Boundary Estimates for Second Order Parabolic Equations and Their Applications

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Communicated by E. HOPF

Introduction. In [5] we derived interior Schauder type estimates for parabolic systems of any order. Those estimates have been used to prove differentiability theorems for both linear and nonlinear parabolic systems. The problem of deriving analogous estimates near the boundary, for general parabolic systems, is still open. However, for second order parabolic equations, these estimates have been derived by RICHARD B. BARRAR (in his unpublished thesis [0]) and later on, for a one-dimensional space variable, by CILIBERTO [1]. In this paper we shall derive these estimates by a simpler method suggested from the analogous elliptic case which was considered in [2] by DOUGLIS & NIRENBERG. Our derivation might serve as a model to derive the analogous boundary estimates for general parabolic systems.

There is one essential difficulty which arises in the parabolic case (for boundary estimates) and which does not arise in the analogous elliptic case. This is explained in the proof of Theorem 1 (Section 2). To overcome that difficulty we prove (in Section 1) Lemma 1. Theorem 1 is analogous to Theorem 1 in [5]. In Section 3 we prove the desired estimates, via Theorem 1. The proof is similar to that of [5]. Finally, in Sections 4 and 5 we give applications of the boundary estimates by deriving existence theorems for linear and some nonlinear parabolic equations. Our method involves a well known continuity argument (see, for instance, SCHAUDER [13], NIRENBERG [11]). Theorem 4 was also proved by R. B. BARRAR by the continuity argument, though in other respects his proof differs from our proof.

1. A FUNDAMENTAL LEMMA

Denote by R the interval $-1 < x < 1$, by S_τ ($S_1=S$) the rectangle $-1 < x < 1$, $0 < t < \tau$, and by ∂S_τ the *normal* boundary of S_τ , that

Prepared under ONR Contract Nonr-908(09), NR 041 037 with Indiana University.

is, the boundary of S , minus the set $-1 < x < 1, t = \tau$. Let $f(x, t)$ be defined and Hölder continuous (exponent α) in $\bar{S}$ (the closure of S) in the sense of [5], that is,

$$(1.1) \quad \frac{|f(x, t) - f(x', t')|}{(|x - x'|^2 + |t - t'|)^{\frac{1}{2}\alpha}} \leq C$$

for all points $(x, t) \in \bar{S}, (x', t') \in \bar{S}$. Denote the smallest possible C by $H[f]$. In what follows we shall use the notation

$$(1.2) \quad V_i(x, t; \xi, \tau) = \frac{1}{(\tau - t)^{\frac{1}{2}i}} \exp \left\{ -\frac{|x - \xi|^2}{4(\tau - t)} \right\}.$$

We can now state the main result of this section.

Lemma 1. *Let $f(\pm 1, 0) = 0$ and define*

$$(1.3) \quad v(\xi, \tau) = \frac{1}{2\pi^{\frac{1}{2}}} \int_0^\tau \int_x V_1(x, t; \xi, \tau) f(x, t) dx dt.$$

For any pair of points $(\xi, \tau), (\xi', \tau')$ in S

$$(1.4) \quad \frac{|v_\tau(\xi, \tau) - v_{\tau'}(\xi', \tau')|}{((\xi - \xi')^2 + |\tau - \tau'|)^{\frac{1}{2}\alpha}} \leq AH[f]$$

where A is a constant depending only on α .

Proof. The proof for $\xi = \xi'$ was essentially given by GEVREY [8; 360–362]. We shall therefore present here only a proof for the case $\tau = \tau'$; the general case follows immediately by combining these two special cases. We can assume in what follows that $0 < \xi' < \xi < 1$.

We shall make use of the formula

$$(1.5) \quad \begin{aligned} 2\pi^{\frac{1}{2}} \frac{\partial^2}{\partial \xi^2} v(\xi, \tau) &= \sum \int_0^\tau \left[\pm \frac{\partial}{\partial \xi} V_1(\pm 1, t; \xi, \tau) \right] f(\xi, t) dt \\ &\quad - \iint_{S_\tau} \left[\frac{\partial^2}{\partial \xi^2} V_1(x, t; \xi, \tau) \right] [f(x, t) - f(\xi, t)] dx dt = w_1 - w_2 \end{aligned}$$

where w_2 denotes the last integral and w_1 denotes the sum of the first two integrals. The proof of (1.5) is elementary and direct (see [8]). Using the fact that $v_{\xi\xi} - v_\tau = f$, we can transform (1.5) into the form

$$(1.6) \quad 2\pi^{\frac{1}{2}} \frac{\partial}{\partial \tau} v(\xi, \tau) = w_1 - w_2 - 2\pi^{\frac{1}{2}} f(\xi, \tau).$$

We proceed to prove the Hölder continuity of $w_1 - w_2$.

Let $\gamma = (\xi - \xi')^2$ and let $B_\gamma = S_\tau - S_{\tau-\gamma}$. We have

$$w_2(\xi, \tau) - w_2(\xi', \tau) = \iint_{B_\gamma} \left[\frac{\partial^2}{\partial \xi^2} V_1(x, t; \xi, \tau) \right] [f(x, t) - f(\xi, t)] dx dt$$

$$\begin{aligned}
 & - \iint_{B_\gamma} \left[\frac{\partial^2}{\partial \xi'^2} V_1(x, t; \xi', \tau) \right] [f(x, t) - f(\xi', t)] dx dt \\
 (1.7) \quad & + \iint_{S_{\tau-\gamma}} \left[\frac{\partial^2}{\partial \xi^2} V_1(x, t; \xi, \tau) - \frac{\partial^2}{\partial \xi'^2} V_1(x, t; \xi', \tau) \right] [f(x, t) - f(\xi, t)] dx dt \\
 & - \iint_{S_{\tau-\gamma}} \left[\frac{\partial^2}{\partial \xi^2} V_1(x, t; \xi', \tau) \right] [f(\xi, t) - f(\xi', t)] dx dt \\
 & = J_1 - J_2 + J_3 - J_4
 \end{aligned}$$

where J_i denotes the i^{th} integral. We proceed to estimate the J_i . In what follows A_i will be used to denote appropriate constants depending only on α .

We first have

$$\begin{aligned}
 (1.8) \quad |J_1| & \leq A_1 H[f] \int_{\tau-\gamma}^{\tau} \int_R V_3(x, 2t; \xi, 2\tau) |x - \xi|^\alpha dx dt \\
 & \leq A_2 H[f] \int_{\tau-\gamma}^{\tau} \frac{dt}{(\tau - t)^{1-\frac{1}{2}\alpha}} = \frac{2}{\alpha} A_2 H[f] \gamma^{\frac{1}{2}\alpha} = \frac{2}{\alpha} A_2 H[f] (\xi - \xi')^\alpha.
 \end{aligned}$$

Here we used the inequality $z \exp \{-\frac{1}{4}z\} \leq \text{const.} \exp \{-\frac{1}{8}z\}$ and (in the first integral) the substitution $|x - \xi| = \rho(\tau - t)^{\frac{1}{2}}$ (for each fixed t).

