A Note on Three Dimensional Simple Lie Algebras¹

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The main purpose of the present note is to establish a simple criterion for complete reducibility of the representations of three dimensional simple Lie algebras of any characteristic other than two. The final result (Theorem 2) giving the structure of a certain associative algebra generated by two elements is effectively stronger than the theorem on complete reducibility. While the present results were developed with a view of application to the proofs of some theorems on nilpotent elements of Jordan algebras, we have decided to publish these separately since they appear to be of independent interest for the following reason. It is known that every Lie algebra (and even every restricted Lie algebra) of characteristic $p \neq 0$ has a non-completely reducible representation (JACOBSON [2] and HOCHSCHILD [1]). One therefore asks for conditions which insure complete reducibility of particular representations. The present result gives such conditions for the three dimensional simple algebras of standard type (split algebras). These may suggest general conditions for the representations of the semi-simple algebras of the Killing-Cartan-Seligman type (K.C.S. algebras). Since the three dimensional subalgebras play a fundamental role for these the present result may be useful in investigating the general class of K.C.S. algebras. A first step in this direction is made here (Theorem 3).

In §1 we give a complete structure theory for three dimensional simple Lie algebras. In spite of the completely elementary character of the proofs, the results appear to be new.

1. Determination of the algebras. Let $\mathfrak{L}$ be a three dimensional Lie algebra over an arbitrary field Φ of characteristic $\neq 2$.² Since one dimensional and two dimensional Lie algebras are solvable, $\mathfrak{L}$ is either solvable or is simple and the

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²The results, except for the one that the derivations are all inner, are valid also for characteristic two. Since the proofs need to be elaborated slightly, we have omitted this case from consideration.

latter condition is equivalent to $\mathfrak{L}' = [\mathfrak{L}\mathfrak{L}] = \mathfrak{L}$. Assume this is the case. Let (x_1, x_2, x_3) be a basis for $\mathfrak{L}$ and define $y_1 = [x_2x_3]$, $y_2 = [x_3x_1]$, $y_3 = [x_1x_2]$. Write $y_i = \sum_{j,k=1}^3 \alpha_{ij}x_j$; then $a = (\alpha_{ij})$ is a non-singular matrix and the Jacobi identity $[x_1y_1] + [x_2y_2] + [x_3y_3] = 0$ is equivalent to the symmetry of a : $a' = a$. Since the above Jacobi identity is the only non-trivial one it is clear that any non-singular symmetric a can be used to construct a three dimensional simple Lie algebra $\mathfrak{L}$. Let $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ be a second basis for $\mathfrak{L}$ and suppose $\bar{x}_i = \sum \mu_{ij}x_j$. Then if we define $\bar{y}_i = [\bar{x}_i\bar{x}_k]$, (i, j, k) a cyclic permutation of $(1, 2, 3)$, we can verify that the matrix $\bar{a}$, such that $\bar{y}_i = \sum \bar{a}_{ij}\bar{x}_j$, $\bar{a} = (\bar{a}_{ij})$, is related to a by the equation $\bar{a} = \mu(m^{-1})'am^{-1}$ where $m = (\mu_{ij})$ and $\mu = \det m$. We recall that two symmetric matrices a and $\bar{a}$ are called multiplicatively cogredient if there exists a non-singular matrix n and a non-zero element ρ in the base field such that $\bar{a} = \rho n' a n$. If this relation holds for 3×3 matrices then we may write $\bar{a} = \rho \sigma^2 (\sigma^{-1}n)' a (\sigma^{-1}n)$ and take $\sigma = \rho \det n$. Then $\bar{a} = \mu(m^{-1})'am^{-1}$ where $m = \sigma n^{-1}$, $\mu = \det m$. We therefore see that with each basis (x_1, x_2, x_3) we may associate a non-singular symmetric matrix a in the manner indicated. We then obtain a 1-1 correspondence between bases and the set of matrices multiplicatively cogredient to a . This implies that the problem of classifying the three dimensional simple Lie algebras over Φ is equivalent to that of classifying non-singular symmetric matrices over Φ relative to multiplicative cogredience. Since a is cogredient to a diagonal matrix we see that the basis (x_1, x_2, x_3) may be chosen so that $[x_1x_2] = x_3$, $[x_2x_3] = \alpha x_1$, $[x_3x_1] = \beta x_2$.

