On Quasi-Linear Parabolic Equations of the Second Order

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Introduction. Let D denote a bounded $(n + 1)$ -dimensional domain and let its points be denoted by $(x, t) = (x_1, \dots, x_n, t)$. Let D be bounded by two hyperplanes $t = 0$ and $t = t_0 > 0$, and by a surface C (C is a closed set) lying between these hyperplanes. Denote by $\bar{D}$ the closure of D , by B the basis of D ($B = \bar{D} \cap \{t = 0\}$), by D_τ the subdomain $D \cap \{0 < t < \tau\}$, by C_τ the intersection $C \cap \{0 \leq t \leq \tau\}$ and, finally, denote by ∂D_τ (∂D) the set $C_\tau \cup B$ ($C \cup B$). Consider the linear operator

$$(1) \quad Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t}$$

with coefficients defined in $\bar{D}$. In Section 1 and occasionally in Section 2 we shall assume that the coefficients of L satisfy the following assumptions:

(A) *The functions a_{ij} have continuous derivatives up to the fifth order with respect to (x, t) in $\bar{D}$; the functions b_i and c have continuous derivatives up to the third order with respect to (x, t) in $\bar{D}$.*

(B) *L is of parabolic type, that is, there exists a positive constant H such that for every point (x, t) in $\bar{D}$ and for every real vector $\xi = (\xi_1, \dots, \xi_n)$*

$$\sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j \geq H \sum_{i=1}^n \xi_i^2.$$

Throughout Section 1 and occasionally in Section 2 we shall assume that the surface C satisfies the following property:

(E') *To every point P of C there exists an $(n + 1)$ -dimensional sphere V with center P such that the intersection $C \cap V$ can be represented, for some i ($1 \leq i \leq n$), in the form*

$$(2) \quad x_i = h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, t)$$

where h has continuous derivatives up to the sixth order in all its arguments.

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Let $f = f(x, t, u, \nabla u)$ (where $\nabla u = (u_{x_1}, \dots, u_{x_n})$) be defined in the cylinder

$$(3) \quad (x, t) \in \bar{D}, \quad -\infty < u < \infty, \quad -\infty < u_{x_i} < \infty \quad (i = 1, \dots, n),$$

and consider the quasi-linear equation

$$(4) \quad Lu = f(x, t, u, \nabla u)$$

with the Dirichlet data

$$(5) \quad u(x, 0) = g(x) \quad \text{on } B, \quad u(x, t) = k(x, t) \quad \text{on } C.$$

In this paper we consider the problems of existence and uniqueness of solutions of the Dirichlet problem (4), (5). An existence theorem for $n = 1$, $Lu = u_{xx} - u_t$ and D a rectangle was proved by CILIBERTO [2], [3] under the condition

$$(6) \quad |f(x, t, u, w)| \leq A + B(|u| + |w|) \quad (A > 0, B > 0),$$

provided the height of the rectangle D is sufficiently small (depending on the constants A, B). He used the method of integral equations with a fundamental solution as a kernel. More recently SATŌ [11] has considered existence and uniqueness problems for weak solutions of the Dirichlet problem corresponding to the elliptic equation $\Delta u = f(x, y, u, u_x, u_y)$. His methods were extended by PINI [10] to parabolic equations of the form $u_{xx} - u_t = f(x, t, u, u_x)$. In particular, he proved the existence of weak solutions provided f satisfies the inequality

$$(7) \quad |f(x, t, u, w)| \leq A + B|u|^\lambda \quad (0 \leq \lambda < 1, A > 0, B > 0).$$

Both papers use the following well known idea: With the aid of Green's function in the whole domain one defines a transformation $v = Tu$ whose fixed points are the desired weak solutions; then one shows that a general fixed point theorem can be applied to the transformation T .

In this paper we define the transformation $v = Tu$ in a different manner: v is a solution of the equation $Lv = f(x, t, u, \nabla u)$ with the appropriate boundary values. Use is made of an existence theorem which we derived in an earlier paper [7]. We also make use of Green's function for rectangular domains constructed in [8].

The results of this paper contain all those mentioned above as very special cases. In §1 we prove an existence theorem; use is made of SCHAUDER's fixed point theorem [12]. In §2 we use the methods of [10], [11] to establish, by simple comparison considerations, a few uniqueness properties. We, further, restate the existence theorem of §1 in a different form. In §3 we mention that the results of §§1 and 2 can be extended to the case of second order elliptic equations.

We finally remark that existence and stability theorems for systems of the form (4), (5) in semi-infinite cylinders were proved for "small" f (for instance, $|f(x, t, u)| \leq \lambda |u|$ and λ is small) by various authors (for a complete treatment see [8]).

I. AN EXISTENCE THEOREM

1.1. The transformation $v = Tu$. Throughout this section it is assumed that f satisfies a Hölder condition in compact subsets of the cylinder (3) and that (A), (B), (E') hold. For any α ($0 < \alpha < 1$) denote by $C_{1+\alpha}$ the set of functions $u(x, t)$ defined and Hölder continuous (exponent α) in the sense of [7] together with their first x -derivatives in $\bar{D}$, and define in $C_{1+\alpha}$ a norm

$$(1.1) \quad |u|_{1+\alpha} = \text{l.u.b.}_{(x,t) \in D} \left\{ |u(x, t)| + \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} u(x, t) \right| \right\} + H[u] + \sum_{i=1}^n H \left[\frac{\partial u}{\partial x_i} \right]$$

where

$$(1.2) \quad H[v] = \text{l.u.b.}_{P, Q \in D} \frac{|v(P) - v(Q)|}{d(P, Q)^\alpha}$$

is Hölder's coefficient of v (if $P = (x, t)$, $Q = (x', t')$, then $d(P, Q) = (|x - x'|^2 + |t - t'|)^{1/2}$). Similarly define $C_{2+\delta}$ to be the set of function $u(x, t)$ with finite norm defined by

$$(1.3) \quad |u|_{2+\delta} = \sum_{i=0}^2 \{ \text{l.u.b.}_{(x,t) \in D} |D_x^i u(x, t)| + H[D_x^i u] \} + \text{l.u.b.}_{(x,t) \in D} |D_t u(x, t)| + H[D_t u]$$

where $H[v]$ in (1.3) is defined as in (1.2) but with α replaced by δ . In (1.3) $D_x^i u$ denotes any i^{th} partial derivative of u ; the summation is understood to be taken also on all the i^{th} partial derivatives of u . Note that $C_{1+\alpha}$ and $C_{2+\delta}$ are Banach spaces; bounded closed sets in $C_{1+\beta}$ are compact in $C_{1+\alpha}$ whenever $\alpha < \beta < 1$.