Similarly we obtain

$$(1.9) \quad |J_2| \leq A_3 H[f] (\xi - \xi')^\alpha.$$

To estimate J_3 we use the equality

$$g(\xi) - g(\xi') = \int_{\xi'}^{\xi} g'(\zeta) d\zeta \quad \text{with} \quad g = \frac{\partial^2 V_1}{\partial \xi^2}.$$

We then have

$$|J_3| \leq A_4 H[f] \int_{\xi'}^{\xi} d\zeta \int_0^{\tau-\gamma} \int_R V_4(x, 2t; \zeta, 2\tau) |x - \xi|^\alpha dx dt.$$

Using the inequality $|x - \xi|^\alpha \leq |x - \zeta|^\alpha + (\xi - \zeta)^\alpha$ and substituting $|x - \zeta| = \rho(\tau - t)^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
 (1.10) \quad |J_3| & \leq A_5 H[f] \int_{\xi'}^{\xi} d\zeta \int_0^{\tau} \frac{dt}{(\tau - t)^{\frac{1}{2}-\frac{1}{2}\alpha}} \\
 & \quad + A_5 H[f] \int_{\xi'}^{\xi} (\xi - \zeta)^\alpha d\zeta \int_0^{\tau-\gamma} \frac{dt}{(\tau - t)^{\frac{1}{2}}} \\
 & \leq A_6 H[f] \int_{\xi'}^{\xi} \frac{d\zeta}{\gamma^{\frac{1}{2}-\frac{1}{2}\alpha}} + 2A_5 H[f] \int_{\xi'}^{\xi} \frac{(\xi - \zeta)^\alpha}{\gamma^{\frac{1}{2}}} d\zeta \\
 & \leq A_7 H[f] (\xi - \xi')^\alpha.
 \end{aligned}$$

We now turn to w_1 . We write

$$\begin{aligned}
 (1.11) \quad w_1(\xi, \tau) - w_1(\xi', \tau) &= \sum \left\{ \pm \int_0^\tau \left[\frac{\partial}{\partial \xi} V_1(\pm 1, t; \xi, \tau) \right. \right. \\
 &\quad \left. \left. \mp \frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) \right] f(\xi, t) dt \right\} \\
 &+ \sum \left\{ \pm \int_{\tau-\gamma}^\tau \left[\frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) \right] [f(\xi, t) - f(\xi', t)] dt \right\} \\
 &+ \sum \left\{ \pm \int_0^{\tau-\gamma} \left[\frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) \right] [f(\xi, t) - f(\xi', t)] dt \right\} \\
 &= K_1 + K_2 + K_3
 \end{aligned}$$

where K_i denotes the i^{th} sum (each sum contains two terms corresponding to $+$ and $-$). Noting that in J_4 $\partial/\partial \xi'$ can be replaced by $-\partial/\partial x$ and using the divergence theorem, we obtain

$$(1.12) \quad K_3 + J_4 = 0.$$

It remains to estimate K_1 and K_2 . We have

$$|K_2| \leq A_8 H[f] \sum \int_{\tau-\gamma}^\tau |\pm 1 - \xi'| V_3(\pm 1, t; \xi', \tau) (\xi - \xi')^\alpha dt.$$

Substituting $z = (\pm 1 - \xi')^2/(\tau - t)$ we get

$$(1.13) \quad |K_2| \leq A_9 H[f] (\xi - \xi')^\alpha \int_0^\infty \frac{1}{z^{\frac{1}{2}}} \exp \left\{ -\frac{1}{2}z \right\} dz \leq A_{10} H[f] (\xi - \xi')^\alpha.$$

It remains to estimate K_1 . We write it in the form

$$\begin{aligned}
 (1.14) \quad K_1 &= \sum \left\{ \pm \int_0^\tau \left[\frac{\partial}{\partial \xi} V_1(\pm 1, t; \xi, \tau) \right. \right. \\
 &\quad \left. \left. - \frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) \right] [f(\xi, t) - f(\pm 1, t)] dt \right\} \\
 &+ \sum \left\{ \pm \int_0^\tau \left[\frac{\partial}{\partial \xi} V_1(\pm 1, t; \xi, \tau) \right. \right. \\
 &\quad \left. \left. - \frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) \right] f(\pm 1, t) dt \right\} = L_1 + L_2
 \end{aligned}$$

where L_i denotes the i^{th} sum. We proceed to estimate the L_i .

$$\begin{aligned}
 |L_1| &\leq A_{11} H[f] \sum \int_{\xi'}^\xi d\xi \\
 &\cdot \int_0^\tau [V_3(\pm 1, t; \xi, \tau) + (\pm 1 - \xi)^2 V_3(\pm 1, t; \xi, \tau)] |\pm 1 - \xi|^\alpha dt.
 \end{aligned}$$

Using the inequality $|\pm 1 - \xi|^\alpha \leq |\pm 1 - \zeta|^\alpha + (\xi - \zeta)^\alpha$ and substituting $z = (\pm 1 - \zeta)^2/(\tau - t)$, we obtain

$$|L_1| \leq A_{12}H[f] \sum \int_{\xi'}^{\xi} \left\{ \frac{1}{|\pm 1 - \zeta|^{1-\alpha}} + \frac{(\xi - \zeta)^\alpha}{|\pm 1 - \zeta|} \right\} d\zeta.$$

Noting that $0 < \xi - \zeta < |\pm 1 - \zeta|$ (here we use the assumption $0 < \xi' < \xi < 1$) we get

$$(1.15) \quad |L_1| \leq A_{12}H[f](\xi - \xi')^\alpha.$$

To evaluate L_2 we introduce the functions

$$g_\pm(t) = \begin{cases} f(\pm 1, t) & \text{if } 0 \leq t < 1 \\ 0 & \text{if } -\infty < t < 0. \end{cases}$$

We then can write

$$(1.16) \quad \begin{aligned} L_2 = \sum \left\{ \pm \int_{-\infty}^{\tau} \left[\frac{\partial}{\partial \xi} V_1(\pm 1, t; \xi, \tau) \right. \right. \\ \left. \left. - \frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) \right] [g_\pm(t) - g_\pm(\tau)] dt \right\} \\ + \sum \left\{ \pm g_\pm(\tau) \int_{-\infty}^{\tau} \left[\frac{\partial}{\partial \xi} V_1(\pm 1, t; \xi, \tau) \right. \right. \\ \left. \left. - \frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) \right] dt \right\} = L_{21} + L_{22}. \end{aligned}$$

We observe that

$$(1.17) \quad L_{22} = 0.$$

Indeed, (1.17) follows from

$$\begin{aligned} \int_{-\infty}^{\tau} \frac{\partial}{\partial \xi} V_1(\pm 1, t, \xi, \tau) dt - \int_{-\infty}^{\tau} \frac{\partial}{\partial \xi'} V_1(\pm 1, t; \xi', \tau) dt \\ = \int_{\xi'}^{\xi} d\zeta \int_{-\infty}^{\tau} \frac{\partial^2}{\partial \zeta^2} V_1(\pm 1, t; \zeta, \tau) dt \\ = - \int_{\xi'}^{\xi} d\zeta \int_{-\infty}^{\tau} \frac{\partial}{\partial t} V_1(\pm 1, t; \zeta, \tau) dt \\ = - \int_{\xi'}^{\xi} \left\{ \lim_{\substack{\epsilon \rightarrow 0 \\ B \rightarrow -\infty}} [V_1(\pm 1, t; \zeta, \tau)]_B^{-\epsilon} \right\} d\zeta = 0. \end{aligned}$$

To estimate L_{21} we make use of the assumption

$$f(\pm 1, 0) = 0$$

from which it follows that $g_{\pm}(t)$ is Hölder continuous (exponent α) in the interval $-\infty < t < 1$. (If $f(\pm 1, 0) \neq 0$, $g_{\pm}(t)$ will not be continuous at $t = 0$ and the inequalities which follow below will not hold near $\tau = 0$.) We thus obtain

$$|L_{21}| \leq A_{13}H[f] \sum \int_{\xi'}^{\xi} d\xi \cdot \int_{-\infty}^{\tau} [V_3(\pm 1, t; \xi, \tau) + (\pm 1 - \xi)^2 V_5(\pm 1, t; \xi, \tau)](\tau - t)^{\frac{1}{2}\alpha} dt.$$

Substituting $z = (\pm 1 - \xi)^2/(\tau - t)$, we obtain

$$(1.18) \quad |L_{21}| \leq A_{14}H[f] \sum \int_{\xi'}^{\xi} |\pm 1 - \xi|^{-1+\alpha} d\xi \leq A_{15}H[f](\xi - \xi')^{\alpha}.$$

Combining (1.18) with (1.17) and (1.16), we obtain

$$(1.19) \quad |L_2| \leq A_{15}H[f](\xi - \xi')^{\alpha}.$$

Combining (1.19) with (1.15), (1.14) and (1.13), we get

$$(1.20) \quad |K_1 + K_2| \leq A_{16}H[f](\xi - \xi')^{\alpha}.$$

Combining (1.20) with (1.12), (1.11), (1.10), (1.9), (1.8) and (1.7) and using (1.6), we finally get

$$(1.21) \quad \left| \frac{\partial}{\partial \tau} v(\xi, \tau) - \frac{\partial}{\partial \tau} v(\xi', t) \right| \leq A_{17}H[f](\xi - \xi')^{\alpha},$$

and the proof is thus completed.

