Some Extremal Problems for a New Class of Univalent Functions

VIKRAMADITYA SINGH

Communicated by C. LOEWNER

1. Much attention in the past has been devoted to the study of various extremum problems for the class V of functions $f(z)$ regular and univalent in the unit circle $|z| < 1$ and normalized so that $f'(0) = 1$ and for its different subclasses, for example, the class V_R of real univalent functions which lead to symmetric domains, the class of functions for which the image domains are convex or star-like and the class of bounded univalent functions whose image domains lie within a fixed circle. In the present paper we consider problems for another class S of regular univalent functions $f(z)$ which are real on the real axis and whose image domains contain the unit circle, the functions being normalized so that $f(0) = 0$ and $f(1) = 1$. Some of the simplest functions belonging to the class S are the identity function z and the Koebe function $K(z) = 4z/(1+z)^2$. We observe that if $f(z) \in S$ then $f(z)/z$ is regular analytic in $|z| < 1$ and hence, by the minimum principle, we have

$$(1) \quad |f(z)| \geq |z|.$$

Thus at a given point $\zeta \in U_z(|z| < 1)$ the identity function $f(z)$ gives the minimum absolute value in the class S .

As a consequence of (1) we have, in the limit when $z \rightarrow 0$, that $|f'(0)| \geq 1$. Further, let $f(z) \in S$ have the following power series expansion in $|z| < 1$:

$$(2) \quad f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

where $a_1, a_2, a_3, \cdots$ are all real. We then see that the function $f(z)/a_1$ belongs to the class V_R of functions $F(z)$ which are normalized so that $F(0) = 0$ and $F'(0) = 1$. Thus an application of the Koebe quarter theorem immediately gives

$$(3) \quad |a_1| = |f'(0)| \leq 4.$$

Reproduction in whole or in part is permitted for any purpose of the United States Government. Prepared under Contract Nonr-225(11) (NR-041-086) for the Office of Naval Research.

The fact that equality in (3) holds for $K(z)$ shows that (3) is sharp. Combining (3) with the earlier result $|a_1| \leq 1$ we can write

$$(4) \quad 1 \leq a_1 \leq 4.$$

The family of functions of the subclass S is normal because of the fact that the whole class V of regular univalent functions is locally uniformly bounded. Further, on account of the inequality (1) the constant function is excluded. Thus, given an arbitrary sequence of functions belonging to S , we can always choose a subsequence $[f_n(z)]$ which in every closed subdomain of $|z| < 1$ converges to a univalent function $f(z)$. In order to show the compactness of the family of functions belonging to S we need to show that $f(z)$ takes all values within $U_w (|w| < 1)$ and that $f(1) = 1$. To prove the former we first observe that, as a consequence of (4), the limit function $f(z)$ maps a neighborhood of the origin in U_z univalently onto a neighborhood N_0 of the origin in U_w . Let $w = w_0$ be any point in U_w and let $|w - w_0| < \rho$ be a small neighborhood $N_{w_0} \subset U_w$ of the point w_0 . Then there exists a region $N \subset U_w$ which contains both N_0 and N_{w_0} . Since the sequence of functions $\{f_n^{-1}(w)\}$, $w \in N$, is uniformly bounded and since in the subregion N_0 it converges to the univalent function $f^{-1}(w)$, it follows from Vitali's theorem that it converges to the univalent function $f^{-1}(w)$ in the whole region N . Thus $f(z)$ takes the value w_0 at some point in U_z and, as w_0 is an arbitrary point in U_w , it follows that $f(z)$ takes all values in U_w .

In order to show that $f(1) = 1$ it is enough to prove that $f(\xi) \neq 1$ for all $-1 < \xi < 1$. Indeed, if it were proved then it would follow that $w = 1$ is a boundary point and, on account of the symmetry of $f(z)$ with respect to the real axis, $f(1)$ would be 1. One easily shows with the help of Hurwitz's theorem [3] that $f(\xi)$ cannot take the value 1 at any interior point on the real axis. This completes the proof of the compactness of the class S .