For any $M > 0$ define a set C_M in the following way:

$$(1.4) \quad C_M = \{u \mid |u|_{1+\alpha} \leq M\}.$$

For each $u \in C_M$ define a transform $v = Tu$ in the following way: v is the solution in D of the equation

$$(1.5) \quad Lv = f(x, t, u, \nabla u)$$

which vanishes on the boundary $\partial D = B + C$ of D . From [7] we conclude that v exists and is uniquely defined; furthermore, for some positive δ (depending on f and M but independent of the specific u)

$$(1.6) \quad |v|_{2+\delta} \leq \text{const.} \quad |\tilde{f}|_\delta \leq \text{const.} < \infty$$

where $\tilde{f}(x, t) = f(x, t, u(x, t), \nabla u(x, t))$. However, since bounded closed sets in $C_{2+\delta}$ need not be compact in $C_{1+\alpha}$ when $\delta < \alpha$, we cannot use (1.6) to prove that the set $\{Tu \mid u \in C_M\}$ is precompact in the norm $|\cdot|_{1+\alpha}$; a proof of this statement will be given later. We proceed to prove that, under some conditions on f , the transformation T maps C_M into itself.

1.2. T maps C_M into itself. One has to prove that

$$(1.7) \quad |Tu|_{1+\alpha} \leq M.$$

1.2.1. Green's function. To prove (1.7) we denote by $|\cdot|_p^{D^0}$ the norm $|\cdot|_p$ taken with respect to the domain D^0 instead of D . Let D^0 be an $(n+1)$ -dimensional rectangle contained in D all of whose edges are parallel to the coordinate axes. In [8] we constructed Green's function $G(x, t; \xi, \tau)$ corresponding to L^* , the adjoint of L , provided L satisfied the assumption (B) and an assumption much weaker than (A). $G(x, t; \xi, \tau)$ is defined for (x, t) and (ξ, τ) in D^0 , $t < \tau$; vanishes on the lateral boundary C^0 of D^0 ; satisfies for every fixed (ξ, τ) the equation $L^* G = 0$, and for every fixed (x, t) the equation $LG = 0$; satisfies the relation

$$(1.8) \quad G(x, t; \xi, \tau) = G^*(\xi, \tau; x, t)$$

where G^* is Green's function corresponding to L in D^0 ; finally, when $\tau \searrow t$ G behaves "essentially" like

$$(1.9) \quad \frac{1}{(\tau - t)^{\frac{1}{2}n}} \exp \left\{ -\frac{\sum A_{ij}(x, t)(x_i - \xi_i)(x_i - \xi_j)}{4(\tau - t)} \right\}$$

where $(A_{ij}(x, t))$ is the matrix inverse to $(a_{ij}(x, t))$.

In what follows we shall need the following inequalities:

$$(1.10) \quad |D_\xi^i D_x^j G(x, t; \xi, \tau)| \leq \frac{A_0}{(\tau - t)^{\frac{1}{2}(n+i+j)}} \exp \left\{ -\frac{A|x - \xi|^2}{\tau - t} \right\} \\ \equiv A_0 V_{n+i+j}(x, t; \xi, \tau) \quad (i, j = 0, 1, 2),$$

$$(1.11) \quad |D_\xi^i D_x^j D_\tau G(x, t; \xi, \tau)| \leq A_0 V_{n+i+j+2} \quad (i, j = 0, 1);$$

here A, A_0 are positive constants. In the sequel A_i will be used to denote appropriate constants.

Proof of (1.10), (1.11). The proofs of (1.10) for $i, j = 0, 1$ and of (1.11) for $i = j = 0$ follow from the explicit form of G (in [8]). From the equation

$$\frac{\partial G}{\partial \tau} = \left(L^* - \frac{\partial}{\partial \tau} \right) G$$

($L^* - \partial/\partial\tau$ involves only ξ -derivatives of the first two orders) we conclude that the proof of (1.11) will be completed if we show that

$$(1.12) \quad |D_x^i D_\xi^j G(x, t; \xi, \tau)| \leq A_0 V_{n+i+j}(x, t; \xi, \tau) \quad (i + j = 3, 4).$$

From the explicit formula of G as a convergent series of fundamental solutions [8] and from the method of EIDELMAN [5] to prove differentiability of fundamental solutions (in particular, [5; formulas (2.14_m), (2.39)]) we conclude that the assumptions (A), (B) are sufficient to ensure the validity of (1.12).