We remark that in the case $\xi = \xi'$ we use the identity

$$(1.22) \quad \begin{aligned} 2\pi^{\frac{1}{2}}v_{\tau}(\xi, \tau) &= - \int_R V_1(x, 0; \xi, \tau)f(x, \tau) dx \\ &\quad - \iint_{S_{\tau}} \left[\frac{\partial}{\partial \tau} V_1(x, t; \xi, \tau) \right] [f(x, t) - f(x, \tau)] dx dt \end{aligned}$$

instead of (1.6) (see [8]).

Generalizations. Lemma 1 can easily be generalized to the case of n dimensions. Let R denote the n -dimensional rectangle $-\beta_i < x_i < \beta_i$ ($i = 1, \dots, n$) (β_i constants) and let S_{τ} ($S_1 = S$) denote the domain $-\beta_i < x_i < \beta_i$ ($i = 1, \dots, n$), $0 < t < \tau$. Let $f(x, t) = f(x_1, \dots, x_n, t)$ be Hölder continuous (exponent α) in $\tilde{S}$ in the sense of [5] (see (1.1)). We then have

Lemma 1'. Assume that $f(x_1, \dots, x_{n-1}, \pm \beta_n, 0) = 0$ and define

$$(1.23) \quad v(\xi, \tau) = \int_0^{\tau} \int_R V_n(x, t; \xi, \tau)f(x, t) dx dt$$

where V_i is defined by (1.2) (with $|x - \xi| = (\sum (x_i - \xi_i)^2)^{\frac{1}{2}}$). Let S_0 be any subdomain of S , each of its limit points lying either in S , or on one of the open

faces of S , or in the intersection of the closures of the faces $x_n = \pm\beta_n$, $t = 0$ of S . Then there exists a constant A such that for each pair of point (ξ, τ) , (ξ', τ') in S_0

$$(1.24) \quad \frac{|v_\tau(\xi, \tau) - v_{\tau'}(\xi', \tau')|}{(|\xi - \xi'|^2 + |\tau - \tau'|)^{\frac{1}{2}\alpha}} \leq AH[f];$$

A depends only on α , S_0 , $\min \{\beta_i\}$ and $\max \{\beta_i\}$.

2. AN AUXILIARY THEOREM

As in [5] we define the distance $d(P, Q)$ between two points $P = (x, t)$, $Q = (x', t')$ by

$$(2.1) \quad d(P, Q) = (|x - x'|^2 + |t - t'|)^{\frac{1}{2}}$$

where $|x - x'|$ is the usual Euclidean distance between $x = (x_1, \dots, x_n)$ and $x' = (x'_1, \dots, x'_n)$. Let $\xi^0 = (\xi_1^0, \dots, \xi_n^0)$ be a fixed point. Denote by R the n -dimensional domain $-d < x_i - \xi_i^0 < d$ ($i = 1, \dots, n-1$), $-d < x_n - \xi_n^0 < \bar{d}$ and by S_τ the topological product of R with the interval $0 < t < \tau$. We shall take $\bar{d} \leq d$ and $\tau \leq \tau^0 \leq d^2$. S_τ will also be denoted by S . The main result of the present section is the following theorem.

Theorem 1. Let $f(x, t)$ be defined and Hölder continuous (exponent α) in the closure $\bar{S}$ of S . Let $u(x, t)$ be defined in $\bar{S}$ and have (in $\bar{S}$) continuous x -derivatives up to the second order and one continuous t -derivative. Assume further that $u(x, t)$ satisfies in $\bar{S}$ the equation

$$(2.2) \quad \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} - \frac{\partial u}{\partial t} = f(x, t)$$

and vanishes on the faces $x_n = \bar{d}$ and $t = 0$ of S . Then $D_x^i u$ ($0 \leq i \leq 2$) is Hölder continuous (exponent α) in $\bar{S}$ and there exists a constant K depending only on α such that, for $i = 0, 1, 2$,

$$(2.3) \quad |D_x^i u(P)| \leq d^{-i} K \text{ l.u.b.}_S |u| + d^{2-i} K \text{ l.u.b.}_S |f| + d^{2-i+\alpha} H_{P,S}[f] \equiv KI,$$

$$(2.4) \quad d^\alpha \frac{|D_x^i u(P) - D_x^i u(Q)|}{d(P, Q)^\alpha} \leq KI + Kd^{2-i+\alpha} H[f] \quad (Q \in S, d(P, Q) < \frac{1}{4}d)$$

where $P = (\xi^0, \tau^0)$.

Here we used the notation of [5]:

$$H_{P,S}[f] = \text{l.u.b.}_{A \in S} \frac{|f(P) - f(A)|}{d(P, A)^\alpha}.$$

Proof. In what follows we shall make use of notations and results contained in [5]. If we use Green's identity with the functions $u(x, t)$ and $\varphi(x, t)\bar{G}(x, t; \xi, \tau)$, where φ is defined in [5] and

$$(2.5) \quad \bar{G}(x, t; \xi, \tau) = cV_n(x, t; \xi, \tau) - cV_n(x, t; \xi', \tau)$$

where V_i is defined by (1.2), c is an appropriate positive constant and ξ' is defined by

$$\xi'_i = \xi_i \quad (i = 1, \dots, n-1), \quad \xi'_n = 2\bar{d} - \xi_n,$$

then we obtain the following formula:

$$(2.6) \quad \begin{aligned} u(\xi, \tau) &= \int_0^\tau \int_R \varphi(x, t) f(x, t) \bar{G}(x, t; \xi, \tau) \, dx \, dt \\ &\quad - \int_0^\tau \int_R u(x, t) L^*[\varphi(x, t) \bar{G}(x, t; \xi, \tau)] \, dx \, dt = H_0 - J_0 \end{aligned}$$

where H_0 and J_0 denote the first and second integrals respectively. Here we used the fact that both u and G vanish on the face $x_n = \bar{d}$ of S and that u vanishes on the face $t = 0$ of S (which we denoted by R).

We now note that the function $\bar{G}(x, t; \xi, \tau)$ and its derivatives satisfy the inequalities

$$(2.7) \quad |D_t^{i_1} D_x^{i_2} D_\tau^{i_3} D_\xi^{i_4} \bar{G}(x, t; \xi, \tau)| \leq \text{const. } V_{n+2(i_1+i_2)+i_3+i_4}(x, 2t, \xi, 2\tau)$$

for any integers i_1, i_2, i_3, i_4 . In proving (2.7) we make use of the inequality $|x - \xi'| \geq |x - \xi|$ which holds for $x \in R$.