The following theorems have been proved in this paper:

Theorem 1. For all functions $f(z)$ belonging to the class S and for any real point ξ in $|z| < 1$ the following inequality holds:

$$(5) \quad |f(\xi)| \leq \frac{4|\xi|}{(1+\xi)^2},$$

equality being attained only for the Koebe function $K(z)$.

Theorem 2. For all $f(z)$ belonging to the class S and having the power series expansion

$$f(z) = a_1 z + a_2 z^2 + \cdots,$$

the following sharp inequality holds

$$(6) \quad \frac{a_2}{a_1} \leq 2(\sqrt{2} - 1)^2,$$

equality being attained for the function

$$(7) \quad f(z) = \frac{-(1-z)^2 + (1+z)(1-2zp+z^2)^{\frac{1}{2}}}{(1-z)^2 + (1+z)(1-2zp+z^2)^{\frac{1}{2}}}, \quad p = \frac{(\sqrt{2}-1)^2}{\sqrt{2}},$$

which maps the unit circle onto the slit-domain bounded by the real axis from 1 to ∞ and -1 to $-\infty$ and a circular arc of width $2\{\pi - 2\cos^{-1}(\sqrt{2}-1)\}$ symmetrically situated at the point -1 .

We note that for the minimum problem related to Theorem 2 we have the well known inequality $a_2/a_1 \geq -2$, the equality being attained for the function $K(z)$. We also observe that for $\zeta \leq 0$ the inequality (5) is a consequence of the Koebe distortion theorem. Hence we need to prove Theorem 1 only for $\zeta > 0$.

Let us define

$$(8) \quad F(z) = \frac{f\left(\frac{z+\zeta}{1+\zeta z}\right) - f(\zeta)}{1 - f(\zeta)}, \quad 1 > \zeta \geq 0.$$

If $f(z) \in S$ then it is easily seen that $F(z)$ is a regular symmetric univalent function for which $F(0) = 0$ and $F(1) = 1$. Thus to prove that $F(z) \in S$ we need only to show that $|F(z)| \geq 1$ for $|z| = 1$. This follows from the observation that, for $\zeta > 0$, $f(\zeta) > 0$ and hence the distance of $f(\zeta)$ from 1 will always be less than its distance from any point on or beyond the unit circumference $|f(z)| = 1$. On applying the inequality (3) to $F(z)$ we have

$$(9) \quad \frac{f'(\zeta)(1-\zeta^2)}{1-f(\zeta)} \leq 4,$$

which on integration from 0 to ζ gives

$$f(\zeta) \leq \frac{4\zeta}{(1+\zeta)^2}.$$

2. Proof of Theorem 2. The proof is based on the differential equation obtained by the method of interior variations [4] for the function which maximizes a_2/a_1 . Since the function is to take all the values within the unit circle and is to be real on the real axis we need to make the variation in such a way that the origin, the unit circumference and the real axis are kept fixed. It has been shown [5] that one such variation can be written in the form

$$(10) \quad w^* = w + a\rho^2 \frac{w(1-w^2)}{(w-w_0)(1-w_0w)} + \bar{a}\rho^2 \frac{w(1-w^2)}{(w-\bar{w}_0)(1-\bar{w}_0w)} + O(\rho^4),$$

where $|w_0| < 1$, and therefore the points w_0 and $\bar{w}_0$ belong to the extremum domain. The point $1/w_0$, and, on account of symmetry, the point $1/\bar{w}_0$ also, may or may not belong to the extremum domain. If both w_0 and $1/w_0$ belong to the extremum domain the variation under (10) of a function $f(z) \in S$ can be written in the form [5]

