condition for S_{ω} to be frame invariant:*

$$(26.10) \quad S'_{\omega} = S_{\omega},$$

is

$$(26.11) \quad F = 0.$$

If this condition is satisfied then $s_{\omega\mu\lambda}$ is frame invariant and (26.5b) reduces to

$$(26.12a) \quad S'_{\omega\mu}{}^{\nu} = S_{\omega\mu}{}^{\nu} + \epsilon q_{\omega\mu}\overset{\nu}{a}_{\mathfrak{z}}^{\lambda}$$

where

$$(26.12b) \quad q_{\omega\mu} \stackrel{\text{def}}{=} s_{\omega\mu\lambda}\overset{\lambda}{a}_{\mathfrak{z}}.$$

Proof. The equations (26.5b) and (26.9) lead to

$$(26.13) \quad S'_{\omega} = S_{\omega} + \frac{3}{2}\epsilon\rho F\overset{3}{a}_{\omega} + \epsilon\rho S_{\omega\mu}\overset{\mu}{a}_{\mathfrak{z}}.$$

On the other hand, we see from (24.3) that

$$(26.14) \quad s_{\omega\mu\lambda} a^\lambda a^\mu = 0,$$

so that, by virtue of (26.6),

$$S_{\omega\mu} a^\mu = -\frac{1}{2} F a_\omega^3$$

and (26.13) reduces to

$$(26.15) \quad S'_\omega = S_\omega + \epsilon \rho F a_\omega^3.$$

This equation proves the necessity and sufficiency of (26.11). The remaining part of the proof is obvious.

Agreement (26.1). *Throughout this chapter we shall assume the validity of (26.11).*

The necessity of this agreement for our problem will be shown in the next section.

27. The Condition $S_\mu = 0$. One of EINSTEIN's conditions, imposed on the connection $\Gamma_\lambda{}^\nu{}_\mu$ is (14.1b), i.e.

$$(27.1) \quad S_\mu = 0.$$

It has been shown* that this condition is equivalent to the first set of Maxwell's equations, which, of course, are frame invariant. Therefore we must assume that (27.1) is frame invariant and this necessitates Agreement (26.1).

In this section we shall find necessary and sufficient conditions for (27.1).

Theorem (27.1). *Necessary and sufficient conditions for (27.1) to be satisfied are the following:*

(1) *The scalar g is constant along each of the X_3 mentioned in Theorem (22.3), i.e.*

$$(27.2a) \quad v^\lambda \partial_\lambda g = 0$$

for

$$(27.2b) \quad v^\lambda a_\lambda^3 = 0.$$

(b) *The nonholonomic components W_{xy} of $W_{\lambda\mu}$ vanish:*

$$(27.3) \quad W_{xy} = 0.$$

(c) *The change of g along a^ν is given by*

$$(27.4) \quad a^\lambda \partial_\lambda g = -4 a^\mu a^\nu \nabla_\mu^0 (g a_\nu^3 + \lambda a_\nu^4).$$

*HLAVATÝ, *Geometry of Einstein's Unified Field Theory* (Noordhoff, Groningen, 1958).

Proof. The condition (27.1) is equivalent to

$$(27.5) \quad s_{\mu\lambda} a^{\lambda} = 0.$$

We know already that, according to Agreement (26.1), (27.5) is frame invariant. Using (7.1a) and (24.3) we may rewrite (27.5) as follows:

$$(27.6) \quad \begin{aligned} (a) \quad s_{xx^*x} &= 0, \\ (b) \quad s_{z(xx^*)} &= 0. \end{aligned}$$

The equation (27.6a) in conjunction with (16.7b) yields

$$(27.7) \quad a^\mu \partial_\mu \lambda = 0.$$

Recalling (16.9) and the relations $\lambda + \lambda = 0$ and $g = 1 - \lambda^2$, we obtain easily (27.2). Consider now (27.6b) for $z = 3$. Then we get, according to (24.8b), (24.10b), (15.6a) and (15.13b),

$$(27.8) \quad 0 = 4gs_{3(xx^*)} = 2L_{3(xx^*)} = 2\overset{0}{K}_{3(xx^*)} - 4W_{xx^*} = -4(\overset{0}{\nabla}_{(x}k_{x^*)3} + W_{xx^*}).$$