1.2.2. Estimation of Tu in the interior. For simplicity we take D^0 to be a semi-cube with edge d and top $P^0 = (x^0, t^0)$, that is, D^0 is defined by

$$-d < x_i - x_i^0 < d \quad (i = 1, \dots, n), \quad d^2 < t - t^0 < 0 \quad (x^0 = (x_1^0, \dots, x_n^0)).$$

We assume that $\bar{D}^0$, the closure of D^0 , is contained in $\bar{D} - \partial D$. Denote by $\varphi(x, t)$ an infinitely differentiable function which is equal to 1 in $\bar{D}_{\frac{1}{2}}$ and vanishes outside of $\bar{D}_{\frac{1}{2}}$; here D_λ ($\lambda > 0$) denotes the semi-cube with edge λd and top P^0 . If we use Green's identity with $v(x, t)$ and $\varphi(x, t) G(x, t; \xi, \tau)$ where

$$(1.13) \quad Lv(x, t) = f(x, t, u(x, t), \nabla u(x, t)) \equiv \tilde{f}(x, t),$$

then we obtain, as in [6], the representation

$$(1.14) \quad \begin{aligned} v(\xi, \tau) &= \int_{t'}^\tau \int_{B^0} \varphi(x, t) \tilde{f}(x, t) G(x, t; \xi, \tau) dx dt \\ &\quad - \int_{t'}^\tau \int_{B^0} v(x, t) L^*[\varphi(x, t) G(x, t; \xi, \tau)] dx dt \equiv H_0 - J_0 \end{aligned}$$

where $t' = t^0 - d^2$, B^0 is the basis of D^0 , and where H_0 and J_0 denote the first and second integrals respectively; (1.14) is valid provided the distance $\{|\xi - x^0|^2 + |\tau - t^0|\}^{\frac{1}{2}}$ from (ξ, τ) to P^0 is less than $\frac{1}{2}d$.

We note that $L^*[\varphi G]$ (in J_0) vanishes on the sets $\bar{D}_{\frac{1}{2}}$ and $D^0 - \bar{D}_{\frac{1}{2}}$; hence the integration in J_0 has to be taken only on the set $\bar{D}_{\frac{1}{2}} - \bar{D}_{\frac{1}{2}}$. Using the fact that $D_\xi^2 D_x G$, $D_\tau D_x G$ and $D_\xi D_\tau D_x G$ are continuous for $(x, t) \neq (\xi, \tau)$, we easily find that

$$(1.15) \quad |J_0|_{1+\alpha}^{D^0} \leq A_1 \text{ l.u.b.}_{D^0} |v|.$$

We proceed to prove that

$$(1.16) \quad |H_0|_{1+\alpha}^{D^0} \leq A_2 \text{ l.u.b.}_{D^0} |\tilde{f}|.$$

1.2.3. Continuation. Proof of (1.16). It will be enough to prove that

$$(1.17) \quad |D_\xi H_0(\xi, \tau) - D_\xi H_0(\xi', \tau')| \leq A_3 (\text{l.u.b.}_{D^0} |\tilde{f}|) \{|\xi - \xi'|^\alpha + |\tau - \tau'|^{\frac{1}{2}\alpha}\}.$$

We shall prove (1.17) in two special cases: first in case $\tau = \tau'$ and then in case $\xi = \xi'$; the proof of the general case is then very easily completed.

Case (i): $\tau = \tau'$. Writing

$$(1.18) \quad D_\xi H_0(\xi, \tau) = \int_{t'}^\tau \int_{B^0} \varphi(x, t) \tilde{f}(x, t) D_\xi G(x, t; \xi, \tau) dx dt$$

and denoting $h^* = (0, \dots, 0, h, 0, \dots, 0)$, we see that all we need to prove is that

$$(1.19) \quad \int_{t'}^\tau \int_{B^0} |D_\xi G(x, t; \xi + h^*, \tau) - D_\xi G(x, t; \xi, \tau)| dx dt \leq A_4 h^\alpha$$

where we assumed (without loss of generality) that $h > 0$.

Let $\gamma = h^2$ and denote by D_γ^0 the domain $D^0 \cap \{t < \gamma\}$ and by S_γ the domain $D_\tau^0 - D_{\tau-\gamma}^0$ (we assume $\tau - \gamma \geq t'$; otherwise we replace $\tau - \gamma$ by t'). Then,

substituting (for each fixed t) $|x - \xi| = (\tau - t)^{\frac{1}{2}}\rho$ and using (1.10), we get

$$\begin{aligned}
 (1.20) \quad & \iint_{S_\gamma} |D_\xi G(x, t; \xi + h^*, \tau) - D_\xi G(x, t; \xi, \tau)| \, dx \, dt \\
 & \leq 2A_0 \iint_{S_\gamma} V_{n+1}(x, t; \xi, \tau) \, dx \, dt \\
 & \leq A_5 \int_{\tau-\gamma}^{\tau} \int_0^{\infty} \frac{e^{-A\rho^2} \rho}{\tau - t} (\tau - t)^{\frac{1}{2}} \, d\rho \, dt \leq A_6 \gamma^{\frac{1}{2}}.
 \end{aligned}$$

To estimate the rest of the integral in (1.19) we shall make use of the formula

$$(1.21) \quad D_\xi G(x, t; \xi + h^*, \tau) - D_\xi G(x, t; \xi, \tau) = \int_{\xi}^{\xi+h^*} D_\xi^2 G(x, t; \zeta, \tau) \, d\zeta$$

where ζ varies in the interval connecting ξ with $\xi + h^*$. We obtain

$$\begin{aligned}
 & \iint_{D^0_{\tau-\gamma}} |D_\xi G(x, t; \xi + h^*, \tau) - D_\xi G(x, t; \xi, \tau)| \, dx \, dt \\
 & \leq \int_{\xi}^{\xi+h^*} d\zeta \iint_{D^0_{\tau-\gamma}} |D_\xi^2 G(x, t; \zeta, \tau)| \, dx \, dt.
 \end{aligned}$$

Using (1.10) we find that the last integral is bounded by

$$A_7 \int_{\xi}^{\xi+h^*} d\zeta \int_{t'}^{\tau-\gamma} \frac{dt}{\tau - t} \leq A_8 h \gamma^{-\epsilon}$$

where ϵ is an arbitrary positive number (A_8 depends on ϵ).

Combining the last result with (1.20) and taking $\gamma = h^2$, $2\epsilon < 1 - \alpha$, the proof of (1.19) is completed.