$$\begin{aligned}
 f^*(z) = f(z) + a\rho^2 & \left[\frac{f(z)(1-f^2(z))}{(f(z)-f(z_0))(1-f(z_0)f(z))} \right. \\
 & + \frac{f(z_0)}{z_0 f'^2(z_0)} \frac{z(1-z^2)f'(z)}{(z_0-z)(1-z_0z)} + \frac{f^3(z_1)}{z_1 f'^2(z_1)} \frac{z(1-z^2)f'(z)}{(z_1-z)(1-z_1z)} \Big] \\
 (11) \quad & + \bar{a}\rho^2 \left[\frac{f(z)(1-f^2(z))}{(f(z)-f(\bar{z}_0))(1-f(z)f(\bar{z}_0))} \right. \\
 & + \frac{f(\bar{z}_0)}{\bar{z}_0 f'^2(\bar{z}_0)} \frac{z(1-z^2)f'(z)}{(\bar{z}_0-z)(1-\bar{z}_0z)} + \frac{f^3(\bar{z}_1)}{\bar{z}_1 f'^2(\bar{z}_1)} \frac{z(1-z^2)f'(z)}{(\bar{z}_1-z)(1-\bar{z}_1z)} \Big] + O(\rho^4),
 \end{aligned}$$

where

$$(12) \quad f(z_0)f(z_1) = 1.$$

If $1/w_0$ and $1/\bar{w}_0$ do not belong to the image domain then the terms involving $f(z_1)$ and z_1 in the right hand side of (11) will not occur.

If $\delta(a_2/a_1)$ denotes the variation in a_2/a_1 , we get from (11)

$$\begin{aligned}
 \delta\left(\frac{a_2}{a_1}\right) = a\rho^2 & \left[-\frac{a_1}{f(z_0)} \left(f(z_0) + \frac{1}{f(z_0)} \right) + \left(\frac{a_2}{a_1} + \frac{1}{z_0} \right) \frac{f(z_0)}{z_0^2 f'^2(z_0)} \right. \\
 (13) \quad & + \left(\frac{a_2}{a_1} + \frac{1}{z_1} \right) \frac{f^3(z_1)}{z_1 f'^2(z_1)} \Big] + \bar{a}\rho^2 \left[\frac{f(\bar{z}_0)}{\bar{z}_0 f'^2(\bar{z}_0)} + \frac{f^3(\bar{z}_1)}{\bar{z}_1 f'^2(\bar{z}_1)} \right] + O(\rho^4).
 \end{aligned}$$

For the extremum of a_2/a_1 the real part of the right hand side of (13) should keep a fixed sign. Hence, since a is arbitrary and z_0 is any point in the unit circle $|z| < 1$, we find, on passing to the limit when $\rho \rightarrow 0$, that the function $f(z)$ which gives the extremum of a_2/a_1 should satisfy the following differential equation:

$$(14) \quad a_1 \left(f(z) + \frac{1}{f(z)} \right) = \frac{f^2(z)}{z^2 f'^2(z)} \left(z + \frac{1}{z} + \frac{a_2}{a_1} \right) + \frac{f^2(z_1)}{z_1^2 f'^2(z_1)} \left(z_1 + \frac{1}{z_1} + \frac{a_2}{a_1} \right),$$

where $f(z)f(z_1) = 1$. One notices immediately that this differential equation is of a different nature from the ones one usually gets in the extremum problems pertaining to other subclasses of univalent functions. It is probably for the first time that the method of interior variations has been applied to the extremum problems belonging to the functions of the class S , and the equation (14) is a representative of the type of functional differential equation one gets in any extremum problem for this class. Therefore, for the sake of completeness, we will prove the following two lemmas which in problems belonging to the subclasses other than S one usually accepts.

Lemma 1. Any function $f(z) \in S$ which satisfies (14) is analytic on $|z| = 1$ except for a singular set on which $|f(z)| = 1$ and a finite number of points $z^1, z^2, \dots, z^a, |z^a| = 1$, where $|f(z^i)| > 1$ and at which it has algebraic singularities of finite order.