Let E be a derivation in $\mathfrak{L}$ and write $x_i E = \sum \epsilon_{ij}x_j$, $e = (\epsilon_{ij})$. Then the conditions $y_i E = [x_i E, x_k] + [x_i, x_k E]$ give the following relations for the matrix $\bar{e}$ of $y_i E = \sum \bar{\epsilon}_{ij}y_j$: $\bar{e} = -e' + (\text{tr } e)1$. On the other hand, since $y_i = \sum \alpha_{ij}x_j$, $\bar{e} = a e a^{-1}$. Hence $a e a^{-1} + e' = (\text{tr } e)1$. Then, taking traces gives $2\text{tr } e = 3\text{tr } e$ so that $\text{tr } e = 0$. Thus we have the condition $a^{-1}e'a = -e$, that is, e is skew relative to the involution $x \rightarrow a^{-1}x'a$. The space of matrices having this property is three dimensional. On the other hand, the set $\text{ad } \mathfrak{L}$ of mappings $\text{ad } u$, u in $\mathfrak{L}$, is a three dimensional Lie algebra of derivations in $\mathfrak{L}$. It follows that all the derivations of $\mathfrak{L}$ are inner and that $\text{ad } \mathfrak{L}$ is the set of linear transformations in $\mathfrak{L}$ which are skew relative to the symmetric bilinear form (x, y) such that $(x_i, x_j) = \alpha_{ij}^*$, $(\alpha_{ij}^*) = a^{-1}$.

We shall say that $\mathfrak{L}$ is *split* if it contains an element $e \neq 0$ such that $\text{ad } e$ is nilpotent. Assume $(\text{ad } e)^{k-1} \neq 0$, $(\text{ad } e)^k = 0$ and let x be an element of $\mathfrak{L}$ such that $x(\text{ad } e)^{k-1} \neq 0$. Then $k > 1$ and $(x(\text{ad } e)^{k-1}, x(\text{ad } e)^{k-1}) = \pm(x, x(\text{ad } e)^{2k-2}) = 0$. Hence the Witt index of (x, y) is positive. It follows that the form whose matrix is a has positive Witt index. It follows that we can find a basis (e, f, h) such that the corresponding matrix is

$$(1) \quad \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus we have the multiplication table

$$(2) \quad [ef] = h, \quad [eh] = 2e, \quad [fh] = -2f.$$

Clearly this implies that any two split algebras are isomorphic. From now on we restrict our attention to split Lie algebras and we suppose that the basis (e, f, h) satisfies (2).

2. Some commutation formulas.³ Let χ denote the characteristic of the base field Φ so that χ is arbitrary but $\neq 2$. Let $\mathfrak{A}$ be an associative algebra over Φ containing three elements e, f, h satisfying (2) where $[xy]$ now denotes $xy - yx$. We have $hf = f(h + 2)$, $h^k f = f(h + 2)^k$, $h^k e = e(h - 2)^k$, $k = 0, 1, 2, \dots$. Hence if $\varphi(\lambda)$ is a polynomial in the indeterminate λ then

$$(3) \quad \varphi(h)f = f\varphi(h + 2), \quad \varphi(h)e = e\varphi(h - 2).$$

If $a \in \mathfrak{A}$ we write a' for $[af]$ and we recall that $a \rightarrow a'$ is a derivation: $a \rightarrow a'$ is linear and $(ab)' = a'b + ab'$. As usual we write a'' for $(a')'$, etc. Then, by (3),

$$(4) \quad \varphi(h)' = f\Delta\varphi(h), \quad \Delta\varphi(\lambda) = \varphi(\lambda + 2) - \varphi(\lambda).$$