We also note that

$$(2.8) \quad \lim_{\epsilon \rightarrow 0} \int_0^{\tau-\epsilon} \int_R \frac{\partial}{\partial \tau} \bar{G}(x, t; \xi, \tau) \, dx \, dt \quad \text{exists.}$$

Indeed, noting that $\partial \bar{G} / \partial \tau = -\partial \bar{G} / \partial t$ and integrating by parts, we find that the integral in (2.8) is equal to

$$\int_R \bar{G}(x, 0; \xi, \tau) \, dx - \int_R \bar{G}(x, 0; \xi, \tau - \epsilon) \, dx,$$

and both integrals are absolutely and uniformly convergent (with respect to ϵ) as one can easily verify by substituting $|x - \xi| = \rho(\tau - t)^{\frac{1}{2}}$.

We finally note that

$$(2.9) \quad D_{\xi_i \xi_i}^2 \int_0^\tau \int_R \bar{G}(x, t; \xi, \tau) \, dx \, dt \quad \text{is bounded if } i \neq n.$$

The proof is similar to that in [5] (corresponding there to G) and will be omitted.

With the aid of (2.7), (2.8), (2.9) and on using the representation (2.6) we can now proceed to prove (2.3) almost word by word as in [5]. We remark, however, that instead of estimating $\partial^2 u(\xi, \tau) / \partial \xi_n^2$ we estimate $\partial u(\xi, \tau) / \partial \tau$ and, finally, estimate $\partial^2 u / \partial \tau^2$ from the estimates of the other derivatives and from (2.2).

Also (2.4), for $i = 0, 1$, can be proved as in [5] (and the $H[f]$ in (2.4) can even be replaced by $H_{q,s}[f]$). For $i = 2$ and for $D_x^2 \neq \partial^2 / \partial x_n^2$ the proof of (2.4) is still very similar to that of [5; Theorem 1]. Therefore it remains to prove (2.4)

with $D_x^2 = \partial^2/\partial x_n^2$ or (since u satisfies (2.2)) it remains to prove that

$$(2.10) \quad d^\alpha \frac{|D_\tau u(P) - D_\tau u(Q)|}{d(P, Q)^\alpha} \leq KI + Kd^\alpha H[f] \quad (d(P, Q) < \tfrac{1}{4}d).$$

This is the only point (for parabolic equations) which presents a complication which does not arise for elliptic equations.

Proof of (2.10). Differentiating (2.6) with respect to τ we obtain

$$(2.11) \quad D_\tau u(\xi, \tau) = D_\tau H_0 - D_\tau J_0 = H - J.$$

The estimation corresponding to J follows as in [5] and we obtain

$$(2.12) \quad d^\alpha \frac{|J(P) - J(Q)|}{d(P, Q)^\alpha} \leq Kd^{-2} \text{l.u.b.}_s |u|.$$

It remains to prove that

$$(2.13) \quad d^\alpha \frac{|H(P) - H(Q)|}{d(P, Q)^\alpha} \leq KI + Kd^\alpha H[f].$$

Denoting $\varphi(P)f(P)$ by $g(P)$ we shall prove that

$$(2.14) \quad d^\alpha \frac{|H(P) - H(Q)|}{d(P, Q)^\alpha} \leq Kd^\alpha H[g]$$

and this will complete the proof of (2.13) since

$$H[g] \leq (\text{l.u.b.}_s |\varphi|)H[f] + \frac{\text{const.}}{d^\alpha} \text{l.u.b.}_s |f|.$$

To prove (2.14) we write H in the form

$$(2.15) \quad H = \frac{\partial}{\partial \tau} \int_0^\tau \int_R g(x, t) c V_n(x, t; \xi, \tau) dx dt - \frac{\partial}{\partial \tau} \int_0^\tau \int_R g(x, t) c V_n(x, t; \xi', \tau) dx dt.$$

We now make the transformation $(x, t) \rightarrow (\bar{x}, \bar{t})$ where

$$(2.16) \quad x = d\bar{x}, \quad t = d^2\bar{t}.$$

This transformation transforms $V_n(x, t; \xi, \tau)$ into $V_n(\bar{x}, \bar{t}; \bar{\xi}, \bar{\tau})/d^n$. Since the edges of the transformed domain $\bar{R}$ ($R \rightarrow \bar{R}$) are of lengths between 1 and 2, we can apply Lemma 1' provided $\bar{g}$ satisfies the conditions imposed in that lemma; here $\bar{g}(\bar{x}, \bar{t}) = g(x, t)$. (Note that Lemma 1' remains true if ξ is replaced by $\bar{\xi}$, the reflection of ξ with respect to $x_n = \beta_n$.) To prove that $\bar{g}$ satisfies the conditions of Lemma 1' we first note that $\varphi(x, t)$ vanishes on all the lateral faces of S with the exception of the face $x_n = \bar{d}$. Hence (since $g = \varphi f$) it is enough to show that $f(x, t)$ vanishes on the intersection of the faces $x_n = \bar{d}$ and $t = 0$ of S .

The last statement follows by recalling that u has second continuous x -derivatives and one continuous t -derivative in $\bar{S}$ and that $u = 0$ on the faces $x_n = \bar{d}$ and $t = 0$ of S , and by using equation (2.2) (which is satisfied in $\bar{S}$).

Having proved that Lemma 1' can be applied, we obtain (on using it)

$$(2.17) \quad \frac{|\bar{H}(\bar{P}) - \bar{H}(\bar{Q})|}{d(\bar{P}, \bar{Q})^\alpha} \leq KH[\bar{g}] \quad (\bar{H}(\bar{P}) = H(P)).$$

Replacing in (2.17) $\bar{P}$ by P , $\bar{Q}$ by Q and $\bar{H}$ by H , (2.14) follows.

We shall now formulate a simple extension of Theorem 1.

Theorem 1'. *Theorem 1 remains true also in the case of the parabolic equation*

$$(2.18) \quad \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial u}{\partial t} = f(x, t),$$

where a_{ij} are constants and (a_{ij}) is a positive matrix. The constant K (in (2.3), (2.4)) depends on the smallest and largest (positive) eigenvalues of the matrix (a_{ij}) and on α .

Proof. Note first that Theorem 1 remains true if the n -dimensional rectangle R is replaced by an n -dimensional rectangle of the form

$$-d\alpha_i < x_i - \xi_i^0 < d\alpha_i \quad (i = 1, \dots, n-1), \quad -d\alpha_n < x_n - \xi_n^0 < \bar{d}\alpha_n$$

where α_i are arbitrary constants. Note also that if we rotate R and thus obtain a new domain R^* , Theorem 1 remains true with respect to the domain R^* . This follows by observing that an appropriate orthogonal transformation keeps both equation (2.2) and the distance function invariant while it transforms R^* into R .

The proof of Theorem 1' can now easily be completed. An appropriate mapping

$$(2.19) \quad x'_i = \sum_{j=1}^n g_{ij} x_j \quad (g_{ij} \text{ constants})$$

transforms equation (2.18) into (2.2) and R into R' . R' is an n -dimensional rectangle of the type R^* considered above. From the previous remarks it follows that we can apply Theorem 1 to the transformed equation in the domain R' . If we use the simple inequality

$$\text{const. } |x - y| \leq |x' - y'| \leq \text{const. } |x - y|$$

where y' is the image of y under (2.19), then the proof of Theorem 1' is completed.

3. SCHAUDER TYPE BOUNDARY ESTIMATES

Let D be a bounded $(n+1)$ -dimensional domain bounded by two planes $t = 0$, $t = t_1 > 0$ and a surface C (C is a closed set) between them. Denote by B the boundary of D on $t = 0$ and let $\partial D = B + C$. We shall assume throughout this section that C satisfies the following regularity property:

Property E. For every point P of C there exists an $(n + 1)$ -dimensional sphere V with center P such that the portion of C in V can be represented, for some i ($1 \leq i \leq n$), in the form

$$(3.1) \quad x_i = h(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n, t)$$

where h has Hölder continuous (exponent α) x -derivatives up to the second order and an Hölder continuous (exponent α) t -derivative of the first order.