Case (ii): $\xi = \xi'$. Let $\gamma > 0$, $h > 0$. Denoting $S_\gamma = D_\tau^0 - D_{\tau-\gamma}^0$, $S_h = D_{\tau+h}^0 - D_\tau^0$ and using (1.18), we get

$$\begin{aligned}
 (1.22) \quad & |D_\xi H_0(\xi, \tau + h) - D_\xi H_0(\xi, \tau)| \leq A_9(\text{l.u.b. } |\tilde{f}|) \\
 & \cdot \left\{ \iint_{D^0_{\tau-\gamma}} |D_\xi G(x, t; \xi, \tau + h) - D_\xi G(x, t; \xi, \tau)| \, dx \, dt \right. \\
 & + \iint_{S_\gamma} [|D_\xi G(x, t; \xi, \tau + h)| + |D_\xi G(x, t; \xi, \tau)|] \, dx \, dt \\
 & \left. + \iint_{S^0_h} |D_\xi G(x, t; \xi, \tau + h)| \, dx \, dt \right\} = A_9(\text{l.u.b. } |\tilde{f}|)(I_1 + I_2 + I_3)
 \end{aligned}$$

where I_j denotes the j^{th} integral on the right side of the inequality. To estimate I_3 we use (1.10) and then substitute $|x - \xi| = \rho(\tau - t)^{\frac{1}{2}}$. We obtain

$$(1.23) \quad I_3 \leq A_{10} h^{\frac{1}{2}}.$$

Similarly we obtain

$$(1.24) \quad I_2 \leq A_{11}(\gamma^{\frac{1}{2}} + h^{\frac{1}{2}}).$$

It remains to estimate I_1 . If we use the formula

$$D_{\xi}G(x, t; \xi, \tau + h) - D_{\xi}G(x, t; \xi, \tau) = \int_{\tau}^{\tau+h} D_{\xi}D_{\sigma}G(x, t; \xi, \sigma) d\sigma$$

and (1.11), then we obtain (on substituting $|x - \xi| = \rho(\sigma - t)^{\frac{1}{2}}$)

$$(1.25) \quad I_1 \leq A_{12}h\gamma^{-\frac{1}{2}}.$$

Combining (1.25) with (1.24), (1.23) and (1.22), and taking $\gamma = h$, the proof of (1.17), in case $\xi = \xi'$, is completed.

Conclusion. Having completed the proof of (1.16) we obtain, on using (1.15) and (1.14), the inequality

$$(1.26) \quad |v|_{1+\alpha}^{\bar{D}_1/\alpha} \leq A_1 \text{ l.u.b.}_{D^0} |v| + A_2 \text{ l.u.b.}_{D^0} |\tilde{f}|.$$

A simple argument shows that for any closed subdomain D_1 of $\bar{D} - \partial D$ the inequality

$$(1.27) \quad |v|_{1+\alpha}^{D_1} \leq A_{13} \text{ l.u.b.}_D |v| + A_{14} \text{ l.u.b.}_D |\tilde{f}|$$

holds with constants A_{13} , A_{14} depending on L and D_1 .

1.2.4. Estimation of Tu near the boundary. It remains to study the behavior of Tu near the boundary $\partial D = B \cup C$ of D . Let P be a point on ∂D . There are three possibilities: either $P \in B$ or $P \in C$ or $P \in B \cap C$. It will be enough to consider the last case; the first two cases can be treated in the same manner. Let V be an $(n+1)$ -dimensional sphere with center P so that $C \cap V$ can be represented in the form (2) with h having six continuous derivatives. For simplicity we assume that $i = n$ in (2).

The transformation

$$(1.28) \quad \begin{aligned} t' &= t, & x'_i &= x_i & (i = 1, \dots, n-1), \\ x'_n &= x_n - h(x_1, \dots, x_{n-1}, t) \end{aligned}$$

maps a certain one-sided closed neighborhood of $C \cap V$, call it E ($E \subset \bar{D}$), onto a semi-cube E' with edges parallel to the x'_i and t' axes. Let E' be defined by

$$\begin{aligned} |x'_i - \beta_i| &\leq d & (i = 1, \dots, n-1), \\ -d < x'_n < 0, & & 0 < t' < d^2. \end{aligned}$$

Equation (1.13) is transformed into $L'w = \tilde{f}'$ where $w(x', t') = v(x, t)$, $\tilde{f}'(x', t') = \tilde{f}(x, t)$ and L' satisfies in E' the assumptions (A), (B) (here we made use of the differentiability properties of h).

If we prove that

$$(1.29) \quad |w|_{1+\alpha}^{\bar{E}'} \leq A_{15} \text{ l.u.b.}_{E'} |w| + A_{16} \text{ l.u.b.}_{E'} |\tilde{f}'|$$

where E'' is the rectangle

$$(1.30) \quad \begin{aligned} |x'_i - \beta_i| &\leq \frac{1}{2}d \quad (i = 1, \dots, n-1), \\ -\frac{1}{2}d &< x'_n < 0, \quad 0 < t' < d^2, \end{aligned}$$

then it will easily follow that

$$(1.31) \quad |v|_{1+\alpha}^{E_0} \leq A_{17} \text{ l.u.b.}_E |v| + A_{18} \text{ l.u.b.}_E |\tilde{f}|$$

where E'' is the image of E_0 under the transformation inverse to (1.28). Combining (1.31) with (1.27) and using the Heine-Borel theorem we conclude that

$$(1.32) \quad |v|_{1+\alpha}^D \leq A_{19} \text{ l.u.b.}_D |v| + A_{20} \text{ l.u.b.}_D |\tilde{f}|.$$

Proof of (1.29). Let $G'(x, t'; \xi', \tau')$ be Green's function corresponding to the adjoint of L' in E' and write Green's identity with $w(x', t')$ and $G'(x', t'; \xi', \tau')$ (not $\varphi G'$!). Noting that G' vanishes on the lateral boundary of E' and that w vanishes on the faces $x'_n = 0$ and $t' = 0$ of E' , we obtain a representation analogous to (1.14) with one important difference: J_0 does not appear and, on the other hand, there appear boundary integrals of the form

$$\int_{S_i} g_i \left[\frac{\partial}{\partial x'} G'(x', t'; \xi', \tau') \right]_{x'=\text{const.}} dS$$

where the g_i are appropriate functions (products of v with coefficients of L') and the S_i are appropriate faces of E' , but not the faces $x'_n = 0$ and $t' = 0$. Noting that $\partial^3 G' / \partial x' \partial \xi'^2$, $\partial^3 G' / \partial x' \partial \tau'$ and $\partial^3 G' / \partial x' \partial \xi' \partial \tau'$ are continuous functions provided $|x' - \xi'|^2 + |t' - \tau'| > 0$, the proof of (1.29) is easily completed.