On the other hand, the equations (12.4c, f) and (22.1) yield

$$-4\overset{0}{\nabla}_{(x}k_{x^*)3} = -4\Lambda_{[x}^3{}_{x^*]}\lambda = 0,$$

so that (27.8) reduces to $W_{xx^*} = 0$. This condition together with (15.16) is condensed in (27.3). Consider now (27.6b) for $z = 4$. Then, according to (24.8c),

$$(27.9a) \quad 0 = 4gs_{4(xx^*)} = 2L_{4(xx^*)} - 6L_{[xx^*3]}\lambda.$$

A simple inspection based on (22.18a) and (17.2a,c) shows

$$(27.9b) \quad 0 = 2L_{4(xx^*)} - 6L_{[xx^*3]}\lambda = 4\overset{0}{\Lambda}_{(x}^3{}_{x^*)}g + 4\overset{0}{\Lambda}_{[x^*}^4{}_{x]}\lambda - a^\lambda \partial_\lambda g$$

or, according to (27.2) and (10.1),

$$(27.9c) \quad \begin{aligned} -a^\lambda \partial_\lambda g &= 4(g\overset{0}{\nabla}_\mu^3 a_\lambda + \lambda\overset{0}{\nabla}_\mu^4 a_\lambda)a^\mu a^\lambda = 4a^\mu a^\lambda \overset{0}{\nabla}_\mu(ga_\lambda^3 + \lambda a_\lambda^4) \\ &= 4a^\mu a^\lambda \overset{0}{\nabla}_\mu(ga_\lambda^3 + \lambda a_\lambda^4). \end{aligned}$$

The question arises whether the conditions (27.4) and (26.11) are compatible. The answer to this question is given in

Theorem (27.2). *The conditions (27.4) and (26.11) are equivalent.*

Proof. Using (22.18a) and (22.1) we obtain easily from (26.5a)

$$(27.10) \quad gF = 2\alpha + \lambda L = -\frac{1}{2}[4\overset{0}{\Delta}_{(12)}^3 g + 4\overset{0}{\Delta}_{(2\ 1)}^4 \lambda - \partial_3 g].$$

The equivalence of (27.4) and (26.11) follows from (27.10) and (27.9b).

Note. We see from (24.4) and (27.3) that the tensor $W_{\lambda\mu} - W_{33} \overset{3}{a}_\lambda \overset{3}{a}_\mu$ is uniquely determined. The nonholonomic component W_{33} is not defined by our problem. The introduction of $W_{\lambda\mu}$ enabled us to find $S_{\lambda\mu}^{\prime\prime}$ and $U_{\lambda\mu}^{\prime\prime}$ *without imposing unnecessary restrictions on $g_{\lambda\mu}$* . If we assign an arbitrary fixed value to W_{33} (for example, $W_{33} = 0$) then $W_{\lambda\mu}$ is *completely* defined. Even in this case, according to (15.13a), one of the tensors $b_{\mu\lambda}$, $u_{\mu\lambda}$ is still *completely arbitrary*.

28. Continuation. In this section we shall deal with the transformation rule for the tensors $u_{\nu\lambda\mu}$ and $U_{\lambda\mu}^{\prime\prime}$ which are defined by (15.10) as well as by (25.7). In the next theorem we shall use the tensors $u'_{\nu\lambda\mu}$ and $U_{\lambda\mu}^{\prime\prime}$ defined by

$$(28.1) \quad \begin{aligned} (a) \quad u'_{\nu\lambda\mu} - \overset{4}{a}'_{\nu} u'_{\lambda\mu} &\stackrel{\text{def}}{=} 2S'_{\nu(\lambda} \overset{\beta}{k}'_{\mu)\beta} = -\frac{1}{g} K_{(\mu}{}^{\beta} L'_{\lambda)\nu\beta}, \\ (b) \quad U_{\lambda\mu}^{\prime\prime} - \overset{4}{a}'_{\lambda} u'_{\mu} &\stackrel{\text{def}}{=} 2a'^{\alpha\nu} S'_{\alpha(\lambda} \overset{\beta}{k}'_{\mu)\beta} = -\frac{1}{g} a'^{\alpha\nu} K_{(\mu}{}^{\beta} L'_{\lambda)\alpha\beta}. \end{aligned}$$