Let T be a portion of ∂D (T is an open set). For any point $R = (\xi, \tau)$ in D let Δ_τ denote the subset of $\partial D - T$ which lies in the half space $t < \tau$. Denote by $\bar{d}_R$ the distance (in the sense of (2.1)) from R to the set Δ_τ . For two points R, Q we define $\bar{d}_{RQ} = \min(\bar{d}_R, \bar{d}_Q)$. We can now define norms analogous to those introduced in [5] by replacing d_R, d_{RQ} by $\bar{d}_R, \bar{d}_{RQ}$:

$$(3.2) \quad \bar{M}_{p,i}[g] = \text{l.u.b.}_{P \in D} \bar{d}_P^{p+i} |D_x^i g(P)|,$$

$$(3.3) \quad \bar{M}_{p,i+\alpha}[g] = \text{l.u.b.}_{P, Q \in D} \bar{d}_{PQ}^{p+i+\alpha} \frac{|D_x^i g(P) - D_x^i g(Q)|}{d(P, Q)^\alpha},$$

$$(3.4) \quad |\bar{g}|_{p,i} = \sum_{i=\max(0, -p)}^i \bar{M}_{p,i}[g],$$

$$(3.5) \quad |\bar{g}|_{p,i+\alpha} = |\bar{g}|_{p,i} + \sum_{i=\max(0, -p)} \bar{M}_{p,i+\alpha}[g].$$

Lemmas 1, 2 of [5] remain true for the new norms.

Let a function ψ be defined on T and assume that there exists a function Ψ defined in $D + T$ which satisfies the following properties:

$$(3.6) \quad \Psi = \psi \quad \text{on} \quad T,$$

$$(3.7) \quad |\Psi|^* = |\bar{\Psi}|_{0,2+\alpha} + \left| \frac{\partial}{\partial t} \bar{\Psi} \right|_{2,\alpha}$$

We then define

$$(3.8) \quad |\psi|_{2+\alpha}^T = \text{g.l.b.} |\Psi|^*;$$

the g.l.b. is taken with respect to all possible extensions Ψ of ψ .

We shall consider a second order equation of the form

$$(3.9) \quad Lu \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + c(x, t)u - \frac{\partial u}{\partial t} = f(x, t)$$

and make the following assumptions:

(A) The functions a_{ij} , b_i and c are Hölder continuous (exponent α) in $D + T$ and

$$(3.10) \quad |\bar{a}_{ij}|_{0,\alpha} \leq K_1, \quad |\bar{b}_i|_{1,\alpha} \leq K_1, \quad |\bar{c}|_{2,\alpha} \leq K_2.$$

(B) For any real vector ξ and for all $(x, t) \in \bar{D}$

$$(3.11) \quad \sum a_{ij}(x, t) \xi_i \xi_j \geq K_2 \sum \xi_i^2 \quad (K_2 > 0).$$

(C) The function $f(x, t)$ is Hölder continuous (exponent α) in $D + T$ and

$$(3.12) \quad |\bar{f}|_{2, \alpha} \text{ is finite.}$$

We can now state our main result concerning the boundary estimates.

Theorem 2. *Let u be a solution in D of equation (3.9) and assume that (A), (B), (C) hold. Assume that u has Hölder continuous (exponent α) x -derivatives in $D + T$ up to the second order and one continuous t -derivative in $D + T$, that $\text{l.u.b.}_D |u| < \infty$ and that $u = \psi$ on T and $|\psi|_{2+\alpha}^T$ is finite. Let T_0 is a portion of ∂D such that $\bar{T}_0 \subset T$ where $\bar{T}_0$ is the closure of the set T_0 . Then*

$$(3.13) \quad |\bar{u}|_{0, 2+\alpha}^0 \leq K(\text{l.u.b.}_D |u| + |\psi|_{2+\alpha}^T + |\bar{f}|_{2, \alpha})$$

where K depends only on K_1, K_2, T_0, T and D , and where $|\bar{u}|_{0, 2+\alpha}^0$ is the norm $|\bar{u}|_{0, 2+\alpha}$ defined with respect to T_0 instead of T .

Proof. Using [5; Theorem 3] we can easily conclude that it is enough to prove (3.13) in the intersection Ω of D with a sphere V whose center P lies on T_0 and whose radius r is arbitrarily small. There are three possibilities: Either P lies on the basis B of D , or P lies only on the boundary surface C of D , or P lies on the intersection of B and C . The first two cases may be considered as special cases of the third case, namely, $P \in B \cap C$. Hence we shall assume that $P \in B \cap C$.

We take r sufficiently small so that the representation (3.1) is valid for $T \cap C \cap V$ and so that $T \cap V = \partial D \cap V$. For simplicity we assume that $i = n$ in (3.1). The transformation

$$(3.14) \quad \begin{aligned} t' &= t, & x'_i &= x_i & (i = 1, \dots, n-1) \\ x'_n &= x_n - h(x_1, \dots, x_{n-1}, t) \end{aligned}$$

maps $T \cap C \cap V$ onto a set which lies on $x'_n = 0$ and, as we may assume, it maps Ω onto a domain Ω' for which $x'_n > 0$; also $t' > 0$. The mapping (3.14) is one-to-one. Moreover,

$$(3.15) \quad \text{const. } d(P', Q') \leq d(P, Q) \leq \text{const. } d(P', Q')$$

since $\partial h / \partial x_i$ and $\partial h / \partial t$ are bounded functions.

Define $v(x', t') = u(x, t)$, $\varphi(x', t') = \psi(x, t)$, $\Phi(x', t') = \Psi(x, t)$. Then v assumes the boundary values of Φ on the boundaries $x'_n = 0$ and $t' = 0$ of Ω' . Equation (3.9) is transformed into the equation

$$(3.16) \quad \begin{aligned} L'v &\equiv \sum a'_{ii}(x', t') \frac{\partial^2 v}{\partial x'_i \partial x'_i} \\ &+ \sum b'_i(x', t') \frac{\partial v}{\partial x'_i} + c(x', t')v - \frac{\partial v}{\partial t'} = f'(x', t') \end{aligned}$$

where $c'(x', t') = c(x, t)$, $f'(x', t') = f(x, t)$, $a'_{\lambda\mu}(x', t')$ is easily expressed in terms of $a_{ii}(x, t)$, $\partial h / \partial x_i$, $\partial^2 h / \partial x_i \partial x_i$, and $b'_i(x', t')$ is easily expressed in terms of $b_i(x, t)$, $\partial h / \partial x_i$, $\partial h / \partial t$. Using the regularity properties of h (see the definition

of Property E) and (3.15) we conclude that the coefficients of L' satisfy the assumption (A) with K'_1 which depends on K_1 and h . Clearly also (B) is valid with K'_2 depending on K_2 and h . Finally, since $f'(x', t') = f(x, t)$, (C) is also valid for f' .

From (3.15) we further conclude that it is enough to prove (3.13) for the equation (3.16) in the domain Ω' . Denoting

$$(3.17) \quad w(x', t') = v(x', t') - \Phi(x', t')$$

we find that w satisfies the system

$$(3.18) \quad L'w = f'',$$

$$(3.19) \quad w(x', t') = 0$$

on the boundary of Ω' which lies on $t' = 0$ and on $x'_n = 0$, where $f'' = f' - L'\Phi$. We have

$$(3.20) \quad |\bar{f}''|_{2,\alpha}^{\Omega'} \leq \text{const. } |\bar{f}|_{2,\alpha} + \text{const. } |\Psi|^*,$$

the constants depending on K_1 , T and T_0 .