We note next

$$(5) \quad (e^m)' = m(h + m - 1)e^{m-1}, \quad e^0 = 1, \quad m = 1, 2, 3, \dots$$

Thus $e' = h$, by (2). Assume (5) for $m = r$ and consider

$$\begin{aligned} (e^{r+1})' &= e'e^r + e(e^r)' = he^r + re(h + r - 1)e^{r-1} \\ &= he^r + r(h + r + 1)e^r = (r + 1)(h + r)e^r. \end{aligned}$$

For $m = 1, 2, 3, \dots$ we define

$$(6) \quad \Pi_{m0} = 1, \quad \Pi_{ms} = \prod_{j=1}^s (h + m - j), \quad s \geq 1.$$

Then

$$(7) \quad \begin{aligned} \Delta\Pi_{m0} &= 0, \quad \Delta\Pi_{m1} = 2, \\ \Delta\Pi_{ms} &= (2h + 2m - s + 1)\Pi_{m, s-2}, \quad s \geq 2. \end{aligned}$$

We shall prove for $1 \leq r \leq n$, $n = 1, 2, \dots$,

$$(8) \quad (e^m)^{(r)} = \sum_{k=0}^{\lfloor \frac{1}{2}r \rfloor} c_{mrk} f^k \Pi_{m, r-2k} e^{m-r+k}$$

where

$$(9) \quad c_{mrk} = \binom{m}{k} \binom{m-k}{r-2k} r!.$$

³Cf. JACOBSON [2].

Since $c_{m10} = m$, (5) shows that (8) holds for $r = 1$. Assume it holds for $r = s$. Then

$$\begin{aligned}
 (e^m)^{(s+1)} &= \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} c_{msk} f^k(\Pi_{m,s-2k})' e^{m-s+k} + \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} c_{msk} f^k \Pi_{m,s-2k} (e^{m-s+k})' \\
 &= \prod_{k=0}^{\lfloor \frac{s}{2} \rfloor} c_{msk} f^{k+1} \Delta \Pi_{m,s-2k} e^{m-s+k} \\
 (10) \quad &+ \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} (m-s+k) c_{msk} f^k \Pi_{m,s-2k} (h+m-s+k-1) e^{m-s+k-1} \\
 &= \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} f^k \varphi_k(h) e^{m-s+k-1}
 \end{aligned}$$

where

$$\begin{aligned}
 \varphi_k(h) &= c_{m,s,k-1} \Delta \Pi_{m,s-2k+2} \\
 &\quad + (m-s+k) c_{msk} \Pi_{m,s-2k} (h+m-s+k-1) \\
 (11) \quad &= (s-2k+2) c_{m,s,k-1} (2h+m-s+2k-1) \Pi_{m,s-2k} \\
 &\quad + (m-s+k) c_{msk} (h+m-s+k-1) \Pi_{m,s-2k} .
 \end{aligned}$$

Now we can verify that

$$\begin{aligned}
 (12) \quad &2(s-2k+2) c_{m,s,k-1} + (m-s+k) c_{m,s,k} = c_{m,s+1,k} , \\
 &(s+1)(s-2k+2) c_{m,s,k-1} + (s+1)(m-s+k) c_{msk} \\
 &= (s-k+1) c_{m,s+1,k} .
 \end{aligned}$$

If we use these relations in (11) we obtain

$$\varphi_k(h) = c_{m,s+1,k} (h+m-s+2k-1) \Pi_{m,s-2k} = c_{m,s+1,k} \Pi_{m,s-2k+1}$$

which proves (8).