Theorem (28.1). *We have*

$$(28.2a) \quad u'_{\nu\lambda\mu} - \overset{4}{a}'_{\nu} u'_{\lambda\mu} = u_{\nu\lambda\mu} - \overset{4}{a}_{\nu} u_{\lambda\mu} + 2\epsilon\rho q_{\nu(\mu} \overset{3}{a}_{\lambda)}^3,$$

$$(28.2b) \quad U_{\lambda\mu}^{\prime\prime} = U_{\lambda\mu}^{\prime\prime} + 2\epsilon\rho[q_{\alpha(\mu} \overset{3}{a}_{\lambda)}^3 a^{\alpha\nu} + r_{\mu\lambda} \overset{4}{a}'^{\nu}]$$

where

$$(28.3) \quad r_{\mu\lambda} \stackrel{\text{def}}{=} a_{\mu}^{\alpha} S_{\alpha(\mu} \overset{\beta}{k}_{\lambda)\beta}^{\beta}.*$$

Proof. Using (26.12a) we get, by virtue of (15.10a) and (2.3),

$$\begin{aligned} 2S'_{\nu(\lambda} \overset{\beta}{k}'_{\mu)\beta} &= 2S'_{\nu(\lambda} \overset{\beta}{k}_{\mu)\beta} = 2S_{\nu(\lambda} \overset{\beta}{k}_{\mu)\beta} + 2q_{\nu(\lambda} \overset{\beta}{a}_{\mu)}^{\beta} \epsilon\rho \\ &= u_{\nu\lambda\mu} - \overset{4}{a}_{\nu} u_{\lambda\mu} + 2q_{\nu(\lambda} \overset{3}{a}_{\mu)}^3 \epsilon\rho. \end{aligned}$$

This equation, in conjunction with (28.1a), leads to (28.2a). Multiplying this latter equation by $a'^{\nu\rho}$ we obtain, by virtue of (7.7a),

$$(28.4a) \quad \begin{aligned} U_{\lambda\mu}^{\prime\prime} - \overset{4}{a}'_{\lambda} u'_{\mu} &= U_{\lambda\mu}^{\prime\prime} - \overset{4}{a}_{\lambda} u_{\mu} + (u_{\alpha\lambda\mu} - \overset{4}{a}_{\alpha} u_{\lambda\mu})(2\epsilon\rho a_{\mu}^{\nu} \overset{3}{a}_{\lambda}^{\alpha} + \rho^2 \overset{4}{a}'_{\mu} \overset{4}{a}'_{\lambda} \overset{\alpha}{a}) \\ &\quad + 2\epsilon\rho q_{\alpha(\mu} \overset{3}{a}_{\lambda)}^3 a^{\nu\alpha} + 2\epsilon\rho q_{\alpha(\mu} \overset{3}{a}_{\lambda)}^3 (2\epsilon\rho a_{\mu}^{\nu} \overset{3}{a}_{\lambda}^{\alpha} + \rho^2 \overset{4}{a}'_{\mu} \overset{4}{a}'_{\lambda} \overset{\alpha}{a}). \end{aligned}$$

Recalling (15.8b), we find easily

* $r_{\lambda\mu}$ is proportional to $\overset{1}{a}_{(\lambda} \overset{2}{a}_{\mu)}$.

$$\begin{aligned}
 (28.4b) \quad (u_{\alpha\lambda\mu} - \overset{2}{a}_{\alpha}u_{\lambda\mu})(2\epsilon\rho a_{\underset{3}{\alpha}}^{\underset{4}{\nu}}a^{\alpha}_{\underset{4}{\nu}} + \rho^2 a_{\underset{4}{\alpha}}^{\nu}a^{\alpha}_{\underset{4}{\nu}}) &= 2S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}(2\epsilon\rho a_{\underset{3}{\alpha}}^{\underset{4}{\nu}}a^{\alpha}_{\underset{4}{\nu}} + \rho^2 a_{\underset{4}{\alpha}}^{\nu}a^{\alpha}_{\underset{4}{\nu}}) \\
 &= 2a_{\underset{4}{\alpha}}^{\nu}a^{\alpha}_{\underset{3}{\lambda}}S_{\alpha(\lambda}{}^{\beta}k_{\mu)\beta}\epsilon\rho = 2\epsilon\rho r_{\lambda\mu}a_{\underset{4}{\alpha}}^{\nu}.
 \end{aligned}$$