To the system (3.18), (3.19) we can apply (on using Theorem 1) the proof of [5; Theorem 3] with d_p , $|\cdot|_{i,\alpha}$, etc. replaced by $\bar{d}_p$, $|\cdot|_{i,\alpha}$, etc. Since that proof can be translated to the present situation almost word by word, we shall omit the details. We only remark that [5; Theorem 1] is slightly stronger than Theorem 1' since (in [5]) in the analogue of (2.4) $H[f]$ is replaced by $H_{Q,s}[f]$. This, however, makes no difference in the proof of our theorem as one can easily see. (Note that $H[f] = \text{l.u.b. } H_{Q,s}[f]$; consequently, an estimate on $H_{Q,s}[f]$ independent of Q in S leads to an estimate on $H[f]$.)

Applying the proof of [5; Theorem 3] we get the estimate

$$(3.21) \quad |\bar{w}|_{\partial,2+\alpha}^{\Omega'} \leq K(\text{l.u.b. } |w| + |\bar{f}''|_{2,\alpha}^{\Omega'})$$

where K depends only on K'_1 , K'_2 and on the domain Ω' . Using (3.17) and (3.20), the proof of Theorem 2 is completed.

We shall use the following notation:

$$(3.22) \quad |v|_{\alpha} \equiv |v|_{\alpha}^D \equiv |\bar{v}|_{0,\alpha} \equiv \text{l.u.b.}_D |v| + H[v] \quad \text{if } T = \partial D,$$

$$(3.23) \quad |v|_{2+\alpha} \equiv |v|_{2+\alpha}^D \equiv \text{l.u.b.}_D \left(\sum_{i=0}^2 |D_x^i v| + |D_x v| \right) + \sum_{i=0}^2 H[D_x^i v] + H[D_x v],$$

$$(3.24) \quad |\psi|_{2+\alpha} \equiv |\psi|_{2+\alpha}^{\partial D} = \text{g.l.b. } |\Psi|_{2+\alpha}^D$$

where the g.l.b. in (3.24) is taken over all Ψ such that $\Psi = \psi$ on ∂D .

We now remark that the proof of Theorem 3 remains true if $T_0 = T = \partial D$.

If we make use of this remark and the maximum principle for parabolic equations [10] (from which it follows that if $Lu = f$ and $u = 0$ on ∂D then

$$\text{l.u.b.}_D |u| \leq \text{const. } (\text{l.u.b.}_D |f|)$$

then we obtain the following theorem.

Theorem 3. Let D be a bounded domain with the lateral boundary C satisfying Property E. Let u be a solution of (3.9) in $\bar{D}$ with boundary values ψ on ∂D and let $|\psi|_{2+\alpha}$ be finite. Assume that L and f satisfy the assumptions (A), (B), (C) (with $T = \partial D$) and that u has Hölder continuous (exponent α) x -derivatives of the second order in $\bar{D}$. Then

$$(3.25) \quad |u|_{2+\alpha} \leq K(|\psi|_{2+\alpha} + |f|_\alpha)$$

where K depends only on K_1 , K_2 and the domain D .

More precisely, the assumption (A) with $T = \partial D$ means

$$|a_{ii}|_\alpha \leq K_1, \quad |b_i|_\alpha \leq K_1, \quad |c|_\alpha \leq K_1,$$

and, similarly, the assumption (C) with $T = \partial D$ means that $|f|_\alpha$ is finite.

4. EXISTENCE THEOREMS FOR LINEAR EQUATIONS

4.1. As a first application of Theorem 3 we shall prove the following theorem.

Theorem 4. Let D be a bounded domain bounded by hyperplanes $t = 0$, $t = t_1 > 0$ and a surface C satisfying Property E. Let ψ be a function defined on ∂D and let $|\psi|_{2+\alpha}$ be finite. Let L satisfy the assumptions (A), (B) (with $T = \partial D$) and let $|f|_\alpha$ be finite. Then there exists one and only one solution of the equation $Lu = f$ which assumes the boundary values ψ on ∂D .

Proof. The proof makes use of a continuity argument used by SCHAUDER [13] which is, by now, well known (see also [11]). It will be enough to sketch this part of the proof. In the first place, it is enough to consider only the case in which $\psi \equiv 0$. We now define Banach spaces $C_{2+\alpha}$ and C_α of all functions v with norms $|v|_{2+\alpha}$ and $|v|_\alpha$ respectively. Clearly $C_{2+\alpha} \subset C_\alpha$. Write

$$L_\mu = \mu L + (1 - \mu)L_0$$

where

$$(4.1) \quad L_0 = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \frac{\partial}{\partial t}$$

and consider the set N of values μ ($0 \leq \mu \leq 1$) for which the transformation L_μ maps $C_{2+\alpha}$ onto C_α . L_μ is a bounded transformation and, by (3.25), it has a bounded inverse. A simple argument involving the Neumann's power-series expansion shows that N is an open set (see also [11; pp. 69–65]). On the other hand, using (3.25) we conclude that N is a closed set. Therefore if N contains $\mu = 0$ then it contains $\mu = 1$ and the proof of Theorem 4 is completed.

Proof that N contains $\mu = 0$. The existence of a solution u of the equation $L_0 u = f$ with zero boundary values is well known (see the proof of Theorem 5). It remains to show that $|u|_{2+\alpha} < \infty$. Since $|u|_{2+\alpha}^G < \infty$ for every rectangular subdomain G which lies in the interior of D [5; Theorem 1], it remains to consider the behavior of u near the boundary. It will be enough to consider the behavior of u

near a portion T of ∂D which contains a point P which lies both on C (the lateral boundary of D) and on B (the basis of D). We shall first assume that the function h in (3.14) is such that h has continuous derivatives of the fifth order with respect to all its arguments.

Making the transformation (3.14) and denoting $u(x, t) = v(x', t')$, we obtain for v the equation $L'_0 v = f$. A certain one-sided neighborhood of T is transformed into a rectangular domain B' , whose edges are parallel to the coordinate axes (x'_i and t'), two of its faces being $t' = 0$ and $x'_n = 0$ (B was assumed to lie on the plane $t = 0$). On these two faces $v(x', t')$ vanishes and we have to study the behavior of $v(x', t')$ near these two faces.

Because of the regularity assumptions made on the function h , the coefficients of L'_0 are such that Green's function for the adjoint of L'_0 , constructed in [7], exists in B' . Denote it by $G(x', t'; \xi', \tau')$. If we show that $v(x', t')$ has Hölder continuous (exponent α) x' -derivatives of the second order near the faces $x'_n = 0$ and $t' = 0$ then the proof that N contains $\mu = 0$ can be easily completed.

We shall prove that

$$(4.2) \quad v(\xi', \tau') = \int_0^{\tau'} \int_R f'(x', t') G(x', t'; \xi', \tau') dx' dt' + \text{boundary integrals};$$

the boundary integrals are of the form

$$\int_{S_i} v a' \frac{\partial G}{\partial \nu} dS$$

where S_i is a face of B' , a' is an appropriate coefficient of L'_0 and ν is the normal to S_i . The precise form of the boundary integrals can be obtained on using Green's formula, assuming that v is continuously differentiable in the closure of S (see [7]). It is important to note that the boundary integrals taken over the faces $x'_n = 0$ and $t' = 0$ of B' do not appear since $v = 0$ on these faces. To prove (4.2) we denote the volume integral in (4.2) by H_0 . Since f' is Hölder continuous, it follows from [3; Theorems 1, 2] that $L'_0 H_0 = f'$. Furthermore, from [7] it follows that the boundary integrals in (4.2) are solutions of $L'_0 w = 0$ and satisfy the same boundary conditions as v . Since H_0 vanishes on R and on the lateral faces of B' (see [7]), we can conclude, on using the maximum principle [10], that both sides of (4.2) are equal.