If $\chi = 0$ and h is an algebraic element then it is known that e and f are nilpotent (JACOBSON [2]). Moreover, h is algebraic if $\mathfrak{A}$ is the algebra of linear transformations in a finite dimensional vector space. In the sequel we shall be interested in the case in which e and f are nilpotent no matter what χ is and we shall require the following result.

Lemma 1. *Let $e^m = 0$ where $m \leq \chi - 1$ if $\chi \neq 0$. Then*

$$(13) \quad \prod_{j=1}^{2m-1} (h+m-j) = 0.$$

Proof. We establish, more generally,

$$(14) \quad \prod_{j=1}^{2r-1} (h+m-j) e^{m-r} = 0, \quad r = 0, 1, \dots, m.$$

For $r = 0$ this is part of our assumption. Assume (14) for $r = 1, \dots, s-1$. Since $(e^m)^{(s)} = 0$, (8) implies that

$$\sum_{k=0}^{\lfloor \frac{1}{2}s \rfloor} c_{m+sk} f^k \Pi_{m, s-2, k} e^{m-s+k} = 0.$$

Multiply this on the left by $\prod_{j=1}^{2s-1} (h+m-j)$ and make use of (14) for $r \leq s-1$. This gives

$$c_{ms0} \prod_{j=1}^{2s-1} (h+m-j) e^{m-s} = 0.$$

Since $m, s \leq \chi - 1$ if $\chi \neq 0$, $c_{ms} = \binom{m}{s} s! \neq 0$. Hence (14) holds.

3. Representations of $\mathfrak{L}$. The term representation will mean representation by linear transformations in a finite dimensional vector space. Let $\mathfrak{L}$ be a split three dimensional simple Lie algebra with basis (e, f, h) satisfying (2). A representation is determined by a mapping $e \rightarrow E, f \rightarrow F, h \rightarrow H$ where E, F, H are linear transformations such that $[EF] = H, [EH] = 2E, [FH] = -2F$. For every dimensionality j of the underlying vector space $\mathfrak{M}$ we can define a representation ρ_j of $\mathfrak{L}$ as follows: Let $(x_1, x_2, \dots, x_i)$ be a basis for $\mathfrak{M}$ and let E, F, H be defined by

$$\begin{aligned} x_i E &= 0, & x_{i+1} E &= i(i-j)x_i, & i &\leq j-1, \\ (15) \quad x_i F &= x_{i+1}, & i &\leq j-1, & x_j F &= 0, \\ x_i H &= (j-2i+1)x_i. \end{aligned}$$

Then one verifies directly that E, F, H satisfy the above commutation relations; hence they define a representation ρ_j of $\mathfrak{L}$.

Lemma 2. *The representation ρ_j is irreducible if $\chi = 0$ or if $\chi \neq 0$ and $j \leq \chi$. Otherwise ρ_j is not completely reducible.*

Proof. Let $\chi = 0$ or $\chi \neq 0$ and $j \leq \chi$. Then the elements $i(i-j)1$, $1 \leq i \leq j-1$, are non-zero elements of the prime field of Φ . Let $\mathfrak{N}$ be a non-zero invariant subspace in $\mathfrak{M}$ and let $v = \alpha_m x_m + \alpha_{m+1} x_{m+1} + \dots + \alpha_j x_j$ be a vector in $\mathfrak{N}$ with $\alpha_m \neq 0$. Then $x_i = \alpha_m^{-1} v F^{i-m} \in \mathfrak{N}$. Since $i(i-j)1 \neq 0$, $1 \leq i \leq j-1$, the elements $x_i E^i$, $i = 0, \dots, j-1$, form a basis for $\mathfrak{M}$. Hence $\mathfrak{M} = \mathfrak{N}$ is irreducible. Next assume $\chi \neq 0$ and $j > \chi$. Then $C = F^\chi \neq 0$ but C is nilpotent. Also we have $[EC] = [EF^\chi] = E \text{ ad } F^\chi = E(\text{ad } F)^\chi = H(\text{ad } F)^{\chi-1} = 2F(\text{ad } F)^{\chi-2} = 0$ since $\chi > 2$. Thus the enveloping associative algebra of the linear transformations E, F, H contains a nilpotent center element $\neq 0$. Hence it cannot be completely reducible.