1.2.5. Conclusion. From the maximum principle for parabolic equations [9] it follows that

$$(1.33) \quad \text{l.u.b.}_D |v| \leq A_{21} \text{ l.u.b.}_D |\tilde{f}|.$$

Using (1.33) in (1.32) we get

$$(1.34) \quad |v|_{1+\alpha} \leq K \text{ l.u.b.}_D |f(x, t, u, \nabla u)| \quad (u = u(x, t) \in C_M)$$

where K depends only on L and D .

To complete the proof of (1.7) we shall have to make the following assumption on f :

(C) *There exists a positive constant M such that*

$$(1.35) \quad K |f(x, t, u, \nabla u)| \leq M$$

holds for all $(x, t) \in \bar{D}$ and $u = u(x, t) \in C_M$ (that is, $|u|_{1+\alpha} \leq M$) where K is the constant of (1.34) (which depends only on L and D).

From (1.34) we conclude: If f satisfies Property (C) then (1.7) holds and, consequently, T maps C_M into itself.

We now observe that the proof of (1.34) holds for any $\alpha < 1$. Taking, in particular, β such that $\alpha < \beta < 1$ we have

$$|v|_{1+\beta} \leq \text{const. l.u.b.}_D |f(x, t, u, \nabla u)| \leq K' \quad (v = Tu)$$

where K' is independent of the specific u in C_M . Hence T maps C_M into a bounded set in $C_{1+\beta}$. Since bounded sets in the norm $|\cdot|_{1+\beta}$ are precompact in the norm $|\cdot|_{1+\alpha}$, we have proved that T maps C_M into a precompact subset of C_M .

1.3. Existence of solutions. Having proved that T maps C_M into a precompact subset of C_M we now note that C_M is a closed bounded convex set in the Banach space $C_{1+\alpha}$. We further note that T is a continuous transformation (if $|u - u_m|_{1+\alpha} \rightarrow 0$ then $|Tu - Tu_m|_{1+\alpha} \rightarrow 0$) and therefore also completely continuous. Hence we can apply SCHAUDER's fixed point theorem [12] and thus conclude that there exists a function u in C_M for which $u = Tu$. Also $u \in C_{2+\delta}$, $Lu = f(x, t, u, \nabla u)$ and u vanishes on ∂D ; the proof of the existence of a solution is thus completed. We have established the following theorem:

Theorem 1. *Let L satisfy the assumptions (A), (B) and let the lateral boundary C of D satisfy the assumption (E'). Let f be defined in the cylinder (3) and satisfy a Hölder condition in compact subsets of it. If f satisfies Property (C) then there exists in D a solution of equation (4) which vanishes on $\partial D = B + C$ (B is the basis of D); moreover, for some $\delta > 0$, $|u|_{2+\delta} < \infty$.*

1.3.1. Examples of functions satisfying Property (C). From Theorem 1 it follows that it is desirable to find simple conditions on f under which Property (C) is satisfied. We shall assume that f satisfies the inequality

$$(1.36) \quad |f(x, t, u, w)| \leq A + B |u|^\lambda + C |w|^\mu,$$

where w is an n -dimensional variable, and find conditions on the positive constants A, B, C, λ, μ under which Property (C) holds.

Case (i): $\lambda < 1, \mu < 1$.

We claim that for any constants A, B, C Property (C) is satisfied. Indeed, from (1.35), (1.36) it follows that all we need to show is that

$$(1.37) \quad KA + KBM^\lambda + KCM^\mu \leq M$$

holds for some positive M ; clearly this is the case if M is sufficiently large.

Case (ii): λ, μ, A arbitrary.

We claim that if B and C are sufficiently small then Property (C) is satisfied. Indeed, take in (1.37) $M = 2KA$. Then (1.37) is reduced to

$$(1.38) \quad BM^\lambda + CM^\mu \leq A$$

which is clearly satisfied for small B and C .

One can easily write down other conditions under which (1.37) is satisfied. We remark that a careful examination of the estimates which led to (1.34) shows that if we replace D by D_τ then K depends on τ and $K/\tau^\beta \rightarrow 0$ when $\tau \rightarrow 0$ for any positive β smaller than $\frac{1}{2}$, provided α is taken to be sufficiently small ($2\beta < 1 - \alpha$). Hence, for any constants A, B, C, λ, μ , (1.37) is satisfied in D_τ provided τ is sufficiently small. Thus if f satisfies (1.36) with any constants A, B, C, λ, μ then there exists a solution of the Dirichlet problem

$$Lu = f(x, t, u, \nabla u) \quad \text{in } D_\tau, \quad u = 0 \quad \text{on } \partial D_\tau,$$

provided τ is sufficiently small.

In the next section we shall replace Property (C) by a different property and then find other simple conditions on f under which there exists a solution of the Dirichlet problem.