We now can show that the second x -derivatives and the first t -derivatives of H_0 (whose explicit form can be given on using [3; Theorems 1, 2]) are continuous functions in the closure of B' . Since the same statement is true for boundary integrals of (4.2) near the faces $x'_n = 0$ and $t' = 0$, it is also true (by (4.2)) for $v(x', t')$ near these two faces of B' .

Using the last result in the equation $L'_0 v = f'$, we conclude that f' vanishes at the points which lie on both faces $x'_n = 0$ and $t' = 0$ of B' . If we show that Lemma 1' is valid with $V_n(x, t; \xi, \tau)$ replaced by $G(x, t; \xi, \tau)$, then our proof can be completed in the following way: In the first place it follows that H_0 has Hölder continuous (exponent α) second x' -derivatives in the closure of B' . We

next notice, by using the regularity properties of the coefficients of L'_0 , the method of EIDELMAN to prove differentiability of fundamental solutions [4; in particular, formulas (2.14_m), (2.39)] and the explicit formula of G in terms of fundamental solutions [7], that $\partial G/\partial \xi'$ has Hölder continuous (exponent α) second x' -derivatives in every closed subdomain of B' where $|x' - \xi'|^2 + |t' - \tau'| > 0$. Hence, a similar statement is true for the boundary integrals in (4.2) in a neighborhood of the faces $x'_n = 0$ and $t' = 0$; this completes the proof that $v(x', t')$ has Hölder continuous (exponent α) x' -derivatives of the second order near the faces $x'_n = 0$ and $t' = 0$ of B' .

It remains to prove Lemma 1' with $V_n(x, t; \xi, \tau)$ replaced by $G(x, t; \xi, \tau)$.

For simplicity let us take $n = 1$. Then the last statement can be proved following step by step the proof of Lemma 1 and making the necessary modifications due to the fact that

$$(4.3) \quad 2\pi^{\frac{1}{2}}G(x, t; \xi, \tau) = \frac{1}{(\tau - t)^{\frac{1}{2}}} \exp \left\{ -\frac{(x - \xi)^2}{4(\tau - t)} \right\} \\ - \frac{1}{(\tau - t)^{\frac{1}{2}}} \exp \left\{ -\frac{(x - \bar{\xi})^2}{4(\tau - t)} \right\} + O(V_0(x, t; \xi, \tau))$$

where $\bar{\xi}$ is the reflection of ξ with respect to the plane $x = 0$. In the case of n dimensions the first term (on the right side of (4.3)), for instance, is replaced by

$$\frac{k(\xi, \tau)}{(\tau - t)^{\frac{1}{2}}} \exp \left\{ -\frac{\sum A_{ij}(\xi, \tau)(x_i - \xi_i)(x_j - \xi_j)}{4(\tau - t)} \right\}$$

where $k(\xi, \tau)$ is twice continuously ξ -differentiable and once continuously τ -differentiable, all these derivatives satisfying the Hölder condition (exponent α), and where $(A_{ij}(\xi', \tau'))$ is the matrix inverse to the matrix of the coefficients of the second derivatives which appear in L'_0 (see [7]).

Having proved that $|u|_{2+\alpha}$ is finite under the additional restrictions (made above) on h , we proceed to prove that $|u|_{2+\alpha}$ is finite if C is only assumed to satisfy Property E.

Approximate C by smooth surfaces C_m such that the corresponding domains D_m satisfy

$$D_m \supset D_{m+1}, \quad D_m \rightarrow D.$$

Extend $f(x, t)$ to a domain D^* which contains all the domains D_m in such a manner that $|f|_{\alpha}^{D^*}$ is finite [9]. From what we have already proved it follows that the system

$$L_0 u_m = f \quad \text{in } D_m, \quad u_m = 0 \quad \text{on } \partial D_m$$

has a solution u_m which satisfies

$$|u_m|_{2+\alpha}^{D_m} \leq |u_m|_{2+\alpha}^{D_m} \leq K_m |f|_{\alpha}^{D_m} \leq K_m |f|_{\alpha}^{D^*}.$$

We can choose C_m such that the corresponding functions h_m will satisfy

$$\sum_{i=0}^2 \{ \text{l.u.b. } |D_x^i h_m| + H[D_x^i h_m] \} + \text{l.u.b. } |D_t h_m| + H[D_t h_m] \leq A$$

where A is a constant independent of m . From the proof of Theorem 2 it then follows that K_m is bounded by a constant independent of m . A simple compactness argument proves the convergence of a subsequence of $\{u_m\}$ to a solution u of $L_0 u = f$ which vanishes on ∂D and for which $|u|_{2+\alpha}^D$ is finite.

Remark. Our proof that N contains $\mu = 0$ does not use the Schauder type interior and boundary estimates but it makes use of Green's function for L in rectangular domains [7]. However, if one wants to make use of those estimates, then it is enough to employ Green's function for L_0 in rectangular domains. Indeed, one first proves Theorem 4 for rectangular domains by the method we used to prove Theorem 4 and uses only Green's function for L_0 in order to conclude that N contains $\mu = 0$. Then one proves Theorem 4 for general domains, using part of the method we used to prove Theorem 4 but applying the result obtained for rectangular domains in order to conclude that N contains $\mu = 0$. This is essentially the method used by BARRAR [0] in proving a theorem similar to Theorem 4.

4.2. We shall now extend Theorem 4 to domains D with C satisfying essentially weaker conditions than Property E and for any continuous boundary values. The method is analogous to that of NIRENBERG [11; pp. 65–68] for elliptic equations. This method is also based on a continuity argument, but it involves the norms $|u|_{0,\alpha}$ (introduced in [5]) and

$$(4.4) \quad |u|_{0,2+\alpha}^* = |u|_{0,2+\alpha} + |D_t u|_{2,\alpha}.$$

We denote the corresponding Banach spaces by E_α and $E_{2+\alpha}$ respectively.

Definition. For each point Q of ∂D , a function w_Q is called a *strong barrier* if it is defined and continuous in $\bar{D}$, if it has continuous second derivatives in D , if $w_Q \geq 0$ in $\bar{D}$ and $w_Q = 0$ at Q and, finally, if $L w_Q \leq -1$ and $L_0 w_Q \leq -1$, where L_0 is defined by (4.1).

If $Q = (x^0, 0)$ then we can take

$$w_Q(x, t) = |x - x^0|^2 + A t$$

with A sufficiently large. If $Q = (x^0, t^0)$ belongs to C and there exists a sphere S such that S and $\bar{D}$ have Q as their only common point then, denoting by (x', t') the center of S and assuming that $x' \neq x^0$, we can take

$$w_Q(x, t) = \frac{1}{R_0^{2k}} - \frac{1}{R^{2k}}$$

where $R^2 = |x - x'|^2 + (t - t')^2$ and $R_0^2 = |x^0 - x'|^2 + (t^0 - t')^2$ and k is sufficiently large.

Other strong barriers were constructed in [7].

We can now formulate an existence theorem which extends Theorem 4.

Theorem 5. Let D be bounded by the hyperplane $t = 0$, $t = t_1 > 0$ and a surface C with the following property: For each point Q of C there exists a sphere S with center P such that the line PQ is not parallel to the t -axis and such that Q is the only common point of S and $\bar{D}$. Let f and L satisfy (A), (B), (C) with T replaced by the empty set. Let ψ be a continuous function defined on ∂D . Then there exists one and only one solution of the equation $Lu = f$ which is continuous in $\bar{D}$ and equal to ψ on ∂D .