Let ρ be a representation of $\mathfrak{L}$ acting in $\mathfrak{M}$ and let E, F, H be the transformations corresponding to e, f, h . If $\chi = 0$, $E^m = 0 = F^m$ for some positive integer m . If $\chi \neq 0$ then we assume from now on that $E^m = 0 = F^m$ for $m \leq \chi - 1$. Then

Lemma 1 implies that H is a root of the polynomial $\mu(\lambda) = \prod_{i=1}^{2m-1} (\lambda + m - j)$. Hence the characteristic roots of H are integers in the prime field. Note also that $0, \pm 1$ are simple roots of $\mu(\lambda)$. Hence if x is a vector in $\mathfrak{M}$ such that $x(H - \eta)^k = 0$ for $\eta = 0, 1$ or -1 then $x(H - \eta) = 0$. If $x(H - i)^k = 0$ then (3) implies that $(xE)(H - i - 2)^k = 0$ and $xF(H - i + 2)^k = 0$. Thus if x is in the characteristic space belonging to i then xE (xF) is in the characteristic space belonging to $i + 2$ ($i - 2$). We shall call a vector x of $\mathfrak{M}$ *maximal* (*minimal*) if $x \neq 0$, $xH = \alpha x$, α in Φ and $xE = 0$ ($xF = 0$). Our condition on the representation implies that $\mathfrak{M}$ contains maximal and minimal vectors. Thus if $y \neq 0$ and $yH = \beta y$ then a suitable vector in the sequence $y, yE, yE^2, \dots$ is maximal and a suitable vector in $y, yF, yF^2, \dots$ is minimal. This remark and the following lemma imply that if ρ is irreducible then ρ is equivalent to one of the ρ_i where $j \leq \chi - 1$ if $\chi \neq 0$.

Lemma 3. *Let ρ be a representation of $\mathfrak{L}$ satisfying $E^{\chi-1} = 0 = F^{\chi-1}$ if $\chi \neq 0$. Then every maximal (minimal) vector x in the representation space $\mathfrak{M}$ generates an irreducible invariant subspace $\mathfrak{N}$ whose associated representation is one of the ρ_i of Lemma 2, where $j \leq \chi - 1$ if $\chi \neq 0$.*

Proof. If x is maximal, set $x_1 = x$, $x_i = xF^{i-1}$ and suppose $x_i \neq 0$ for $i \leq j$ but $x_{j+1} = 0$. Since $F^{\chi-1} = 0$ if $\chi \neq 0$, $j \leq \chi - 1$ in this case. Since $x_1H = \alpha x_1$, $x_iH = (\alpha - 2i + 2)x_i$. Since the elements $(\alpha - 2i + 2)1$ are distinct in Φ , the x_i are linearly independent. We have $x_1E = 0$ and we can prove by induction that $x_{i+1}E = i(i - \alpha - 1)x_i$ for $1 \leq i \leq j - 1$. These relations show that the space $\mathfrak{N}$ spanned by the x_i is invariant. Since $H = [EF]$, $\text{tr}_{\mathfrak{N}} H = 0$ where $\text{tr}_{\mathfrak{N}}$ denotes the trace of the induced transformation in $\mathfrak{N}$. Hence $j\alpha - j(j - 1)1 = 0$ and $\alpha = (j - 1)1$. It follows that we have the relations (15) for E, F, H . Hence the representation in $\mathfrak{N}$ is equivalent to ρ_j . Since $j \leq \chi - 1$ if $\chi \neq 0$, $\mathfrak{N}$ is irreducible.