By the same token, according to (26.12b) and (24.3),

$$(28.4c) \quad q_{\alpha(\mu}a_{\lambda)}^3(2\epsilon\rho a_{\underset{3}{\alpha}}^{\underset{4}{\nu}}a^{\alpha}_{\underset{4}{\nu}} + \rho^2 a_{\underset{4}{\alpha}}^{\nu}a^{\alpha}_{\underset{4}{\nu}}) = 0.$$

On the other hand,

$$(28.4d) \quad a_{\underset{4}{\alpha}}^{\nu}u'_{\lambda\mu} = a_{\underset{4}{\alpha}}^{\nu}u_{\lambda\mu}.$$

Substituting from (28.4b–d) into (28.4a) one obtains (28.2b).

Theorem (28.2). *We have*

$$(28.5a) \quad S'_{\lambda\mu}{}^{\nu} + U''_{\lambda\mu}{}^{\nu} = S_{\lambda\mu}{}^{\nu} + U'_{\lambda\mu}{}^{\nu} + \epsilon\rho P_{\lambda\mu}{}^{\nu}$$

where

$$(28.5b) \quad P_{\lambda\mu}{}^{\nu} = a_{\underset{4}{\lambda}}^{\nu}(q_{\lambda\mu} + 2r_{\lambda\mu}) + 2q_{\alpha(\lambda}a_{\mu)}^3a^{\alpha\nu},$$

$$(28.5c) \quad P_{[\lambda\alpha]}{}^{\alpha} = 0.$$

The proof follows from (28.2b) and (26.12a,b).

We shall use these results in the next section.

29. The Connection $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$.

Theorem (29.1). *The system (14.1a,b) admits the following solution:*

$$\begin{aligned}
 (29.1a) \quad \Gamma_{\lambda}{}^{\nu}{}_{\mu} &= \overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu} + \frac{1}{2g} a^{\alpha}{}^{\nu}[L_{\lambda\mu\alpha} + 3K_{\lambda}{}^{\beta}K_{\mu}{}^{\gamma}L_{[\beta\gamma\alpha]}] \\
 &\quad - \frac{1}{g} K_{(\mu}{}^{\beta}L_{\lambda)\alpha\beta}a^{\alpha\nu} + W_{\lambda\mu}a_{\underset{4}{\alpha}}^{\nu}.
 \end{aligned}$$

Proof. Substituting into (14.8) from (13.3) we obtain

$$(29.2) \quad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = \overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu} + S_{\lambda\mu}{}^{\nu} + U'_{\lambda\mu}{}^{\nu} + a_{\underset{4}{\lambda}}^{\nu}b_{\mu}.$$

Replacing $S_{\lambda\mu}{}^{\nu}$ and $U'_{\lambda\mu}{}^{\nu}$ by the values taken from (25.6) and (25.7b) and using (15.13a) one obtains (29.1a).

Note. The equation (29.1a) reduces for the second class to

$$(29.1b) \quad \Gamma_{\lambda}{}^{\nu}{}_{\mu} = \overset{0}{\Lambda}_{\lambda}{}^{\nu}{}_{\mu} + \frac{1}{2g} L_{\lambda\mu\alpha}a^{\alpha\nu} - \frac{1}{g} K_{(\mu}{}^{\beta}L_{\lambda)\alpha\beta}a^{\alpha\nu} + W_{\lambda\mu}a_{\underset{4}{\alpha}}^{\nu} \quad (g = 1),$$

as is easily seen from (25.6b). However, we may regard (29.1a) as valid for

both classes with the proviso that for the second class

$$K_{\lambda}^{\beta} K_{\mu}^{\gamma} L_{[\beta\gamma\alpha]} = 0.$$

In the next theorem we shall use the relations (28.5) and (22.24).