Proof. For $|\zeta| = 1$ let $f(\zeta) = \omega$, such that $|\omega| > 1$. Then the point ζ_1 for which $f(\zeta_1)f(\zeta) = 1$ will be an interior point of $|z| = 1$ and $f(z)$ will be regular, analytic and univalent in the neighborhood of ζ_1 . Hence we can write $z = g(1/w)$, for points z in the neighborhood of ζ_1 , where $g(1/w)$ is a regular analytic function of w in the neighborhood of $1/\omega$. Thus, putting $f(z) = w$, we can write the differential equation in the form

$$\left(z + \frac{1}{z} + \frac{a_2}{a_1}\right) = w'^2 F\left(\frac{1}{w}\right),$$

where $F(1/w)$ is analytic in $1/w$. This is an ordinary differential equation with analytic coefficients. Hence its solution is also analytic except at points at which $F(1/w) = 0$ and $z + (1/z) + (a_2/a_1) = 0$. The latter expression will vanish only at two points and $F(1/w) = 0$ will have only a finite number of roots because $F(1/w)$ is a non-trivial analytic function in $|1/w| < 1$. The zeros of these expressions form the set $\{z^i\}_1^n$.

One observes that for points on $|z| = 1$, at which $|f(z)| = 1$, the corresponding points z'_i for which $f(z'_i)f(z_i) = 1$ will, on account of symmetry of the domain with respect to the real axis, also lie on the boundary $|z| = 1$. Hence the analyticity of F at such points cannot be ensured and the above argument is not adequate to yield any definite statement. This proves the lemma.

As a consequence of this lemma we conclude that the boundary of the extremum domain consists of arcs which are analytic everywhere, even at infinity. The end points of these analytic arcs can have points of accumulation only on the unit circumference.

Lemma 2. *The extremum domain belonging to the function $f(z) \in S$ which maximizes a_2/a_1 does not possess any exterior points.*

Proof. Let us assume that the image domain has some exterior points and let w_1 be one such exterior point. Then there will be some region R about w_1 all of whose points will be exterior points of the image domain. Let the reflected image of R in the unit circle $|w| = 1$ be denoted by R_0 . If the point $w_0 = 1/w_1$ in the variation (10) belongs to R_0 then it is easily seen that the condition for the extremum of a_2/a_1 is

$$(15) \quad a_1 \left(w_0 + \frac{1}{w_0} \right) - \frac{w_0^2}{z_0^2 w_0^2} \left(z_0 + \frac{1}{z_0} + \frac{a_2}{a_1} \right) = 0,$$

where $w_0 = f(z_0)$. The left hand side of (15) is an analytic function of z_0 and since (15) holds in a region containing z_0 it will hold everywhere in the region of its analyticity, which is the unit circle $|z| < 1$. Further, there must be some points in the exterior of $|w| \leq 1$ which belong to the image domain of the extremum function. Indeed, if this were not true then the identity function $f(z) = z$ would be the extremum function and the extremum value of a_2/a_1 would be non-positive. But this is not true because the function in (7) belongs to S and

has positive a_2/a_1 . For those points w_0 for which $w_1 = 1/w_0$ also belongs to the image domain the following condition will hold:

$$(16) \quad a_1 \left(w_0 + \frac{1}{w_0} \right) - \frac{w_0^2}{z_0^2 w_0^2} \left(z_0 + \frac{1}{z_0} + \frac{a_2}{a_1} \right) - \frac{w_1^2}{z_1^2 w_1^2} \left(z_1 + \frac{1}{z_1} + \frac{a_2}{a_1} \right) = 0.$$

Again, (16) holds in a region containing z_0 and is analytic there. Hence it can be continued analytically in the whole of $|z| < 1$ where it is analytic. Thus (15) and (16) hold in the common region $|z| < 1$. But this implies

$$\frac{w_1^2}{z_1^2 w_1^2} \left(z_1 + \frac{1}{z_1} + \frac{a_2}{a_1} \right) \equiv 0,$$

which is not possible because at $z_1 = 0$ the left hand side has a pole. This proves Lemma 2.

Having shown that the boundary of the extremum domain consists of analytic slits we are now in a position to make some further variations to get more information about the extremum domain.