Consequences. (1) Any two irreducible representations of the same degree (satisfying our condition) are equivalent. (2) The linear transformation H has either 0 or 1 as characteristic root. (3) If $\mathfrak{M}$ is irreducible then it is spanned by characteristic vectors belonging to distinct characteristic roots of H . Moreover, if x is a characteristic vector of H which is not maximal (minimal) then $xEF = \beta x \neq 0$ ($xFE = \beta x \neq 0$). These results are immediate from (15).

We can now prove the following theorem.

Theorem 1. *Any representation of $\mathfrak{L}$ such that $E^{\chi-1} = 0 = F^{\chi-1}$ if $\chi \neq 0$ is completely reducible.*

Proof. Let $\mathfrak{M}$ be the representation space and let $\mathfrak{M}_1$ be the subspace spanned by the characteristic roots of H . Then $\mathfrak{M}_1$ is invariant. We assert that $\mathfrak{M}_1 = \mathfrak{M}$. Otherwise there is an x in $\mathfrak{M}$ such that $x(H - \alpha)^2 = 0$, $x(H - \alpha) \neq 0$. Then $x(H - \alpha) \equiv 0 \pmod{\mathfrak{M}_1}$ and if we replace x by a suitable vector in the sequence $x, xE, xE^2, \dots$ we may suppose also that $xE \equiv 0 \pmod{\mathfrak{M}_1}$. By Lemma 3, the coset of x generates an irreducible invariant subspace of $\mathfrak{M} - \mathfrak{M}_1$. Then

there exists a k such that $(x^{F^k})H \equiv \eta(x^{F^k}) \pmod{\mathfrak{M}_1}$ where $\eta = 0$ or 1 and $x^{F^k} \notin \mathfrak{M}_1$. On the other hand, $(x^{F^k})(H - \alpha + 2k)^2 = 0$; hence $(x^{F^k})(H - \eta)^2 = 0$. Since η is a simple root of the minimum polynomial of H , $(x^{F^k})(H - \eta) = 0$. This means that $x^{F^k} \in \mathfrak{M}_1$, contrary to the above. Hence $\mathfrak{M} = \mathfrak{M}_1$. Next let $\mathfrak{N}$ be the sum of the irreducible invariant subspaces of $\mathfrak{M}$. If $\mathfrak{N} \neq \mathfrak{M}$ there exists a characteristic vector x of H not contained in $\mathfrak{N}$. As before, we may assume that $xE \equiv 0 \pmod{\mathfrak{N}}$. Also $xE \neq 0$ since otherwise x is maximal, hence generates an irreducible invariant subspace, contrary to $x \notin \mathfrak{N}$. Write $\mathfrak{N} = \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \cdots \oplus \mathfrak{N}_r$ where the $\mathfrak{N}_i$ are irreducible invariant subspaces and let $(x_1^{(i)}, \dots, x_{i_i}^{(i)})$ be a basis for $\mathfrak{N}_i$ chosen as in (15). If x belongs to the characteristic root α of H then xE belongs to $\alpha + 2$ and $xE = \sum \alpha_i x_{k_i}^{(i)}$ where $x_{k_i}^{(i)}$ belongs to $\alpha + 2$ if $\alpha_i \neq 0$. Since $xE \neq 0$ some of the α_i are $\neq 0$. If $x_{k_i}^{(i)}$ is not minimal then $x_{k_i}^{(i)} FE = \beta_i x_{k_i}^{(i)} \neq 0$. Set $y = x - \sum' \beta_i^{-1} \alpha_i x_{k_i}^{(i)} F$ where the summation is taken over the i such that $x_{k_i}^{(i)}$ is not minimal. Then $yEF = 0$ and y has the other properties of x : y is a characteristic vector of H , $y \notin \mathfrak{N}$, $yE \in \mathfrak{N}$, $yE \neq 0$. Since yE is minimal we have an irreducible invariant subspace with the basis $(yE, yE^2, \dots, yE^m)$ where $yE^{m+1} = 0$. Then (15) shows that yE belongs to the characteristic root $-(m-1)1$. On the other hand, the coset of $y \pmod{\mathfrak{N}}$ is a maximal vector so that this coset generates an irreducible invariant subspace with a basis consisting of the cosets of $(y, yF, \dots, yF^{m-1})$ where $yF^m \in \mathfrak{N}$. Then y belongs to the characteristic root $(n-1)1$ of H . Hence $(n-1) \cdot 1 + 2 \cdot 1 = -(m-1) \cdot 1$. If $\chi = 0$ this is impossible and if $\chi \neq 0$ then this and the fact that $n, m \leq \chi - 1$ imply that $m + n = \chi$. If $n = 1$, $m = \chi - 1$ and $yE^m \neq 0$ contrary to assumption. Hence $n > 1$ and $yFE = y(EF - H) = -yH = -(n-1)y \neq 0$. By induction, $yF^k E = k(k-n)yF^{k-1} \neq 0$, $1 \leq k \leq n-1$. It follows that $(yF^{n-1})E^{m+n-1} \neq 0$ which again contradicts $E^{x-1} = 0$. This proves that $\mathfrak{M} = \mathfrak{N}$ is completely reducible.