1.4. Remarks and extensions. If D is a rectangular domain then Theorem 1 can be proved without referring to the results of [7]. Indeed, we then define the transformation $v = Tu$ in the following way:

$$(1.39) \quad v(\xi, \tau) = \iint_{D_\tau} G(x, t; \xi, \tau) f(x, t, u(x, t), \nabla u(x, t)) \, dx \, dt$$

where G is Green's function in the domain D . This transformation we define on the set F_M of functions $u(x, t)$ with finite norms

$$(1.40) \quad |u|_1 = \text{l.u.b.}_D (|u| + |\nabla u|) \leq M$$

and prove that, for any positive $\alpha < 1$,

$$(1.41) \quad |v|_{1+\alpha} \leq \text{const. l.u.b.}_D |f|.$$

Then by SCHAUDER's fixed point theorem [12] it follows that there exists a function u in F_M for which $u = Tu$. If f is Hölder continuous in compact subsets of the cylinder (3), then it follows (on using the explicit formula for G and [4; Theorems 1, 2]) that u is a solution of (4). From [8] we also conclude that u vanishes on ∂D .

We remark that in deriving (1.41) we find that the constant which appears on the right side can be taken to be of the form $K |D|^\gamma$ where K depends on γ, L and the diameter $|B|$ of the basis B , $|D|$ is the diameter of D and γ is any positive constant smaller than $\frac{1}{2}$ (see also [10], [11]).

1.4.1. Weak solutions. If we consider the fixed points of the transformation $v = Tu$ defined by (1.39) as weak solutions of the Dirichlet problem, then in order to prove the existence of weak solutions it is sufficient to show that the set F_M defined by (1.40) is mapped by T into a set of functions in F_M which are equally continuous together with their first x_j -derivatives. (By ARZELA's theorem, such a set is precompact in the norm $|\cdot|_1$.)

To prove the equal-continuity of the set $\{Tu \mid u \in F_M\}$ we need to make

use only of (1.10) with $i, j = 0, 1$. The proofs of these inequalities follow from [8] under the assumptions (A'), (B) on L ; assumption (A') is defined as follows:

(A') *The functions*

$$(1.42) \quad a_{ij}, \quad \frac{\partial}{\partial x_\lambda} a_{ij}, \quad \frac{\partial^2}{\partial x_\lambda \partial x_\mu} a_{ij}, \quad \frac{\partial}{\partial t} a_{ij}, \quad b_i, \quad \frac{\partial}{\partial x_\lambda} b_i, \quad c$$

are Hölder continuous in $\bar{D}$.

We have thus proved: *If D is a rectangular domain and if the assumptions (A'), (B), (C) hold and f is continuous in the cylinder (3), then there exists a weak solution $u(x, t)$ of the equation (4) which vanishes on the boundary ∂D .*

The last part of the theorem (that $u = 0$ on ∂D) follows from [8].

1.4.2. The case $f = f(x, t, u)$. If $f = f(x, t, u)$ and D is a general domain as in Theorem 1, then to prove existence of solutions (in the usual sense) we can start with C_M defined in the following way:

$$(1.43) \quad C_M = \{u \mid |u|_\alpha \leq M\}.$$

We then need to make use only of (1.10) with $i + j \leq 3$ and (1.11) with $i + j \leq 1$. Consequently, we can replace the assumptions (A), (E') by weaker assumptions. It is enough, for instance, to assume that a_{ij} have four continuous derivatives in $\bar{D}$, b_i and c have continuous second derivatives in $\bar{D}$ and the functions h (in the local representation of C) have five continuous derivatives in all their arguments.

2. UNIQUENESS THEOREMS. ANOTHER EXISTENCE THEOREM

In this section, unless the contrary is explicitly stated, $f(x, t, u, w)$ (w is n -dimensional) is assumed to be merely continuous in the cylinder (3), the coefficients of L are assumed to be merely continuous in $\bar{D}$ and the coefficient $c(x, t)$ of u in L is assumed to be non-positive. A function $v(x, t)$ defined in $\bar{D}$ is said to be *regular* if it is continuous in $\bar{D}$ and if the functions

$$\frac{\partial v}{\partial x_\lambda}, \quad \frac{\partial^2 v}{\partial x_\lambda \partial x_\mu}, \quad \frac{\partial v}{\partial t}$$

are continuous in D . Unless the contrary is explicitly stated, all functions in this section are assumed to be regular.

We shall now prove a few simple theorems concerned with uniqueness of solutions of the equation $Lu = f$. Later on we shall combine some of these theorems with Theorem 1 and obtain a new existence theorem (Theorem 2).

2.1. *Let the functions $u, \bar{u}$ satisfy $Lu = f$, $L\bar{u} = \bar{f}$ where $\bar{f}$ satisfies the same properties as f , and assume that*

$$f(x, t, u, w) > \bar{f}(x, t, \bar{u}, w) \quad \text{in } D \quad \text{whenever } \bar{u} < u.$$

If $\bar{u} \geq u$ ($\bar{u} > u$) on ∂D , then $\bar{u} \geq u$ ($\bar{u} > u$) in D .

Proof. It will be enough to consider the case $\bar{u} \geq u$ on ∂D . If the assertion does not hold, then $v = \bar{u} - u$ assumes its negative minimum at an interior point (x^0, t^0) of D . At this point $\nabla u_x = \nabla \bar{u}_x$ and $\bar{u} < u$; consequently,

$$Lv(x^0, t^0) = \bar{f}(x^0, t^0, \bar{u}(x^0, t^0), \nabla u(x^0, t^0)) - f(x^0, t^0, u(x^0, t^0), \nabla u(x^0, t^0)) < 0$$

which involves a contradiction since $Lv \geq 0$ at every interior point where v assumes its negative minimum.

2.2. Let $\bar{u}$, $\underline{u}$ and u satisfy the following properties: $L\bar{u} < f(x, t, u, \nabla \bar{u})$ whenever $\bar{u} < u$, $L\underline{u} > f(x, t, u, \nabla \underline{u})$ whenever $\underline{u} > u$ and $Lu = f(x, t, u, \nabla u)$. If $\bar{u} \geq u \geq \underline{u}$ ($\bar{u} > u > \underline{u}$) on ∂D , then $\bar{u} \geq u \geq \underline{u}$ ($\bar{u} > u > \underline{u}$) in D .

The proof is similar to that of 2.1.