Let us take a boundary curve Γ of the image domain and make at each point $\omega \in \Gamma$ a shift along a vector $\delta\omega$ which points in the direction of the interior normal at ω and varies continuously along Γ . On account of the symmetry of the domain a similar variation needs to be made on the curve Γ^* which is the mirror image of Γ in the real axis. We thus obtain a domain which is mapped by means of a schlicht function $f^*(z)$. According to JULIA [1] the variational law in the case of symmetric domains can be written as

$$(17) \quad \delta f(z) = z f'(z) \frac{1}{2\pi i} \oint_{\Gamma} \frac{\zeta(1-z^2)}{(\zeta-z)(1-\zeta z)} \frac{\delta\omega d\omega}{\zeta^2 f'^2(\zeta)}, \quad \omega = f(\zeta).$$

From (17) one easily finds that the corresponding variation in a_2/a_1 is

$$\delta \left(\frac{a_2}{a_1} \right) = -\frac{1}{2\pi i} \oint_{\Gamma} \left(\zeta + \frac{1}{\zeta} + \frac{a_2}{a_1} \right) \frac{\delta\omega d\omega}{\zeta^2 f'^2(\zeta)}.$$

As $(1/i)\delta\omega d\omega/\zeta^2 f'^2(\zeta)$ is the real positive quantity $\delta n ds$ we see that the necessary condition for the maximum of a_2/a_1 is that

$$(18) \quad \zeta + \frac{1}{\zeta} + \frac{a_2}{a_1} \geq 0, \quad |\zeta| = 1.$$

If $\zeta = e^{i\theta}$ (18) can be written in the form

$$(19) \quad 2 \cos \theta + \frac{a_2}{a_1} \geq 0.$$

Now it is well known that, for functions of the class V, $a_2/a_1 \leq 2$, where equality holds only in the case of the function $z/(1-z)^2$. As this function does not belong to the subclass S, a_2/a_1 must be < 2 . Hence there will exist two real values $\pm\alpha$ of θ for which $2 \cos \theta + (a_2/a_1) = 0$.

These will be simple zeros; hence, for $-\alpha < \theta < \alpha$,

$$(20) \quad 2 \cos \theta + \frac{a_2}{a_1} \leq 0, \quad -\alpha \leq \theta \leq \alpha.$$

This violates the condition (19) and corresponds to the fact that the circular arc $-\alpha \leq \theta \leq \alpha$ of $|z| = 1$ is mapped on the inner part of a circular arc of $|f(z)| = 1$. Indeed, on account of the slit character of the extremum domain and the fact that the image domain contains the unit circle $|w| < 1$ one cannot make a Julia variation on this part of the boundary. Further, because this circular arc is an analytic curve the function $f(z)$ is analytic on $-\alpha < \theta < \alpha$ and it, being the image under analytic mapping of a single connected piece of $|z| = 1$, is also connected. On account of the symmetry of the image domain this arc will therefore lie symmetrically either at the point 1 or -1 . Since $\max(a_2/a_1)$ is positive it is clear that it will be situated at $w = -1$.

We can now assert that the boundary of the extremum domain must contain (i) the positive real axis from 1 to ∞ , (ii) the negative real axis from -1 to $-\infty$ and (iii) a circular arc of $|w| = 1$ symmetrically situated at $w = -1$. That (i) holds follows from the fact that the point $w = 1$ is a boundary point and the image domain is a symmetric slit domain. (iii) has already been proved in the preceding paragraph. (ii) follows from the fact that this circular arc should be connected with the positive real axis in such a way that the image remains a symmetric slit domain. Let us denote this system of curves by C_0 and let D_0 be the simply connected domain whose boundary is C_0 . Also let A and $\bar{A}$ denote the end points of the circular arc of C_0 and let the inner arc of C_0 be denoted by C_0^i .

We now state the theorem which is proved in the next section and on which the proof of Theorem 2 depends.