Theorem (29.2). *The connection $\Gamma_{\lambda}^{\nu}{}_{\mu}$ given by (19.1) is not frame invariant. If we put*

$$\begin{aligned} \Gamma_{\lambda}^{\nu}{}_{\mu} &\stackrel{\text{def}}{=} \overset{0}{\Lambda}_{\lambda}^{\nu}{}_{\mu} + S_{\lambda\mu}^{\nu} + U''_{\lambda\mu} + a^{\nu} b_{\lambda\mu}' \\ (29.3) \quad &= \overset{0}{\Lambda}_{\lambda}^{\nu}{}_{\mu} + \frac{1}{2g} a'^{\nu\alpha} (L'_{\lambda\mu\alpha} + 3K_{\lambda}^{\beta} K_{\mu}^{\gamma} L_{[\beta\gamma\alpha]}) \\ &\quad - \frac{1}{g} K_{(\mu}^{\beta} L'_{\lambda)\alpha\beta} a'^{\alpha\nu} + W'_{\lambda\mu} a^{\nu} \end{aligned}$$

then

$$(29.4a) \quad \Gamma_{\lambda}^{\nu}{}_{\mu} = \Gamma_{\lambda}^{\nu}{}_{\mu} + \epsilon_{\rho} T_{\lambda\mu}^{\nu}$$

where the tensor

$$(29.4b) \quad T_{\lambda\mu}^{\nu} \stackrel{\text{def}}{=} P_{\lambda\mu}^{\nu} - a^{\nu} (\alpha h_{\lambda\mu} + \sigma a_{\lambda}^3 a_{\mu}^3)$$

satisfies the condition

$$(29.5a) \quad T_{\lambda\omega}^{\alpha} g_{\alpha\mu} + T_{\omega\mu}^{\alpha} g_{\lambda\alpha} = 0.$$

$$(29.5b) \quad T_{[\lambda\alpha]}^{\alpha} = 0.$$

Proof. The equations (29.4) follow easily from (28.5a), (11.4) and (22.24). From the manner in which we obtained the solution $\Gamma_{\lambda}^{\nu}{}_{\mu}$, it is clear that $\Gamma_{\lambda}^{\nu}{}_{\mu}$ given by (29.3) is also a solution of (14.1a,b). This fact, together with (28.5c) and Theorem (15.1), yields the proof of the remaining part of our theorem.

Note I. According to Theorems (29.2) and (15.1) the dependence on the frame of $\Gamma_{\lambda}^{\nu}{}_{\mu}$ is inherent in our problem. Let us stress again that although one of the tensors $u_{\lambda\mu}$ and $b_{\lambda\mu}$ (which figure in the definition of $W_{\lambda\mu}$) is arbitrary, the tensor $W_{\lambda\mu} - W_{33} \overset{3}{a}_{\lambda} \overset{3}{a}_{\mu}$ is uniquely defined. Therefore the right hand term in (29.1a) contains only one arbitrary tensor, namely a multiple of $\overset{3}{a}_{\lambda} \overset{3}{a}_{\mu}$. If we assign a fixed value to W_{33} then the right hand term in (29.1a) contains no undetermined term. It is important to note that the connection

$$*\Gamma_x^{\nu}{}_{\mu} = \Gamma_{\lambda}^{\nu}{}_{\mu} - a^{\nu} W_{\lambda\mu}$$

is, in general, not a solution of (14.1). This is evident from the fact that

$$W_{\lambda\omega} a^{\alpha} g_{\alpha\mu} + W_{\omega\mu} a^{\alpha} g_{\lambda\alpha} = 2W_{\omega[\mu} \overset{3}{a}_{\lambda]} \neq 0.$$

On the other hand, if $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$ is a solution of (14.1) then

$$*\tilde{\Gamma}_{\lambda}{}^{\nu}{}_{\mu} = \Gamma_{\lambda}{}^{\nu}{}_{\mu} + W a_{\lambda}^3 a_{\mu}^3 a^{\nu}{}_3$$

is also a solution of (14.1) regardless of the value of W , as follows from the fact that

$$a_{\lambda}^3 a_{\omega}^3 a^{\alpha}{}_4 g_{\alpha\mu} + a_{\omega}^3 a_{\mu}^3 a^{\alpha}{}_4 g_{\lambda\alpha} = 2 a_{\omega}^3 a_{[\mu}^3 a_{\lambda]}^3 = 0.$$

This is also the reason why $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$, given by (29.1a), is a solution of (14.1) for any value assigned to W_{33} .

Note II. In the next paper we shall consider the corresponding field equations

$$R_{(\lambda\mu)} = 0, \quad \partial_{[\omega} R_{\mu\lambda]} = 0,$$

where $R_{\mu\lambda}$ is the contracted curvature tensor of $\Gamma_{\lambda}{}^{\nu}{}_{\mu}$.

Graduate Institute for Mathematics
and Mechanics
Bloomington, Indiana