2.3. Let $f(x, t, u, 0)$ be monotone increasing in u and let $f(x, t, 0, 0) = 0$. Then zero is the only solution of the equation $Lu = f(x, t, u, \nabla u)$ which vanishes on ∂D .

Proof. Apply 2.2 with $\underline{u} = 0$ and with $\bar{u} = 0$.

2.4. Assume that $f(x, t, u, w) < f(x, t, \bar{u}, w)$ whenever $u < \bar{u}$. Then there exists at most one solution of the Dirichlet problem $Lu = f$ with non-homogeneous boundary values $u = u_0$ on ∂D .

Proof. Suppose there exist two solutions u^1 and u^2 and let $u = u^1 - u^2$. If $u \not\equiv 0$ then we may assume that u takes its positive maximum at an interior point (x^0, t^0) of D (the case of a negative minimum can be treated in the same way). Since $\nabla u^1 = \nabla u^2$ and $u^1 > u^2$ at (x^0, t^0) we get

$$\begin{aligned} Lu(x^0, t^0) &= f(x^0, t^0, u^1(x^0, t^0), \nabla u^1(x^0, t^0)) \\ &\quad - f(x^0, t^0, u^2(x^0, t^0), \nabla u^1(x^0, t^0)) > 0. \end{aligned}$$

On the other hand, we have $Lu \leq 0$ at every interior point where u takes its positive maximum and we have thus arrived at a contradiction.

2.5. The result of 2.4 is valid under the weaker assumption that $f(x, t, u, w) \leq f(x, t, \bar{u}, w)$ whenever $u < \bar{u}$.

Proof. The transformation $v = e^{-\alpha t} u$ ($\alpha > 0$) transforms the equation $Lu = f$ into $Lv = \tilde{f}$ where $\tilde{f} \equiv fe^{-\alpha t} + \alpha v$. We can now apply 2.4 to v and thus conclude that v (and therefore u) is uniquely determined from its boundary values on ∂D .

2.6. If for some constant M

$$|f(x, t, u, w) - f(x, t, \bar{u}, w)| \leq M |u - \bar{u}|,$$

then there exists at most one solution of $Lu = f$ with prescribed boundary values on ∂D .

Proof. The function $v = ue^{-Bt}$ satisfies

$$Lv = f(x, t, u, \nabla u)e^{-Bt} + Bv \equiv \tilde{f}(x, t, v, \nabla v).$$

If we take $B > M$ then our assertion follows from 2.4.

2.7. Let $w, \bar{w}$ be functions which satisfy

$$(2.1) \quad w(x, t) < 0 < \bar{w}(x, t) \quad \text{in } \bar{D}.$$

Let $\tilde{C}$ be the set of points (x, t, u, w) which satisfy

$$(2.2) \quad (x, t) \in \bar{D}, \quad w(x, t) \leq u \leq \bar{w}(x, t), \quad -\infty < w_i < \infty \quad (i = 1, \dots, n),$$

where $w = (w_1, \dots, w_n)$.

If L satisfies (A), (B) and C satisfies (E'), if f is Hölder continuous in the cylinder (3) and satisfies Property (C), and if

$$(2.3) \quad Lw > f(x, t, w, \nabla w), \quad L\bar{w} < f(x, t, \bar{w}, \nabla \bar{w})$$

then the Dirichlet problem $Lu = f$ in D , $u = 0$ on ∂D , has a solution in D which satisfies the inequality

$$(2.4) \quad w(x, t) \leq u(x, t) \leq \bar{w}(x, t) \quad \text{in } D.$$

Proof. Define $F(x, t, u, w) = f(x, t, u', w)$ where u' is defined for every (x, t) in $\bar{D}$ in the following way:

$$u' = \begin{cases} w(x, t) & \text{if } u < w(x, t) \\ u & \text{if } w(x, t) \leq u \leq \bar{w}(x, t) \\ \bar{w}(x, t) & \text{if } u > \bar{w}(x, t). \end{cases}$$

It is elementary to show that, together with f , F is also Hölder continuous. Furthermore,

$$(2.5) \quad \begin{aligned} Lw &> F(x, t, u, \nabla w) \quad \text{whenever } u < w(x, y), \\ L\bar{w} &< F(x, t, u, \nabla \bar{w}) \quad \text{whenever } u > \bar{w}(x, y). \end{aligned}$$

Now, since f satisfies Property (C), also F satisfies it and, consequently, there exists a solution of $Lu = F(x, t, u, \nabla u)$ which vanishes on ∂D . (2.4) is a consequence of 2.2 and (2.5).

2.8. An existence theorem.

Theorem 2. Let L satisfy the assumptions (A), (B) and let C satisfy condition (E'). Assume that there exist two regular functions w and $\bar{w}$ satisfying (2.1) and (2.3), and let $f(x, t, u, w)$ be bounded in the domain $\tilde{C}$ defined by (2.2) and Hölder continuous in compact subsets of $\tilde{C}$. Then there exists a solution $u(x, t)$ in D of the equation (4) which vanishes on ∂D ; u satisfies (2.4) and, for some positive δ , $|u|_{2+\delta} < \infty$.

Proof. The proof follows from 2.7 by noting that the function F constructed in 2.7 satisfies Property (C).

We remark that Theorem 2 remains true if the assumption that $c(x, t) \leq 0$ is removed. Indeed, a transformation of the form $v = ue^{\beta t}$ reduces the case of general c to the case of non-positive c .

2.9. Examples of functions f for which Theorem 2 is valid. It will be enough to consider the case $c(x, t) \leq 0$. Assume that f satisfies the inequality

$$(2.6) \quad |f(x, t, u, w)| \leq A + B |u|^\lambda$$

in the cylinder (3). Take

$$(2.7) \quad \bar{w}(x, t) = \varphi(t) > 0, \quad w(x, t) = -\varphi(t) < 0.$$

Then

$$L\bar{w} = c\varphi - \varphi' \leq -\varphi', \quad Lw = -c\varphi + \varphi' \geq \varphi'.$$

Hence (2.3) is valid if φ satisfies

$$(2.8) \quad \varphi'(t) - B[\varphi(t)]^\lambda - A > 0.$$

Clearly, if (2.8) holds then, in the domain $\tilde{C}$ (defined in terms of $\varphi(t)$), f will be bounded and, consequently, Theorem 2 is valid.