Theorem 3. *The extremum function $f(z)$ which belongs to the class S and maximizes a_2/a_1 maps the unit circle $|z| < 1$ onto the domain bounded by the curve system C_0 .*

Theorem 2 follows from Theorem 3. Indeed, if the end points of C_0 are $e^{*i\beta}$, $0 < \beta < \pi$, then the function $f(z)$ which maps $|z| < 1$ onto D_0 is given by

$$(21) \quad f(z) = \frac{-(1-z)^2 + (1+z)[1+2pz+z^2]^{\frac{1}{2}}}{(1-z)^2 + (1+z)[1+2pz+z^2]^{\frac{1}{2}}},$$

where

$$p = 1 - \frac{8r^2}{(1+r^2)^2}$$

and

$$r = \left[\frac{1 - \sin \frac{1}{2}\beta}{1 + \sin \frac{1}{2}\beta} \right]^{\frac{1}{2}}.$$

One easily finds that in (21)

$$\frac{a_2}{a_1} = 2 \cos \frac{1}{2}\beta \frac{(1 - \cos \frac{1}{2}\beta)}{1 + \cos \frac{1}{2}\beta},$$

and that the end points of C_0 are the images of the points $e^{*i\theta}$ where $\cos \theta = -(1 - 2 \cos \frac{1}{2}\beta)$. Since the equality in (19) is attained for this value of θ and a_2/a_1 we obtain that $\cos \frac{1}{2}\beta = \sqrt{2} - 1$. The remaining constants of Theorem 2 can now be obtained from (21) by substituting this value of $\cos \frac{1}{2}\beta$.

3. Proof of Theorem 3. Let us assume that the boundary C of the extremal domain D contains, besides the curve system C_0 , some more curves which do not destroy the slit character and the simple connectedness of the domain. In each domain $|w| > q > 1$ these curves are analytic except for a finite number of branch points, but at points on the unit circumference they may either have branch points of infinite order or a point of accumulation of the branch points or both. We consider the class $\mathcal{C}$ of all simply connected symmetric domains whose boundary contains C_0 and is contained in C , and let S_1 be the subclass of S such that $f(z) \in S_1$ maps $|z| < 1$ onto a domain of the class $\mathcal{C}$. Let D_ω and D_n be any elements of $\mathcal{C}$ and let $f_\omega(z) \in S_1$ and $f_n(z) \in S_1$ be the functions which map $|z| < 1$ onto D_ω and D_n respectively. We say that $D_\omega \subset D_n$ if $f_n^{-1}[f_\omega(z)]$ maps the unit circle $|z| < 1$ into itself. We obtain, in this way, a partially ordered set of domains by inclusion. By Zorn's lemma, we can choose from this partially ordered set a simply ordered subset $\{D_n\}$, such that $D_n \supset D_{n+1}$, which converges to D . Let $f_n(z) \in S_1$ be the function which maps $|z| < 1$ onto D_n and let $a_1^{(n)}$ and $a_2^{(n)}$ be its first and second coefficients in the power series expansion near the origin. By Caratheodory's kernel theorem the sequence $\{f_n(z)\}$ converges uniformly in each closed subdomain of $|z| < 1$ to the extremum function $f(z)$. Hence the ratio $a_2^{(n)}/a_1^{(n)}$ converges to a_2/a_1 .

Let us assume, at first, that in a sufficiently small neighborhood of the points A and $\bar{A}$ there are no points of C which do not belong to C_0 . Let Γ_n be the arc of $|z| = 1$ which is mapped by $f_n(z) \in S_1$ onto C_0^i . By our assumption, there will be no branch points on the outer arc of C_0 , in a small neighborhood of A and $\bar{A}$. Hence $f_n(z) \in S_1$ will be analytic in an arc γ_n of $|z| = 1$, which contains Γ_n in its interior. Further, because $f_n^{-1}[f_{n+1}(z)]$ is a regular univalent function which maps the unit circle into itself and the arc Γ_{n+1} onto Γ_n , we have by Löwner's lemma [2] that $\Gamma_n \supseteq \Gamma_{n+1}$ and $\gamma_n \supseteq \gamma_{n+1}$. Let Γ be the arc of $|z| = 1$ which is mapped by $f(z)$ onto C_0^i and let γ be the corresponding arc of $|z| = 1$ which includes Γ and on which $f(z)$ is analytic. Then Γ is contained within all Γ_n 's and all the $f_n(z)$ are regular analytic on γ . By Schwarz's reflection principle they can be continued analytically through γ beyond the unit circle $|z| < 1$ and, accordingly, the sequence $\{f_n(z)\}$ will converge uniformly on a fixed arc γ which includes Γ in its interior.