Let $\mathfrak{E}^{(i)}$ denote the associative algebra of all linear transformations of a j -dimensional vector space over Φ . Then the main result of the present considerations is the following theorem.

Theorem 2. *Let $\mathfrak{F}$ be the free associative algebra (with an identity 1) generated by three elements e, f, h and let $\mathfrak{S}$ be the ideal in $\mathfrak{F}$ generated by the elements $[eh] - 2e$, $[fh] + 2f$, $[ef] - h$, e^m , f^m where $m \leq \chi - 1$ if $\chi \neq 0$. Then $\mathfrak{U} \equiv \mathfrak{F}/\mathfrak{S} \cong \mathfrak{E}^{(1)} \oplus \mathfrak{E}^{(2)} \oplus \cdots \oplus \mathfrak{E}^{(m)}$.*

Proof. The commutator relations in $\mathfrak{U}$ imply that every coset in $\mathfrak{U}$ is a linear combination of the cosets of the monomials $e^{k_1} f^{k_2} h^{k_3}$, $k_i \geq 0$. Since $e^m \equiv 0 \pmod{\mathfrak{S}}$, Lemma 1 shows that $\prod_{i=1}^{2m-1} (h + m - j) = 0 \pmod{\mathfrak{S}}$. It follows that every coset is a linear combination of the cosets of the monomials $e^{k_1} f^{k_2} h^{k_3}$ with $0 \leq k_1$, $k_2 \leq m-1$, $0 \leq k_3 \leq 2m-2$. Hence $\mathfrak{U}$ is finite dimensional. It is clear from the definition of $\mathfrak{U}$ that a finite dimensional representation of $\mathfrak{U}$ determines a representation of the split Lie algebra $\mathfrak{g}$ satisfying $E^m = 0 = F^m$ and conversely. Hence every representation of $\mathfrak{U}$ is completely reducible, and $\mathfrak{U}$ is semi-simple. Moreover, we have a 1-1 correspondence between the simple components of

$\mathfrak{U}$ and the set of inequivalent irreducible representations of $\mathfrak{L}$ satisfying our condition if $\chi \neq 0$. Lemmas 2 and 3 provide such a representation for every degree j with $1 \leq j \leq m$ and these are the only ones up to equivalence. It is easy to verify directly that the linear transformations E, F, H of (15) with $1 \leq j \leq m$ generate $\mathfrak{G}^{(j)}$. Hence the simple component corresponding to the irreducible representation of degree j is isomorphic to the $\mathfrak{G}^{(j)}$. Hence $\mathfrak{U} \cong \mathfrak{G}^{(1)} \oplus \mathfrak{G}^{(2)} \oplus \dots \oplus \mathfrak{G}^{(m)}$.