Case 1: $\lambda < 1$.

If we take $\varphi(t) = H(t + 1)$ then (2.8) becomes

$$H - BH^\lambda(t + 1)^\lambda - A > 0$$

which is clearly satisfied for any A, B if H is sufficiently large.

Case 2: $\lambda = 1$.

(2.8) is satisfied by any positive solution of the equation

$$(2.9) \quad \varphi'(t) - B\varphi(t) - A' = 0$$

where A' is any fixed number larger than A . For H positive and sufficiently large

$$\varphi(t) = He^{Bt} - \frac{A'}{B}$$

is a positive solution of (2.9).

Case 3: $\lambda > 1$.

Taking $\varphi(t) = H(t + \epsilon)$ where $\epsilon > 0$, (2.8) is reduced to

$$(2.10) \quad H - BH^\lambda b^\lambda - A > 0$$

(since ϵ may be taken arbitrarily small). Here b is the height of D . Taking $H = 2A$ we reduce (2.10) to

$$A > B(2A)^\lambda b^\lambda.$$

Hence (2.8) is valid provided

$$b < \frac{1}{2} \left(\frac{A^{1-\lambda}}{B} \right)^{1/\lambda}.$$

Note that Case 1 is included in §1, Case 2 is not included in §1 and Case 3 gives a more precise condition than the condition which one can derive from §1 when $\mu = 0$. Case 1 was also considered by PINI [10] using different functions for $\bar{w}$ and ψ .

2.10. Non-homogeneous boundary values. In Theorems 1, 2 we proved the existence of solutions of the Dirichlet problem $Lu = f$ with vanishing boundary values. To discuss the case of the general boundary conditions (5) we assume that there exists a function $\psi(x, t)$ having second x -derivatives and one t -derivative all of which are Hölder continuous in $\bar{D}$ and such that $\psi = g$ on B and $\psi = k$ on C . We then can prove the existence of a solution of the problem (4), (5).

Indeed, writing $v = u - \psi$, our problem is reduced to that of finding a solution of the equation

$$(2.11) \quad Lv = f(x, t, v + \psi, \nabla v + \nabla \psi) - L\psi \equiv \tilde{f}(x, t, v, \nabla v)$$

with zero boundary values. Since $\tilde{f}$ is Hölder continuous in all its arguments in the cylinder (3), we have reduced our present problem to the one considered in Theorems 1, 2.

2.11. Uniqueness. Theorems 1, 2 consider the problem of existence of solutions. We recall that the question of the uniqueness of solutions was considered in 2.4, 2.5 and 2.6. One can easily formulate conditions on f such that the problem (4), (5) has one and only one solution.

3. ELLIPTIC EQUATIONS

The results of §§1, 2 can be extended to second order elliptic equations. Writing such an equation in the form

$$(3.1) \quad \bar{L}u = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f(x, u, \nabla u),$$

we make the following assumptions:

- (i) The coefficients of $\bar{L}$ are defined and Hölder continuous (exponent λ) in the closure $\bar{B}$ of a bounded domain B ; $c(x) \leq 0$;
- (ii) $\bar{L}$ is elliptic at each point of $\bar{B}$;
- (iii) For each point P on ∂B there exists a neighborhood V such that $V \cap \partial B$ can be written, for some i , in the form

$$x_i = h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

where h has Hölder continuous (exponent $\sigma > \lambda$) second x -derivatives.

Under conditions weaker than the above ones, CORDES [1; p. 307] (for $n > 2$) and BERS & NIRENBERG [0; pp. 6–7] (for $n = 2$) have proved that if u is a solution of the equation $Lu = g(x)$ in B , which has second continuous x -derivatives in $\bar{B}$ and which vanishes on ∂B , then there exists an $\alpha > 0$ depending only on L and B such that

$$(3.2) \quad |u|_{1+\alpha} \leq K \text{ l.u.b.}_B |g|,$$

where K depends only on L and B (not on g) and where

$$|u|_{1+\alpha} = \text{l.u.b.}_B (|u| + |\nabla u|) + \text{l.u.b.}_B \frac{|\nabla u(x) - \nabla u(y)|}{|x - y|^\alpha}$$

If we use this result, together with the Schauder theory, then we can extend Theorem 1 (and then also Theorem 2) to elliptic equations; moreover, the assumptions on L and B are weaker than those made in the analogous parabolic case. In particular, we have the following theorem:

Assume that (i), (ii), (iii) hold. Let $f(x, u, \nabla u)$ be defined in the cylinder

$$x \in \bar{B}, \quad -\infty < u, u_{x_i} < \infty,$$

and be Hölder continuous in compact subsets of it. Let the inequality

$$(3.3) \quad |f(x, u, \nabla u)| \leq A + B |u|^\beta + C |u|^\gamma$$

be satisfied for some non-negative constants A, B, C, β, γ . There exists a solution of the equation (3.1) which vanishes on ∂B , provided one of the following two conditions hold:

(a) A, B, C are arbitrary; $\beta < 1, \gamma < 1$;

(b) A, β, γ are arbitrary; B and C are sufficiently small.

Moreover, for some $\delta > 0$, $|u|_{1+\delta} + |\nabla u|_{1+\delta} < \infty$.

Added in proof. When we prepared Sections 1, 2 of this paper, we were not familiar with the estimate of CORDES [1] mentioned above. We plan to try to prove an analogous estimate for parabolic equations. With such an estimate at hand, Theorems 1, 2 will hold under very weak assumptions on L and D , analogous to the assumptions (i), (ii), (iii) of the present section.

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