For the proof of Theorem 3 we need the following:

Lemma 3. *There exists an n_0 such that for $n > n_0$ and for each $f_\omega(z)$ between*

$f_n(z)$ and $f_{n+1}(z)$ there exists another $f_\Omega(z)$ such that their image domains satisfy the inclusion relation $D_n \supset D_\omega \supset D_\Omega \supset D_{n+1}$ and for which

$$\frac{a_2^{(\omega)}}{a_1^{(\omega)}} > \frac{a_2^{(\Omega)}}{a_1^{(\Omega)}}.$$

Proof. We observe that on the boundary of D_ω there will be some point at which either a new slit will originate or the original slit will proceed. At any such point, let us either add an infinitesimal slit or extend the slit infinitesimally in such a way that the resulting domain belongs to the class $\mathcal{C}$. This is a domain D_Ω which satisfies the required inclusion relation. The variation of $f_\omega(z)$ can be expressed by Löwner's [2] differential equation

$$(22) \quad \frac{\partial f_\omega(z, t)}{\partial t} = -\frac{z}{2} \left[\frac{\kappa + z}{\kappa - z} + \frac{\bar{\kappa} + z}{\bar{\kappa} - z} \right] \frac{\partial f_\omega(z, t)}{\partial z},$$

where $|\kappa| = 1$, and the variation $\delta(a_2^{(\omega)}/a_1^{(\omega)})$ is then given by

$$(23) \quad \delta \left(\frac{a_2^{(\omega)}}{a_1^{(\omega)}} \right) = - \left(\kappa + \frac{1}{\kappa} + \frac{a_2^{(\omega)}}{a_1^{(\omega)}} \right), \quad |\kappa| = 1.$$

The point κ will lie in the complement of γ with respect to the unit circumference. We now assert that we can find an index n_0 such that, for z lying in the complement of γ and $|z| = 1$,

$$(24) \quad z + \frac{1}{z} + \frac{a_2^{(\omega)}}{a_1^{(\omega)}} > 0$$

holds for all $f_\omega(z)$ for which $D_{n_0} \supset D_\omega$. Indeed, otherwise we could find a sequence of domains D_n^* , of the class $\mathcal{C}$, satisfying $D_1^* \supset D_2^* \supset D_3^* \supset \dots$ such that the corresponding sequence of functions $f_n^*(z)$ converges uniformly to $f(z)$. But for such a sequence $a_2^{*(n)}/a_1^{*(n)}$ converges to a_2/a_1 and the above inequality must be fulfilled since on account of (18), in the complement of γ .

$$z + \frac{1}{z} + \frac{a_2}{a_1} \geq \delta > 0$$

holds. Thus the right hand side of (23) is negative. This proves Lemma 3.

As a consequence of Lemma 3 we have

Corollary. For $n > n_0$

$$\frac{a_2^{(n)}}{a_1^{(n)}} > \frac{a_2^{(n+1)}}{a_1^{(n+1)}}.$$

Proof. We consider the partially ordered set of functions $f_\omega(z)$ between $f_n(z)$ and $f_{n+1}(z)$. These functions form a normal and compact family, and hence the existence of the minimum of a_2/a_1 , in this class, is assured. The minimum cannot be attained by $f_n(z)$ or any intermediate function because, by Lemma 3, to any such function there exists another function satisfying the required inclu-

sion relation, which has a smaller value of a_2/a_1 . Hence the minimum is attained only by the function $f_{n+1}(z)$.