Some remarks. Our results include the well known results on the representations of $\mathfrak{L}$ for Φ of characteristic 0. The new results for characteristic $\chi \neq 0$ are essentially the best which can be obtained. Thus the proof of the second part of Lemma 2 shows that any representation of $\mathfrak{L}$ in which e is represented by a nilpotent linear transformation E such that $E^\chi \neq 0$ is not completely reducible. We note also that a result of HOCHSCHILD [1] implies that there exist representations of $\mathfrak{L}$ which are not completely reducible but for which $E^\chi = 0 = F^\chi$. Theorem 1 implies that if $\rho^{(1)}$ and $\rho^{(2)}$ are representations of the three dimensional Lie algebra $\mathfrak{L}$ such that $E_i = E^{\rho^{(i)}}$ and $F_i = F^{\rho^{(i)}}$ satisfy $E_i^{\frac{1}{2}(\chi-1)} = 0 = F_i^{\frac{1}{2}(\chi-1)}$ then the Kronecker product $\rho_1 \otimes \rho_2$ is completely reducible.

We suppose now that $\mathfrak{L}$ is an arbitrary Lie algebra having the following properties: $\mathfrak{L}$ has (1) a commutative Cartan subalgebra $\mathfrak{H}$ and (2) a corresponding Cartan decomposition into root spaces $\mathfrak{L} = \mathfrak{H} + \sum_{\alpha \neq 0} \mathfrak{L}_\alpha$ where the $\mathfrak{L}_\alpha$ are one dimensional. (3) If α is a root then $-\alpha$ is a root and $\mathfrak{L}_\alpha + \mathfrak{L}_{-\alpha} + [\mathfrak{L}_\alpha \mathfrak{L}_{-\alpha}]$ is a split three dimensional simple Lie algebra. (4) $\mathfrak{H} = \sum_{\alpha \neq 0} [\mathfrak{L}_\alpha \mathfrak{L}_{-\alpha}]$. These properties are possessed by the semi-simple Lie algebras of the type considered by SELIGMAN [4], which we have called the K.C.S. algebras. The result on the three dimensional simple Lie algebra suggests the following conjecture: Complete reducibility holds for any representation ρ of a K.C.S. algebra such that, for every root $\alpha \neq 0$, if $e_\alpha \in \mathfrak{L}_\alpha$ and $e_\alpha \rightarrow E_\alpha$ then $E_\alpha^{\chi-1} = 0$. We take a first step in this direction by proving the following theorem.⁴

Theorem 3. *Let $\mathfrak{L}$ be a Lie algebra having the properties (1), (2), (3) listed above and suppose we have a representation of $\mathfrak{L}$ such that if $\chi \neq 0$ then the representing linear transformation E_α of every $e_\alpha \in \mathfrak{L}_\alpha$ satisfies $E_\alpha^{\chi-1} = 0$ then $\mathfrak{H}$ "acts diagonally," that is, the representation space is a direct sum of one dimensional subspaces invariant under $\mathfrak{H}$.*

Proof. Our conditions imply that the induced representation of every three dimensional algebra $\mathfrak{L}_\alpha + \mathfrak{L}_{-\alpha} + [\mathfrak{L}_\alpha \mathfrak{L}_{-\alpha}]$ is completely reducible into representations equivalent to the ρ_i defined before. It follows that every $[\mathfrak{L}_\alpha \mathfrak{L}_{-\alpha}]$ acts diagonally. Since $\mathfrak{H}$ is commutative and $\mathfrak{H} = \sum [\mathfrak{L}_\alpha \mathfrak{L}_{-\alpha}]$ this implies that $\mathfrak{H}$ acts diagonally.

⁴Further partial results have been obtained by SELIGMAN and by C. W. CURTIS (to appear in a forthcoming paper). Questions similar to the ones raised here for Lie algebras can also be formulated for rational representations of the unimodular group of degree two and for simple algebraic groups in general.

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