Proof. Without loss of generality we may assume $c(x, t) \leq 0$. We first consider the case $\psi \equiv 0$. Define the operator

$$L_\mu = \mu L + (1 - \mu)L_0$$

where L_0 is defined by (4.1) and consider the system

$$(4.5) \quad L_\mu u = f, \quad u = 0 \quad \text{on} \quad \partial D.$$

From the interior estimates of [5] we have, for a function u satisfying (4.5),

$$|u|_{\delta, 2+\alpha}^* \leq K'(\text{l.u.b.}_D |u| + |f|_{0, \alpha}).$$

Using the maximum principle [10] we get

$$(4.6) \quad |u|_{\delta, 2+\alpha}^* \leq K'' |f|_{0, \alpha}.$$

Denote by N the set of numbers μ , $0 \leq \mu \leq 1$, with the following property: For each f with finite norm $|f|_{0, \alpha}$ the equation (4.5) has a solution u . The proof of Theorem 5 for $\psi \equiv 0$ will be completed if we prove the following statements:

- (a) $\mu = 0$ belongs to N ,
- (b) N is an open set,
- (c) N is a closed set.

The proof of (a) is easy. The function

$$v(\xi, \tau) = c \iint_{D_\tau} \frac{1}{(\tau - t)^{\frac{1}{2}n}} \exp \left\{ -\frac{|x - \xi|^2}{4(\tau - t)} \right\} f(x, t) d\xi d\tau,$$

where c is an appropriate number and $D_\tau = D \cap \{t < \tau\}$, is a solution of $L_0 v = f$ (see, for instance, [8] or [3]). Let w be a solution of $L_0 w = 0$ with the boundary values $-v$ on ∂D ; its existence can be established in various ways (see [8] or see [7] where the Perron method is employed). The function $u = v + w$ is the desired solution of (4.5) with $\mu = 0$.

The proof of (b) follows as in the case of Theorem 4 and it makes use of (4.6) (see also [11; pp. 64–65]); it will be omitted.

The decisive step is the proof of (c). Suppose a sequence $\{\mu_m\}$ converges to μ and $\mu_m \in N$. We shall prove that $\mu \in N$.

Using the interior estimates [5] we conclude that $\{u_m\}$ ($L_{\mu_m} u_m = f$, $u_m = 0$ on ∂D) converges in D to a solution u of $Lu = f$. It remains to show that $u(P) \rightarrow 0$ as $P \rightarrow Q$ ($P \in D$), for any point Q of ∂D .

We now make use of the strong barriers w_Q defined and constructed above. Since $L_0 w_Q \leq -1$, $L w_Q \leq -1$ the function $v = (\text{l.u.b. } |f|) w_Q$ satisfies

$$L_{\mu_m} v \leq -1. \text{u.b. } |f|.$$

Hence the maximum principle yields

$$|u_m(P)| \leq (1. \text{u.b. } |f|) w_Q(P).$$

Taking $m \rightarrow \infty$ we get

$$|u(P)| \leq (1. \text{u.b. } |f|) w_Q(P).$$

Taking $P \rightarrow Q$, $P \in D$, we conclude that u is continuous in $\bar{D}$ and $u = 0$ on ∂D .

Having completed the proof of Theorem 5 for the case $\psi \equiv 0$ we can easily extend it to the case of boundary values ψ with $|\psi|_{2+\alpha} < \infty$.

Consider now the case that ψ is only continuous on ∂D . Extend ψ by a function Ψ continuous in $\bar{D}$. Approximate Ψ uniformly in $\bar{D}$ by polynomials Ψ_m . Let u_m be the solution of $Lu_m = f$ which coincides on the boundary ∂D with Ψ_m . From the interior estimates [5] it follows that the sequence $\{u_m\}$ converges in closed subsets of D to a solution u of $Lu = f$. From the maximum principle [10] it follows that the sequence $\{u_m\}$ converges uniformly in $\bar{D}$. Thus u is continuous in $\bar{D}$ and, consequently, $u = \psi$ on ∂D ; the proof of Theorem 5 is thus completed.

Added in proof. The proof of Theorem 5 can be simplified, following the pattern of the elliptic case in [11; revised issue]. Let $\{D_m\}$ be a sequence of subdomains of D which converges to D . Let u_m be the solution of $Lu_m = f$ with zero boundary values on ∂D_m . As before, $\lim u_m = u$ and $Lu = f$. By the maximum principle, $|u_m| \leq (1. \text{u.b. } |f|) |w_Q|$, from which the proof is easily completed.

Note that, using the last proof, it is enough to assume in the definition of a strong barrier that $Lw_Q \leq -1$ (and not $Lw_Q \leq -1$, $L_Q w_Q \leq -1$).

5. EXISTENCE THEOREMS FOR NONLINEAR EQUATIONS

In this section we shall consider nonlinear equations of the form

$$(5.1) \quad Lu = f(x, t, u).$$

Theorem 6. Assume that L satisfies the assumptions (A), (B) (with $T = \partial D$) and that the lateral boundary C of D possesses Property E. Assume that f is defined and continuously differentiable in the cylinder

$$(5.2) \quad (x, t) \in \bar{D}, \quad -\infty < u < \infty,$$

and satisfies, for every pair of points $\pi^i = (x^i, t^i, u^i)$ ($i = 1, 2$) in that cylinder (w^i is $(n+1)$ -dimensional),

$$(5.3) \quad |f(\pi^1) - f(\pi^2)| \leq A(|x^1 - x^2|^2 + |t^1 - t^2|)^{\frac{1}{2}\alpha} + \lambda |u^1 - u^2|$$

where A and λ are positive constants and $0 < \alpha < 1$. There exists a constant K depending only on L and D such that if

$$(5.4) \quad K\lambda < 1$$

then there exists a solution u of the equation $Lu = f$ in D which vanishes on ∂D .

Proof. Let M be a positive number and denote by C_M the set of functions u for which $|u|_\alpha \leq M$. For each $u \in C_M$ define a transformation $v = Tu$ in the following way: v vanishes on ∂D and satisfies in D the equation

$$(5.5) \quad Lv = f(x, t, u).$$

The existence and uniqueness of v follows from Theorem 4. Moreover, on using Theorem 3 (formula (3.25)) and the assumption (5.3), we obtain

$$(5.6) \quad |v|_{2+\alpha} \leq KA + K\lambda |u|_\alpha^p \leq KA + KM\lambda.$$

Therefore if (5.4) holds and if M is sufficiently large, then $|v|_{2+\alpha}^p \leq M$ and the transformation $v = Tu$ maps C_M into itself. Moreover, since the set $\{Tu \mid u \in C_M\}$ is bounded in the norm $|\cdot|_{2+\alpha}$, it is necessarily compact in the norm $|\cdot|_\alpha$. C_M is a closed convex set in the norm $|\cdot|_{1+\alpha}$ and $v = Tu$ is a continuous transformation on it ($|u - u_m|_\alpha \rightarrow 0$ implies $|Tu - Tu_m|_\alpha \rightarrow 0$; this follows with the aid of the maximum principle [10] or on using Theorem 3). From these remarks it follows that one can apply a fixed point theorem due to SCHAUDER [12] and thus conclude that there exists a function u in C_α such that $Tu = u$; the proof of Theorem 6 is thus completed.

By slightly modifying the proof of Theorem 6 we can prove the following result.

Theorem 6'. *Theorem 6 is true if in the assumption (5.3) $|u^1 - u^2|$ is replaced by $|u^1 - u^2|^\epsilon$, where ϵ is any positive number smaller than 1; the λ in (5.3) may then be taken arbitrarily large (it need not satisfy (5.4)).*

Remark. In a forthcoming paper [6] we shall consider the existence of solutions of the equation (5.1) under very weak assumptions on f .

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