We have thus shown that if the boundary of the extremum domain contains some other analytic arcs besides the curve system C_0 , then there exists a uniformly convergent sequence of functions $[f_n(z)]$ for which the ratio $a_2^{(n)}/a_1^{(n)}$, $n > n_0$, is monotonic decreasing and converges to a_2/a_1 . Hence the limit $a_2^{(n)}/a_1^{(n)}$ cannot give the maximum. This proves Theorem 3 when in a sufficiently small neighborhood of A and $\bar{A}$ there are no points of C which do not belong to C_0 .

Let us now suppose that the points A and $\bar{A}$ are branch points of finite or infinite order. As these branches go in the direction away from the unit circle $|w| \leq 1$, they are analytic. Let us denote by I^* and $\bar{I}^*$ any two analytic arcs, belonging to C , which start at A and $\bar{A}$ respectively. We choose $\bar{I}^*$ to be the mirror image of I^* in the real axis. This is permissible because the image domain is symmetric with respect to the real axis. Let i_n and $\bar{i}_n$ denote the arcs of $|z| = 1$ which are mapped by $f_n(z)$ onto I^* and $\bar{I}^*$ respectively. Then the totality of the arcs at A and $\bar{A}$ fall in two classes according as their inverse images lie in the connected circular arcs N or N_1 of $|z| = 1$ which, respectively, contain the point 1 or -1 . Let us denote by C_1 the curve system C_0 together with the arcs I^* , $\bar{I}^*$ and all those arcs at A and $\bar{A}$ which have their inverse images in N_1 . Let us denote by C_1^* the arcs at A and $\bar{A}$ which belong to the curve system C_1 . One can then say that C_1^* consists of those arcs at A and $\bar{A}$ which lie between the unit circumference $|w| = 1$ and the arcs I^* and $\bar{I}^*$. Let Γ_n^* denote the arc of $|z| = 1$ which is mapped by $f_n(z)$ onto C_0^* and C_1^* . If Γ^* is the corresponding arc associated with $f(z)$ then one easily sees that $\Gamma \subset \Gamma^* \subset \Gamma_n^*$ and that the points of the unit circumference, $|z| = 1$, which go into the boundary curves which do not belong to C_1 lie in the complement of Γ . Using the above argument one can show that the boundary of the extremum domain does not contain any curves other than those belonging to C_1 . In particular, the boundary of the extremum domain possesses at the points A and $\bar{A}$ only such analytic arcs which lie between the unit circle $|w| = 1$ and the arcs I and I^* . Since I or I^* is arbitrary it follows that there can at most be one arc at each of the points A and $\bar{A}$. These arcs will be the mirror images of each other in the real axis. Again using Löwner's variation one can show that by cutting these arcs one can increase the value of a_2/a_1 . Hence the boundary of the extremum domain consists only of the curve system C_0 . This completes the proof of Theorem 3 and, hence, the proof of Theorem 2 also.

ACKNOWLEDGEMENT

My thanks are due to Prof. M. SCHIFFER for his many valuable suggestions and keen interest in the progress of the paper.

REFERENCES

- [1] G. JULIA, Sur une equation aux dérivées fonctionnelles liées à la représentation conforme, *Ann. École Norm* (3), **39** (1922), pp. 1-28.

- [2] K. LÖWNER, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, *Math. Ann.*, **89** (1923), pp. 103–121.
- [3] Z. NEHARI, *Conformal Mapping*, McGraw-Hill (1952), pp. 139–140.
- [4] M. SCHIFFER, A method of variation within the family of simple functions, *Proc. London Math. Soc.*, **44** (1938), pp. 432–449.
- [5] V. SINGH, Interior variations and some extremal problems for certain classes of univalent functions, *Pacific Journal of Mathematics*, **7** (1957), pp. 1485–1504.

Stanford University
Stanford, California
and
D. A. V. College
Kanpur, V. P.